Math 131 - Topology I: Topological Spaces and the Fundamental Group

Taught by Clifford Taubes
Notes by Dongryul Kim

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In 2015, Math 131 was taught by Professor Clifford Taubes. We met on Mondays, Wednesdays, and Fridays from 12:00 to 1:00 every week, and used *Topology* by James Mukres as a textbook. There were projects instead of exams; we had to write notes for hypothetical one-hour lectures we would give to the class.

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Topology is the theory of shapes and relationships.

1.1 Definition

Definition 1.1. Let X be a set. A **topology** on X is a distinguished collection $\mathcal{T}$ of subsets of X such that

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$,
2. any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$,
3. and any intersection of finitely many elements from $\mathcal{T}$ is in $\mathcal{T}$.

Example 1.2. *The set of open sets in $\mathbb{R}^n$ is a topology.*

What is an open set?

Definition 1.3. A set A in $\mathbb{R}^n$ is **open** if for each $p \in A$ there exists $r > 0$ so that the open disk $\{x : |x - p| < r\}$ is in A.

It is easy to check that the first two conditions are satisfied, and you can check the third one.

1.2 Basis of a topology

Definition 1.4. $\mathcal{B}$ is a **basis** for a topology

1. if $p \in X$, then $B \in \mathcal{B}$ and $p \in B$ for some B,
2. and if $p \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then exists $B_3 \in \mathcal{B}$ so that $p \in B_3$ and $B_3 \subset B_1 \cap B_2$.

From a basis $\mathcal{B}$, we can make up a topology as follows: Let a set A be open if for each $p \in A$, there is a $B \in \mathcal{B}$ for which $p \in B$ and $B \subset A$. Basically the open sets in this topology are those which can be represented as a union of some elements in $\mathcal{B}$.

Example 1.5 (Zariski topology)**.** *Take any finite $\{p_1, \dots, p_n\}$ and let $B = \mathbb{R} \setminus \{p_1, \dots, p_n\}$. These kind of sets whose complement if finite form a topology on $\mathbb{R}$. This is called the Zariski topology on $\mathbb{R}$.*

1.3 An application: Knot theory

Trefoil is not a knot, i.e. an unknot. You can also try to change a trefoil to the mirror image of itself, but it would not be easy. And it also would not be easy to prove that it is impossible.

How do we understand knots mathematically? This is where topology comes in. If K is a knot in $\mathbb{R}^3$, the topology of $\mathbb{R}^3 \setminus K$ does not change. (We will define later what "not changing" means.) In fact, this connects with graph theory in combinatorics, and other areas in algebra.

2 September 4, 2015

Today, we will be playing with topologies. Let us recall what a topology is.

For a set X, a topology $\mathcal{T}$ is a distinguished collection of subsets, and these subsets are called open sets. There are three requirements.

- $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
- Any union of subsets from $\mathcal{T}$ is also in $\mathcal{T}$.
- Finite intersections of subsets from $\mathcal{T}$ are in $\mathcal{T}$.

The **standard topology** on $\mathbb{R}^n$ is given as the following: a set A is open if for each $p \in A$, there exists a $r > 0$ such that $\{x : |x - p| < r\}$ is entirely in A.

A collection of subsets $\mathcal{B}$ is a basis for a topology if

- For any $p \in X$, there exists a $B \in \mathcal{B}$ with $p \in B$.
- For any $B_1, B_2 \in \mathcal{B}$ with $p \in B_1 \cap B_2$, there exists another $B_3 \in \mathcal{B}$ with $p \in B_3$.

In the topology $\mathcal{T}$ generated by $\mathcal{B}$, a set A would be open if for any $p \in A$, there exists $B \in \mathcal{B}$ with $p \in B$ and $B \subset A$. You can check that these open sets actually forms a topology.

2.1 Some examples

Example 2.1. *Consider the real line $\mathbb{R}$. The basis for the standard topology is $\mathcal{B} = \{(a, b) : a < b\}$.The open sets are those whose each point in the set has a whole interval around in the set.*

Example 2.2. *There is also the **lower limit topology** on $\mathbb{R}$. Consider the topology generated by the basis $\mathcal{B} = \{[a, b) : a < b\}$. It is possible to check that if two basis element have nonempty intersection, the intersection is again an element of the basis.*

In this topology, a set A is open if, given any $p \in A$, there is an interval $[a, b)$ containing p and $[a, b) \subset A$.

Example 2.3. *There is also an **upper limit topology** . A basis for this topology is $\mathcal{B} = \{(a, b] : a < b\}$.*

Example 2.4. *What about the upper & lower limit topology? Let's check if $\mathcal{B} = \{[a, b] : a < b\}$ is a basis. Consider two basis elements $[0, 2]$ and $[2, 4]$. The intersection is $[0, 2] \cap [2, 4] = \{2\}$, but this does not contain any basis element. So there is no upper & lower limit topology.*

Example 2.5. *I'm going to give two bases on the plane. The first one is*

$$\mathcal{B}_1 = \{\{x : |x - p| < r\} : p \in \mathbb{R}^2 \textit{ and } r > 0\},$$

and the second one is

$$\mathcal{B}_2 = \{(a, a + r) \times (c, c + r) : a, c \in \mathbb{R} \textit{ and } r > 0\}.$$

Using pictures, it is possible to check that both are bases. The question is: do they generate the same topology?

You can see that the answer is yes, using the following lemma.

Lemma 2.6. *Two bases $\mathcal{B}$ and $\mathcal{B}'$ give the same topology when:*

- *If $B' \in \mathcal{B}'$ and $p \in B'$, then there is a $B \in \mathcal{B}$ with $p \in B$ and $B \subset \mathcal{B}'$.*
- *If $B \in \mathcal{B}$ and $p \in B$, then there is a $B' \in \mathcal{B}'$ with $p \in B'$ and $B' \subset \mathcal{B}$.*

The first condition actually is saying that every open set in the set generated by $\mathcal{B}'$ is also open in the topology generated by $\mathcal{B}$. If this is the case, we say that the topology generated by $\mathcal{B}$ is **finer** than the topology generated by $\mathcal{B}'$.

Example 2.7. *The lower limit topology and the upper limit topology are finer that the standard topology on $\mathbb{R}$.*

As we have seen, the upper & lower topology did not work. So, let's introduce a new definition.

2.2 Subbasis of a topology

Definition 2.8. A **subbasis** S can be any collection of subsets. You can generate a topology $\mathcal{T}$ from S, first by adding X and $\emptyset$, and then adding any unions and finite intersections to the collection of open sets.

By this new definition, the upper & lower topology can be resurrected. The problem was the intersection $[a, b] \cap [b, c] = \{b\}$. So we can allow all one-point sets to the basis to form a new basis

$$\mathcal{B}' = \{[a, b] : a \leq b\}.$$

But actually, the topology generated by this basis is the set of all subsets of $\mathbb{R}$, which is not so useful.

3 September 9, 2015

There are some ways to make new topologies from old topologies.

3.1 Product topology

For two sets X and Y, the Cartesian product $X \times Y$ is

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

For example, $\mathbb{R} \times \mathbb{R}$ is the 2-dimensional Euclidean space. The n-dimensional Euclidean space is defined as $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. You can even think spaces like $S^1 \times S^1$.

Let's define a topology on the product

Definition 3.1. For two topological spaces X and Y, the **product topology** on $X \times Y$ is defined as the topology generated by the basis

$$\mathbb{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

Example 3.2. *A basis for the topology on* $\mathbb{R} \times \mathbb{R}$ *is*

$$\{(a, b) \times (c, d) : a < b \text{ and } c < d\}.$$

The interesting thing is that the topology generated by this basis is exactly the same as the standard topology on $\mathbb{R}^2$*. This can be proved by Lemma 2.6. Also, the product topology on* $\mathbb{R}^p \times \mathbb{R}^{n-p}$ *is identical to the standard topology.*

Actually, you just need the bases for topologies on X and Y to construct a basis of the product topology.

Proposition 3.3. *Let* $\mathbb{B}_X$ *be a basis for the topology on* X *and* $\mathbb{B}_Y$ *be a basis for the topology on* Y*. Then the topology generated by the basis* $\{U \times V : U \in \mathbb{B}_X, V \in \mathbb{B}_Y\}$ *is same as the product topology.*

3.2 Subspace topology

Let us define a topology on $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Definition 3.4. A set in S^1 is open if it is the intersection with an open set in $\mathbb{R}^2$.

If $\mathbb{B}$ is a basis for topology $\mathbb{R}^2$, then a basis for the subspace topology on S^1 is $\{B \cap S^1 : B \in \mathbb{B}\}$. We can generalize this to define the subspace topology for any topological space.

Definition 3.5. Let X be a topological space, and let Y be a subset of X. The **subspace topology** on Y is defined as

$$\mathbb{T} = \{A \subset Y : A = Y \cap V \text{ for some open set } V \subset X\}.$$

It can be verified that this is indeed a topology. Let $\mathcal{A}$ is a collection of open sets in Y. For each $A \in \mathbb{A}$, let V_A be the open set for which $A = V_A \cap Y$. Then

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (V_A \cap Y) = \Big(\bigcup_{A \in \mathcal{A}} V_A \Big) \cap Y.$$

Example 3.6. *Consider the subspace topology on* $Y = [0, 1) \subset \mathbb{R}$. *Then the interval* $[0, 1/2)$ *is an open set since* $[0, 1/2) = [0, 1) \cap (-1/2, 1/2)$.

Example 3.7. *A basis for the subspace topology on* $S^1 \subset \mathcal{R}$ *is the set of open "intervals" on the circle.*

Example 3.8. *Consider the map*

$$f(t) = \begin{cases} (1, t) & -2\pi < t \le 0 \\ (\cos t, \sin t), & 0 \le t < 2\pi \end{cases}$$

and the subspace topology generated on the image $f((-2\pi, 2\pi))$. *This looks similar to the subspace topology on just the interval* $(-2\pi, 2\pi)$, *but because* $f(t)$ *gets near* $(1, 0)$ *when* $t \to 2\pi$, *every open set containing* 0 *must also contain an open interval* $(a, 2\pi)$ *for some* $a < 2\pi$.

3.3 Infinite product topology

Consider the infinite product

$$\prod_{n=1}^{\infty} \mathbb{R}.$$

This looks like the set of infinitely long vectors

$$(x_1, x_2, x_3, \dots, x_n, \dots).$$

Or more generally, we can think of the product set $\prod_{n=1}^{\infty} X_n$ where each X_n is a topological space. How do we define the topology on this space? There are two ways.

The first one is called the box topology, and the second is called the product topology. It turns out that the box topology is bad, and the product topology is good.

Definition 3.9. The **box topology** on $\prod_{n=1}^{\infty}$ is the topology generated by the basis

$$\{U_1 \times U_2 \times \cdots : U_n \text{ is a open set in } X_n\}.$$

The **product topology** is the topology generated by the basis

$$\{U_1 \times \cdots \times U_n \times X_{n+1} \times X_{n+2} \times \cdots : U_n \text{ is a open set in } X_n\}.$$

We will talk more next class.

4 September 11, 2015

I want to start with reviewing the things we did last Wednesday. Given a topological space X(for example, $\mathbb{R}^n$) and a subset $Y \subset X$, there is a subspace topology on Y which comes from the topology on X. A set $A \subset Y$ is open in the subspace topology if it is the intersection of Y with an open set in X. This is useful because anything which can fit in the Euclidean space can be given a natural topology.

I also talked about the product topology. The product of n spaces $X_1, \dots, X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \dots, x_n) : x_1 \in X_1, \dots, x_n \in X_n\}.$$

The topology defined on this the one generated by $U_1 \times \cdots \times U_n$ where $U_k \subset X_k$ is open in each topology. But what if there is a countable number of topological spaces $X_1, X_2, \dots$?

4.1 Infinite products

One motivating example is the Fourier decomposition of a function. For a function f defined on $[-\pi, \pi]$, Fourier decomposition of the function would be

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt.$$

For example,

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin nt.$$

This actually means that a function $f : [-\pi, \pi] \to \mathbb{R}$ corresponds to a sequence

$$(a_0, b_1, a_1, b_2, \dots) \in \prod_{n=1}^{\infty} \mathbb{R}.$$

The labeling set in this product is $\mathcal{J} = [-\pi, \pi]$.

Another way to look at the space is function is thinking that a real number is assigned for each $t \in [-\pi, \pi]$. This can be thought as a vector of uncountable length. This lead to the following definition.

Definition 4.1. Let $\mathcal{J}$ be a labeling set and for each $\alpha \in \mathcal{J}$ assign any set X_α. Let $\mathfrak{X} = \bigcup_{\alpha \in \mathcal{J}} X_\alpha$. Then

$$\prod_{\alpha \in \mathcal{J}} X_\alpha$$

is the set of maps from $\mathcal{J}$ to $\mathcal{X}$ which send any $\alpha \in \mathcal{J}$ to a point in X_α.

The product can be complex, for instance, if $\mathcal{J}$ is a torus and $X = S^2$, any smashing of a doughnut on an apple is an element of the product $\prod_{\alpha \in \mathcal{J}} X$.

There are two ways to put a topology in the product $\prod_{\alpha \in \mathcal{J}} X_\alpha$, as I have said last class.

Definition 4.2. Let X_α be a topological space for each $\alpha \in \mathcal{J}$. The **box topology** on $\prod_{\alpha\in\mathcal{J}} X_\alpha$ is the topology generated by the sets of form

$$\prod_{\alpha\in\mathcal{J}} U_\alpha$$

where U_α is a open set in X_α. The **product topology** is the topology generated by the sets of form

$$\prod_{\alpha\in\mathcal{J}} U_\alpha$$

where U_α is a open set in X_α, and only finitely many U_α are not X_α.

Example 4.3. *In the box topology, an open set in $\prod_{t\in[-\pi,\pi]} \mathbb{R}$ looks like*

$$\{f : f(t) \in (|t|, t^2 + |t| + 1)\}.$$

In the product topology, an open set looks like

$$\{f : f(1) \in (-1, 1), f(2) \in (0, 1), f(-1/2) \in (1, 2)\}.$$

4.2 Some set theory

The axiom of choice state that for any collection of nonempty sets, you can choose one elements from a each set, even if there are uncountably many sets.

Axiom of Choice. *If $\mathcal{J}$ is a nonempty set, and if X_α is a nonempty set for each α, then*

$$\prod_{\alpha\in\mathcal{J}} X_\alpha \neq \emptyset.$$

Definition 4.4. A set J is called countably infinite if there exists a bijection

$$\phi : \{1, 2, 3, \dots\} \to J.$$

Proposition 4.5.

- *A subset of a countably infinite set is finite our countably infinite.*
- *A countable union of a countable set is countable.*

The second statement can be proved by labeling the sets as $X_1, X_2, \dots$ and then labeling the elements of the sets as $X_i = \{X_{i,1}, X_{i,2}, \dots\}$. Then count the elements in the lexicographical order of $(k + l, l)$.

Are there uncountable sets? We can prove that $\mathbb{R}$ is uncountable. Suppose that there exist a bijection $f : \{1, 2, \dots, n, \dots\} \to \mathbb{R}$. Let the decimal expansion of $f(1) = (\dots).a_1(\dots)$ and $f(2) = (\dots).\bullet a_2(\dots)$ and $f(3) = (\dots).\bullet\bullet a_3(\dots)$ and so on. Then $0.a_1'a_2'\dots$ is not in the image of f, where $a_k \neq a_k$ for all k. Thus we arrive at a contradiction.

In fact, there is a whole hierarchy of uncountable sets. If we denote $\mathcal{P}(X)$ be the set of all subsets of X, the sets

$$\mathbb{Z}_+, \mathbb{R}, \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathcal{P}(\mathbb{R})), \dots$$

have all distinct cardinality. One can prove that actually for any set X, the power set $\mathcal{P}(X)$ has cardinality greater than X.

5 September 14, 2015

We are going to do some more set theory.

5.1 More set theory

Definition 5.1. We say that two sets X and Y have the same **cardinality** if there exists a bijection $f : X \to Y$.

How many cardinalities are there? The set $\mathbb{Z}_+ = \{1, 2, \dots\}$ is countable, and $\mathbb{R}$ has a cardinality of continuum. In fact there are infinitely many cardinalities, because of the following theorem.

Theorem 5.2. *For any set X, the power set $\mathcal{P}(X)$ always has a cardinality greater than X.*

This mean that there is no surjective map $X \to \mathcal{P}(X)$.

Let us try some examples. If $X = \emptyset$, then $\mathcal{P} = \{\}$. For a one-element set $X = \{x\}$, we have $\mathcal{P}(X) = \{\emptyset, \{x\}\}$. Because we have verified it for $|X| = 0, 1$, we may—in a very unmathematical manner—say that $\mathcal{P}$ always has a cardinality greater than X.

All proofs of these kind comes down to the Cretan barber paradox. If in a village, a barber shaves everyone who, and only who, does not shave themselves, who is going to shave the barber?

Proof. Let $f : X \to \mathcal{P}(X)$ be a map. Let

$$A = \{x \in X : x \notin f(x)\}.$$

Suppose that $A = f(x)$ for some x. If $x \in A = f(x)$, this contradicts to the definition of A. Therefore $x \notin A = f(x)$, but this also lead to a similar contradiction. Thus A is not in the image of f. $\square$

Actually, with this theorem, we can prove that the question "How many cardinalities are there?" is unanswerable in the language of cardinalities. Suppose there is indexing set I and a collection of sets $\{s_i\}_{i \in I}$ which represent all cardinalities. Then we can consider the union

$$S_{big} = \bigcup_{i \in I} S_i.$$

But then because there is no maximal cardinality, since $\mathcal{P}(X)$ is bigger than X, the new set S_{big} is always bigger than S_i for any $i \in I$. This means that $\{s_i\}_{i \in I}$ did not have all cardinalities in the first place.

5.2 Closed sets, interior, closure, and limit points

Definition 5.3. A set A is a **closed set** when $X \setminus A$ is an open set.

Do not make the mistake that all sets are either open or closed. For example, the set $[0, 1)$ is neither open nor closed since $[0, 1)$ and $(-\infty, 0) \cup [1, \infty)$ are both not open.

Proposition 5.4. *A few observations:*

- *Infinite, arbitrary intersections of closed sets are closed.*
- *Finite unions of closed sets are closed.*

Note that we are not allowed to take infinite unions, since the union of $[1/n, 1]$ for $n = 1, 2, \ldots$ is $(0, 1]$, which is not an open set.

Definition 5.5. For a set A, we define the **interior** of A

$$\begin{aligned}\mathrm{Int}(A) &= \text{Union of all open sets in } A \\ &= \text{Union of all basis sets in } A\end{aligned}$$

Example 5.6. *Consider the set* $A = \{(x, y) : x^2 + y^2 \leq 1\}$. *For any point* (x, y) *for which* $x^2 + y^2 < 1$, *there exists a neighborhood inside A, but for any point* (x, y) *for which* $x^2 + y^2 = 1$, *an open ball goes outside the set. Hence*

$$\mathrm{A} = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Definition 5.7. For a set A, we define the **closure** of A

$$\bar{A} = \text{Intersection of all closed sets that contain } A.$$

Example 5.8. *What is the closure of* $(0, 1)$*? The closed interval* $[-1, 3/2]$ *is a closed set containing* $(0, 1)$, *and we can recede the size to* $[0, 1]$. *In fact, the closure of* $(0, 1)$ *is* $[0, 1]$.

If A is closed, the closure is itself, i.e., $\bar{A} = A$. The closure of a non-closed set is the union of A and some other set. A point x in $\bar{A} \setminus A$ is a point such that every open set (or basis) that contains x intersects A and $X \setminus A$.

Definition 5.9. A point x is a **limit point** of A if every open set containing x has points from $A \setminus \{x\}$.

Example 5.10. *For the set* $A = \{0\} \cup (1, 2)$, *the limit points are* $[1, 2]$.

In the plane, it is clear what a convergence of a sequence $\{x_k\}_{k=1}^{\infty}$ means. The sequence $\{x_k\}$ converges to x if given any $\epsilon > 0$, $|x - x_k| < \epsilon$ if k is large. There is an analogue of this notion in topological spaces.

Definition 5.11. Let x be a point in a topological space X. A sequence $\{x_k\}_{k=1}^{\infty}$ in X is said to be **converge** to x if, given any open set U containing x, all $x_k \in U$ if x is sufficiently large.

Now we can say the difference between the box topology and the product topology. Consider $\prod_{k=1}^{\infty}[0,1]$, and a sequence of points

$$\{x_i = (x_{0i}, x_{1i}, x_{2i}, \dots)\}.$$

What does it mean for this sequence to converge to $(0, 0, \dots)$? A natural definition would be each term converging to 0. Then the sequence for which

$$x_{ij} = \begin{cases} 0 & i \geq j \\ 1 & i < j. \end{cases}$$

should converge in a "good" topology. In the box topology, this sequence does not converge to $(0, 0, \dots)$, because no point it contained in the open set $(1/2, -1/2) \times (1/2, -1/2) \times \cdots$. But we can easily prove that it converges in the product topology.

5.3 Hausdorff spaces

Definition 5.12. A topological space X is said to be **Hausdorff** when given any $p \neq q$, there is an open set U containing p and V containing q with $U \cap V = \emptyset$.

There were many variations, but this proved out to be most useful. Note that the Euclidean space is Hausdorff, because given two distinct points, we can easily give open balls around each of them.

Example 5.13. *Recall that in the Zariski topology, open sets are sets of form*

$$\mathbb{R} - \{p_1, \dots, p_k\}$$

whose complement are finite sets. This is not Hausdorff, because any two open sets intersect.

We will talk about functions next class.

6 September 16, 2015

For two sets X, Y, we can think of a function $f : X \to Y$. For example, there is the function which maps S^1 to a knot K embedded in $\mathbb{R}^3$. There is also a Gauss map, which maps each point in a smooth surface in $\mathbb{R}^3$ to a unit vector $\vec{n} \in S^2$ perpendicular to the surface at that point.

Recall that X and Y has the same cardinality if there exists a bijection between them. This is a notion of equivalence in sets. What would be the analogue of this for topological sets?

6.1 Homeomorphisms

Definition 6.1. Consider two topological spaces X, Y, A function $f : X \to Y$ is called a **homeomorphism** if it satisfies the following.

- f is a bijection.
- $f(U)$ is open for each open $U \subset X$.
- $f^{-1}(V)$ to be open in X for open $V \subset Y$.

Example 6.2. *The real line $\mathbb{R}$ and the subspace topology it imposes on $(-1, 1)$ is homeomorphic by the map*

$$f(t) = \frac{t}{\sqrt{1+t^2}}.$$

Likewise, the interior both disc $\{(x, y) : x^2 + y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2$ be the map

$$\vec{x} \mapsto \frac{\vec{x}}{\sqrt{1+|\vec{x}|^2}}.$$

We can generalize it to the fact that $\mathbb{R}^n$ is homeomorphic to the open ball in it.

Homeomorphism is a equivalence relation, because if $f : X \to Y$ and $g : Y \to Z$ are homeomorphism, $g \circ f : X \to Z$ is a homeomorphism.

Example 6.3. *The upper limit topology and the lower limit topology are homomorphic by the map $t \mapsto -t$.*

How do you tell two spaces are not homeomorphic? One way is to think about the connectedness.

Definition 6.4. A space is **connected** if it can't be decomposed to $U_1 \cup U_2$ where U_1 and U_2 are non-empty open sets and $U_1 \cap U_2 = \emptyset$.

Until next Friday, try to figure out if $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic or not.

Another question: what topological spaces are homeomorphic to a subspace of a Euclidean space?

In the Euclidean space, the basis for the standard topology were the balls of radius $\epsilon > 0$ centered on $\vec{x}$. These kind of spaces are called the metric space.

Definition 6.5. A space X with a distance function $d : X \times X \to [0, \infty)$ satisfying the following conditions is called a **metric space**.

- $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, y) + d(y, z) \geq d(x, z)$.

In these spaces, you can give a topological space by giving a basis

$$B_\epsilon(p) = \{q \in X : d(p, q) < \epsilon\}.$$

These are most useful in spaces of functions.

Example 6.6. *The space $\mathbb{R}^\mathbb{R}$ of functions $f : \mathbb{R} \to \mathbb{R}$ can be given the following metric.*

$$\operatorname{dist}(f, f') = \min\left\{1, \sup_{t\in\mathbb{R}} |f(t) - f'(t)|\right\}$$

Let me give yet another question: Is every topology a metric space topology? The answer is no. This is because all metric spaces are Hausdorff. If $\operatorname{dist}(p, q) = r$, then the ball $B_{r/4}(p)$ and $B_{r/4}(q)$ are two disjoint open sets.

This rises yet another question: Is every Hausdorff space a metric space? We will all answer these questions in a week or two.

6.2 Continuous maps

One last definition. There is a notion of continuity which is half of homeomorphism.

Definition 6.7. A map $f : X \to Y$ is **continuous** if $f^{-1}(U)$ is open in X for any open set $U \subset Y$.

The definition uses inverse image because if we use the image, then the constant map $f(x) = 1$ would not be continuous.

7 September 18, 2015

Let X and Y be topological space, and consider the functions $f : X \to Y$. Continuous functions are a set of distinguished functions, such that f^{-1}(open set in Y) is open in X. The classical case is $f : X \to \mathbb{R}$. For example, we can let $X = \mathbb{R}^3$ and walk around with a thermometer. In some cases, there are no non-constant continuous function. If $X = \mathbb{R}$ with the Zariski topology, this is the case. For $f(t_1) = a$ and $f(t_2) = b$, then we can give two open sets $a \in O_1$ and $b \in O_2$ which are disjoint. Then $f^{-1}(O_1)$ and $f^{-1}(O_2)$ cannot both be open, because they are disjoint. This is a really impoverished example.

Theorem 7.1. *Suppose that X is not Hausdorff. Let $p, q \in X$ be distinct points such that every open set containing p contains q. Then for any $f : X \to \mathbb{R}$ which is continuous, $f(p) = f(q)$. Also, one can replace $\mathbb{R}$ with any Hausdorff space Y.*

In analysis, we gave the following definition for a continuous function.

Definition 7.2. Consider a function $f : \mathbb{R} \to \mathbb{R}$. The function continuous at x if given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ if $x - x' < \delta$.

In the language of topology, what it means is: "You give me an open set $\mathcal{O}$ around $f(x)$ and I can give you an open set $\mathcal{O}'$, around x such that $f(\mathcal{O}') \subset \mathcal{O}$." And after playing around, we see that this is actually equivalent to f^{-1}(open set) is open.

7.1 Convergence of a sequence

Definition 7.3. A sequence $\{x_i\}$ in X **converges** to x, if given any open set $U \ni x$ then $\{x_i\}_{i>>1}$ are all in U.

To be on the safe side, let us just assume all spaces are Hausdorff.

Does the continuity of $f : X \to Y$ implies that $\{f(x_i)\}$ converges to $f(x)$? Does the converse hold? I tried to figure out, but fell asleep last night, so I don't know yet. Bonus problem.

7.2 Topologies induced by maps

Note that for any spaces X and Y, finer the topology on X is, the easier the function $f : X \to Y$ to be continuous. Conversely, the finer the topology on Y is, the harder the function to be continuous. For example the step function is not continuous if X has the standard topology, but it is continuous if X has the lower limit topology. In fact, given a function, you can define your own topology on X to make it continuous.

Definition 7.4. Let $X \to Y$ where only Y is a topological space. Then a set $U \in X$ is open in the **induced topology** on X if and only if U is an inverse of an open set in Y.

Definition 7.5. Let $X \to Y$ where only X is a topological space. Then a set $U \in Y$ is open in the **quotient topology** on Y if and only if $f^{-1}(U)$ is open.

You can glue two topological spaces by the quotient topology. Let us construct a wormhole in Interstellar. Let us take a ball B_1 in $\mathbb{R}^3$, which is one end of the wormhole. An think of a second ball B_2 which is the other end of the wormhole. We take the complement of two balls. Now think of another space, which is the ball with a smaller hole inside it. We glue the inner side of the hole to B_1 and the outer side of the ball to B_2. Then we get a wormhole!

8 September 21, 2015

8.1 Using quotient spaces

Last time I pointed out that if there is a set X and a topological space Y and a map $f : X \to Y$, then there we can give a induced topology on X which makes f continuous; $U \subset X$ is open when it is $f^{-1}(\mathcal{O})$. Note that his is the coarsest topology on X which makes f continuous.

You can do it the other way round. If X is a topological space and Y is set, then the quotient topology on Y can be defined; $\mathcal{O} \subset Y$ is open when $f^{-1}(\mathcal{O})$ is open in X. This is the finest topology on Y which makes f continuous.

This quotient topology is very useful. Let $A \subset X$ a subset, and let C be some one-point set. Then we can collapse all points in A to C.

$$\begin{tikzcd} X \arrow[dr, "f"] & \supset & A \arrow[d, "f_A"] \\ & & C \end{tikzcd}$$

Let Y be the set of equivalence classes made by the relation $x \sim x'$ if and only if $x, x' \in A$. Then if $x \in X \setminus A$, then it is its own equivalence class, and A is another equivalence class. Using the map f induced by this equivalence classes, we can define a quotient topology on Y. For instance, if $X = S^2$ and A is a circle on the sphere, then Y would look like some kind of a snowman.

Example 8.1. *Let $X = X_1 \cup X_2$, and let $A_1 \subset X_1$, $A_2 \subset X_2$ which are homeomorphic to each other, and set $A = A_1 \cup A_2$. Let $C = A_2$ and $f_A : A_1 \to A_2$ be the homeomorphism which can be extended to $f : X \to (X_1 \setminus A_1) \cup X_2$. Then in the quotient topology, X_1 is glued to X_2 on A_2.*

8.2 Connected spaces

Definition 8.2. A topological space X is not **connected** when X can be written as a union of two open sets $A \cup B$, both open non-empty and $A \cap B = \emptyset$.

Theorem 8.3. *The open interval (a, b) is connected.*

Proof. Suppose that it is not connected. Then there exist open sets A, B such that $A \cup B = (a, b)$. Let $c \in A$ be any element. Since A is an open set, there exists a small interval around c which is inside A. Since $B \neq \emptyset$, either $[c, b) \cap B \neq \emptyset$ or $(a, c] \cap B \neq \emptyset$, but without loss of generality, let $[c, b) \cap B \neq \emptyset$.

Let

$$d = \inf([c, b) \cap B).$$

Then $c < d$ since there was a open interval around c in A. If $d \in B$, then since B is open, d cannot be a lower bound. On the other hand, if $d \in A$, then d cannot be the greatest lower bound. Thus we get a contradiction. □

Example 8.4. *In the lower limit topology, every open set is disconnected. This is because $[a, b) = [a, c) \cup [c, b)$ and $(a, b) = (a, c) \cup [c, b)$.*

Example 8.5. *Recall that the infinite product $\prod_{n=1}^{\infty} \mathbb{R}$ has two topologies: the product topology and the box topology. This space is not connected in the in box topology, because the sets*

$$A = \{(a_1, a_2, \dots) : \{|a_k|\} \text{ bounded}\}, \quad B = \{(a_1, a_2, \dots) : \{|a_k|\} \text{ not bounded}\}$$

are both open. But it is connected in the product topology.

8.3 Path connected spaces

Definition 8.6. Let X be a topological space. A **path** in X from $x_0 \in X$ to $x_1 \in X$ is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The topological space X is **path connected** when when any two points are joined by a path.

For example, $[0,1]$ is path connected.

Theorem 8.7. *Suppose that X is a space, disconnected so that $X = A \cup B$ where A, B are open, non-empty, and disjoint. Let $\gamma : [0,1] \to X$, then the image of γ is either all in X or all in B.*

Corollary 8.8. *The only continuous maps from the $[0,1]$ to $(-\infty, \infty)$ in the lower limit topology are constant maps.*

9 September 23, 2015

I want to say more about path connected sets. As a reminder, X is not connected if it can be written as $X = A \cup B$ where A and B are nonempty open sets and $A \cap B = \emptyset$. I showed last time that the interval (a, b) is connected, and that the space $\prod_{n\in\mathbb{Z}} \mathbb{R}$ with the box topology is not connected, and also that $\mathbb{R}_{LL}$ with the lower limit topology is not connected.

9.1 More on connectedness

Theorem 9.1. *Suppose $A \subset X$ and A is connected. Then it closure $\bar{A}$is connected. In fact, $A \cup \{$some limit points$\}$ is also connected.*

Proof. Suppose that $\bar{A} = C \cup B$ where B and C are nonempty disjoint open sets. Since A is connected, either $A \subset C$ or $A \subset B$. If we assume $A \subset C$ without loss of generality, then B must contain only limit points. But because B is open, B should intersect with A. Hence we get a contradiction. $\square$

Because (a, b) is connected, other kinds of intervals $[a, b]$ or $[a, b)$ are also connected.

Another notion of connectedness is 'being able to go from anywhere to anywhere.' For example, that the Science center is connected might mean that I can go from room 507 to room 112. This is the notion of path connectedness.

Proposition 9.2. *No two distinct points in $\mathbb{R}_{LL}$ are connected by a continuous path.*

Proof. Let $a < b$ any two points, and let $A = (-\infty, c)$ and $B = [c, \infty)$ for some $a < c < b$. Suppose that γ is a path from a to b. Since A and B are open and disjoint, $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are also open and disjoint. But then this means that $[0, 1]$ is disconnected, which is false. Thus we arrive a contradiction. $\square$

Actually copying the proof, it is also possible to prove the following lemma.

Lemma 9.3. *If $X = A \cup B$ where A and B are nonempty disjoint open sets, and $\gamma : [0, 1] \to X$ is continuous, then $\gamma([0, 1])$ is either entirely in A or B. This shows that disconnected implies not path connected.*

Example 9.4. *The converse is not true. Consider the topologist's sine curve, which is the subspace topology on*

$$\{(x, y) : x > 0, y = \sin(1/x)\} \cup \{(x, y) : x = 0\}.$$

Then because the curve part is path connected, and the line part is path connected, any connected open set should either contain or not intersect both parts. But since any open set containing the origin should intersect the curve part, the whole set is connected. But it is not path connected.

Example 9.5. *The circle S^1 and the interval $(a, b) \subset \mathbb{R}$ are not homeomorphic. This is because if you erase one point from each set, the circle $S^1 \setminus \{p\}$ is connected, while $(a, b) \setminus \{p\}$ is not connected.*

Example 9.6. *Consider a two circles sharing one point. This is not homeomorphic to a circle because we can pluck out the shared point to make the two circles not connected. If three circles are glued by one point, then this is homeomorphic to neither the circle nor the two-circle, because if you pluck out the shared point, there are three components.*

Definition 9.7. A **path component** is a equivalence class made by the equivalence relation $x \sim x'$ if and only if there is a path from x to x'.

The hardest thing to check is that the relation is transitive. This can be done by walking twice fast along the paths.

Proposition 9.8. *Consider the set*

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

This is not homeomorphic to $\mathbb{R}^2$, and even to any open subset of $\mathbb{R}^2$.

9.2 Sequential compactness

Definition 9.9. A space X is called **sequentially compact**, if for any sequence $\{x_k\}_{k=1,2,\dots} \subset X$, there is a subsequence

$$\{x_{k_1}, x_{k_2}, x_{k_3}, \dots\}$$

with $k_1 < k_2 < \dots$ which converges.

Theorem 9.10. *A subspace X of $\mathbb{R}$ is sequentially compact if and only if it is closed and bounded. (A is bounded means that $|\vec{x}|$ are bounded.)*

Example 9.11. *For example, the sequence $\{0, 1, 0, 1, \dots\}$ in the interval $[0, 1]$ has a converging subsequence; the sequence of 0s converges.*

Example 9.12. *Let θ be a irrational multiple of 2π, and let $\{x_k = k\theta\}$. Then every point in S_1 is a limit point.*

10 September 25, 2015

For a sequence $\{x_n\}_{n=1,2,\dots} \subset X$, a subsequence is $x_{n_1}, x_{n_2}, x_{n_3}, \dots$. A X is said to be sequentially compact, if every sequence has a converging subsequence. This is equivalent to saying that every sequence has a limit point. Or is it? There is the issue of actually choosing the subsequence, even if there is a limit point. If X is a metric space, you can just choose the distance to converge to zero, but if X is not a metric space, we might not be able to choose a collection which converge to it.

Anyways, we have the following.

Proposition 10.1. *If X is sequentially compact, and $f : X \to Y$ is a continuous map, then $f(X)$ is also sequentially compact.*

Proof. Let $y_1, \dots, y_n, \dots$ be a sequence in the image $f(X)$. Then there is a sequence $\{x_n\}$ for which $f(x_n) = y_n$. Since X is sequentially compact, there is a subsequence $x_{n_1}, x_{n_2}, \dots$ which converges to come x. Then since f is continuous, $f(x_{n_1}), f(x_{n_2}), \dots$ converges to $f(x)$. This means that $y_{n_1}, \dots, y_{n_k}, \dots$ converges to $f(x)$. □

Note that in proving this, we used the following lemma, which is one direction of the extra credit problem.

Lemma 10.2. *Let $f : X \to Y$ be a continuous function and $\{x_1, x_2, \dots\}$ converges to x. Then $f(x_1), f(x_2), \dots$ converges to $f(x)$.*

One implication of this proposition is that if $f : X \to Y$ homeomorphism, then X is sequentially compact if and only if Y is sequentially compact.

10.1 Bolzano-Wierstrass theorem

Theorem 10.3 (Bolzano-Wierstrass)**.** *A subset in $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

Proof. For instance, let us prove that S^2 is sequentially compact. We draw a box $[-2,2] \times [-2,2] \times [-2,2]$ containing the sphere. Because there are infinitely many points, we can divide the cube in half, and say that one half should contain infinitely many point. If the top half $[-2,2] \times [-2,2] \times [0,2]$ contains infinitely many points, then let x_1 be a point in it. Next divide it into the left half and the right half, and choose the second point inside the part with infinitely many point. If we continue to do this, then the points $x_1, x_2, \dots$ gets closer and closer. In fact,

$$|x_p - x_{p+k}| < Lc^{-p}$$

for some L and $c < 1$. This mean that the points are a Cauchy sequence. Because we are working in the Euclidean space, this converges. □

Corollary 10.4. *The sphere S^2 is not homeomorphic to $\mathbb{R}^2$. In fact, the image of S^2 in $\mathbb{R}^2$ should always be sequentially compact.*

10.2 Continuity and convergence of a sequence

Does the converse of Lemma 10.2 above hold? Suppose that for any $\{x_n\} \subset X$ converging to x, the sequence $\{f(x_n)\} \subset Y$ converges to $f(x)$. Is f necessarily continuous?

The answer is yes if X is a metric space, and moreover, if X is "first countable."

Definition 10.5. A space X is **first countable** if for any $p \in X$, there is a sequence of open sets U_n containing p such that for any neighborhood V of p, there is a U_n which is contained in V.

Proof of converse of Lemma 10.2. Let $\mathcal{O} \subset Y$ be an open set and suppose that $f^{-1}(\mathcal{O})$ is not open. If $x \in f^{-1}(\mathcal{O})$ such that every neighborhood of x is not completely contained in $f^{-1}(\mathcal{O})$. Since X is first countable, there exist open neighborhoods $U_1, U_2, \ldots$ of x such that every neighborhood V of x contains U_n for some V. Now pick a point $p_k \in U_k \setminus \mathcal{O}$ for each k. Then the sequence $\{p_1, p_2, \ldots\}$ converges to x, but the sequence $\{f(p_1), f(p_2), \ldots\}$ does not converge to $f(x)$ because none of them are in $\mathcal{O}$. Thus we get a contradiction. $\square$

Example 10.6. *Let $X = \prod_{t\in\mathbb{R}}[0,1]$ with the box topology, and let $Y = [0,1]$. Then there is a counterexample for the converse of the lemma, in this non-first-countable space.*

11 September 28, 2015

A topological space X was said to be sequentially compact, if every sequence $\{x_n\}$ has a convergent subsequence. We proved that a subset $X \subset \mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

11.1 Compact spaces

Although this definition is more intuitive, there is also the notion of compactness, which is not intuitive but more useful in many cases. A collection $\mathcal{U}$ is called a **cover** of X if every point is in some some set from $\mathcal{U}$, and is called an **open cover** if its sets are open.

Definition 11.1. A space X is said to be **compact** if and only if every open cover has a finite subcover.

Theorem 11.2. *Compactness implies sequential compactness. In a metric space, these two notions are equivalent.*

Proof. Assume that a metric space X is compact, and let $A = \{x_n\}_{n=1,2,\dots}$ be a sequence in X. We need to show that there exists a point x and an infinite subsequence that converges to x. In a metric space, it means that given any ball $B(x)$ with radius $1/k$, I can find an element of the sequence in the ball. If we assume that there is no convergent subsequence, A is a closed set. Moreover, for each k I can find a open ball U_k which contains only x_k from A. Then we get a open cover

$$\{U_1, U_2, \dots, U_k, \dots, X \setminus A\}.$$

But we cannot have a finite subcover, because it will leave out point of A. Thus we arrive at a contradiction, and hence X is should be sequentially compact.

Now assume that for a metric space X, every sequence has a convergent subsequence. Why is it compact? Actually we need a new notion of a Lebesgue number.

Definition 11.3. For a open cover $\mathcal{U}$, a number $\epsilon \geq 0$ is called a **Lebesgue number** if the radius ϵ ball about each point is in some open set from $\mathcal{U}$.

Theorem 11.4 (Lebesgue number theorem)**.** *Every open cover $\mathcal{U}$ in a sequentially compact metric spaces has a positive Lebesgue number $\epsilon > 0$.*

Proof. If not, for each $n = 1, 2, \dots$ there is $p_n \in X$ such that the ball of radius $1/n$ at p_n is not entirely in any set from cover. Then the sequence $\{p_n\}$ have a convergent subsequence which converges to p. The point p must be contained in some $U \in \mathcal{U}$, and we can find a little ball $B(p, \delta)$ which is in U. Since the subsequence was convergent to p, there is a sufficiently large n for which p_n is in the ball $B(p, \delta/2)$. If n is sufficiently large so that $1/n < \delta/2$, the ball $B(p_n, 1/n)$ is in $B(p, \delta)$, which is in U. Then we have a contradiction. $\square$

Now we need to prove that every open cover of a sequentially compact space with positive Lebesgue number has a finite subcover. The proof is essentially going to be something like "you can't fill infinitely many m&m's inside a cup."

Every point has a ball of radius ϵ entirely in some subset of U. Let p_1 be any point, and let $U_1 \supset B_\epsilon(p_1)$. If U_1 is not the whole space, then there is point $p_2 \notin U_1$, and a set $U_2 \supset B_\epsilon(p_2)$. If $U_1 \cup U_2$ is not the whole space, again there is a point $p_3 \notin U_1 \cup U_2$. We proceed in this way to make a sequence.

Now because the X is sequentially convergent, there must be a subsequence $p_{k_1}, p_{k_2}, \ldots$ which converges to p. Then $\mathrm{dist}(p, p_{k_m}) \to 0$ as $m \to \infty$, and also $\mathrm{dist}(p_{k_m}, p_{k_{m+1}}) \to 0$ as $m \to \infty$. But since the ball $B_\epsilon(p_n) \subset U_n$, and the points $p_{n+1}, p_{n+2}, \ldots$ are not in U_n, those points cannot be really close to p_n. Therefore, we have a contradiction. $\square$

Now I have a few minutes, and let me state the following theorem.

Theorem 11.5. *Let X be a compact space and Y a Hausdorff space. If a map $f : X \to Y$ is continuous bijection, it is a homeomorphism!*

12 September 30, 2015

12.1 Properties of compact spaces

Let me state some properties of a compact space.

1. A closed subspace in a compact space X is compact.

   Proof. Let $Y \subset X$ be a closed space, and let $\mathcal{U}_Y$ be an open cover of Y. If $U \in \mathcal{U}_Y$ then there is a open $\mathcal{O}_U \subset X$ with $Y \cap \mathcal{O}_U = U$. Then
   $$\{\mathcal{O}_U : U \in \mathcal{U}_Y\} \cup \{X - Y\}$$
   is a open cover of X, and thus have a finite subcover. Then $\mathcal{O}_U$ should also have a finite subcover of Y. $\square$

2. A compact subspace of a Hausdorff space X is closed.

   Proof. Let $Y \subset X$ be compact, and show that $X \setminus Y$ is closed. Consider any $p \in X \setminus Y$. For any $y \in Y$, there exist two open sets $\mathcal{O}_y$ in X containing y, and $\mathcal{O}_{p,y}$ in X containing p such that $\mathcal{O}_{p,y} \cap \mathcal{O}_y = \emptyset$. Since the set of $\mathcal{O}_y$s is a open cover of Y, it must have a finite subcover. Let $\mathcal{O}_{y_1}, \dots, \mathcal{O}_{y_n}$ cover Y. Then $\mathcal{O}_{p,y_1} \cap \cdots \cap \mathcal{O}_{p,y_n}$ is a open set containing p, and disjoint from Y. $\square$

3. Suppose $f : X \to Y$ is continuous and X is comapct. Then $f(X)$ is compact.

   Proof. Take an open cover $\mathcal{U}$ be an open cover of $f(X)$. Then $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X, and hence there is a finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$ which covers X. Then $U_1, \dots, U_n$ covers $f(X)$. $\square$

4. Let X be a compact space and Y be a Hausdorff space. Then any continuous bijection $f : X \to Y$ is a homeomorphism.

   Proof. We need to prove that $f(\text{open})$ is open, or alternatively, $f(\text{closed})$ is closed. X is compact, so closed sets are compact. So $f(\text{closed})$ is compact. Since Y is Hausdorff, $f(\text{closed})$ are closed. $\square$

5. Let $f : X \to \mathbb{R}$ be a continuous map defined on a compact space X. Then f achieve its maximum and minimum value. That is, there are $x_a, x_b \in X$ such that $f(x_a) \le f(x) \le f(x_b)$ for all $x \in X$.

   Proof. Let $A = f(X) \subset \mathbb{R}$. Since A is compact, it is closed and bounded. So it must have a maximum and the minimum value $a, b \in \mathbb{R}$ such that $a \le t \le b$ for all $t \in A$. There is a x_a with $f(x_a) = a$ and x_b with $f(x_b) = b$. $\square$

12.2 Locally compact spaces

Definition 12.1. X is **locally compact** if for each $p \in X$ there is an open set $p \in U \subset X$ which lies in a compact set.

Theorem 12.2. *Let X be a locally compact space. Then there is a unique compact Y and $p \in Y$ such that $Y \setminus \{p\}$ is homeomorphic to X. This Y is called the* ***one-point compactification*** *of X.*

We can define a topology on $X \cup \{p\}$ where p is a point from nowhere. For $x \in X$, the basis for open sets containing x are the open sets in X. Also, the open neighborhoods of p are the complements of compact sets in X.

Example 12.3. *The one-point compactification of the real line $\mathbb{R}$ is the circle S^1. The 'north pole' is the p, and the homeomorphism is the stereographic projection. Likewise, the one-point compactification of $\mathbb{R}^2$ is S^2.*

Why is it unique? Suppose that we have two spaces Y and Y'. Because $Y \setminus \{p\}$ and $Y' \setminus \{p'\}$ are both homeomorphic to X, we have the diagram

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad f \quad} & & Y' \\ \cup & & & & \cup \\ Y \setminus \{p\} & \xrightarrow{g} & X & \xleftarrow{g'} & Y' \setminus \{p'\} \end{array}$$

and define f using g and g'. It is possible to check that f is a homeomorphism, especially for the neighborhood of p and p'.

12.3 Metrizable spaces

Definition 12.4. A topological space X is called **metrizable** if the topology on X can be defined using a metric.

To ask what spaces are metrizable is to ask what the key properties of metric spaces are. We try to list some properties.

- It is Hausdorff.
- Given $p, q \in X$, there is a function which is zero at p and one at q, namely

  $$x \mapsto \frac{\operatorname{dist}(p, x)}{\operatorname{dist}(q, x)}.$$

  This is quite interesting, because it says that there is some function separating any two points.

13 October 2, 2015

13.1 Separability and countability axioms

Definition 13.1. X has a **locally countable base** if for any point $p \in X$, there are open neighborhoods $\mathcal{U}_1, \mathcal{U}_2, \dots$ such that for each open $p \in V$, V is contained in some $\mathcal{U}_n$.

Definition 13.2. X has a **countable base** if and only if $\{U_n\}_{n=1,2,\dots}$ is open such that for any point p in an open set V, there exists U_n containing p, entirely in V.

Definition 13.3. X is **regular** when If $p \in X$ and A is a closed subset $p \notin A$, there is a pair of disjoint open sets such that p is in one, and A is in the other.

Theorem 13.4 (Urysohn's theorem). *If X is regular, and it has a countable base, then it is metrizable.*

One thing to point about this theorem is that metric spaces are regular. Let p be a point not in a closed set A. First, there is a $\delta > 0$ such that the radius δ ball centered at p is disjoint from A. This is because the complement of A is open. Then we set U_1 as at the radius $\delta/2$ ball at p. Then we set U_2 as the complement of the closure of the radius $\frac{3}{4}\delta$ ball at p. Then $p \in U_1$ and $A \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

However, metric spaces not necessarily have a countable base. So Urysohn's theorem does not give a completely equivalent condition. One thing to note is that $\mathbb{R}$ has a countable base, because the set of $\{(a,b) : a, b \in \mathbb{Q}\}$ is a countable base. The product topology on $\mathbb{R}^\omega$ also has a countable base, which consists of open set of $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots$ with $a_1, \dots, a_n, b_1, \dots, b_n$ rational.

Theorem 13.5 (Nagata-Smirnov metrization theorem). *A space X is metrizable if and only if it is regular, and has a countably locally finite base. This means a $\mathcal{U} = \bigcup_{n=1}^\infty \mathcal{U}_n$ such that given $p \in X$, only finitely man sets from each $\mathcal{U}_n$ contain p.*

Definition 13.6. A spaces X is **normal** if for any disjoint closed sets A and B, there are disjoint open sets U_A and U_B containing A and B.

Metric spaces are also normal.

13.2 Flow chart of the proof of Urysohn's theorem

Lemma 13.7 (Urysohn's lemma). *Let X be a normal space. Then for closed A, B disjoint, there is $f : X \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$.*

This is brilliant, and is the key part in the theorem.

$$
\begin{array}{c}
X \text{ is regular} + \text{countable basis} \\
\downarrow \\
X \text{ is normal} \\
\downarrow \\
\text{Urysohn's lemma} \\
\downarrow_{\text{+countable bases}} \\
\{f_n\}_{n=1,2,\ldots} \\
\downarrow \\
F = (f_1, f_2, \ldots) : X \to \prod_{n=1,2,\ldots}[0,1] \\
\downarrow \\
F \text{ is continuous and homeomorphism onto } F(X) \text{ as } X \subset \prod_{n=1,2,\ldots}[0,1]
\end{array}
$$

Each step took Urysohn about two years.

14 October 5, 2015

The question was "when does a topology come from a metric?" We introduced the notion of regularity; a space is regular when a closed set can be separated with a point.

Theorem 14.1. *If X is regular and has a countable base, it is metrizable.*

14.1 Metrizability of the product topology

As I have mentioned, in the proof, we construct functions $f_n : X \to [0, 1]$ which separate all the points, and this gives a embedding $F : X \to \prod_{k=1,2,\dots}[0, 1]$. Then F is homeomorphic to a subspace of a $\prod_{k=1,2,\dots}[0, 1]$. Now what we need to prove is that $\prod_{k=1,2,\dots}[0, 1]$ is a metric space.

For two elements $t = (t_1, t_2, \cdots)$ and $s = (s_1, s_2, \cdots)$, we define the distance as

$$D(t, s) = \sup_{k=1,2,\dots} \frac{1}{k}|t_k - s_k|.$$

(If we wanted to show that $\prod_{k=1,2,\dots} \mathbb{R}$ is a metrizable, we should have used $\hat{D}(t, s) = \sup \frac{1}{k} \min\{|t_k - s_k|, 1\}$.) First, I need to show that this is a metric, and then that it gives the product topology. You can check that this is a distance function. Let us show that this distance induces the product topology. A basis element of $\prod_{k=1,2,\dots} \mathbb{R}$ is $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots$. If $c_1 \in (a_1, b_1), \dots, c_n \in (a_n, b_n)$, then there is an $\epsilon > 0$ for which $a_k < c_k - \epsilon$ and $c_k + \epsilon < b_k$ for each $k = 1, \dots, n$. Then for any t such that $\hat{D}(c, t) < \epsilon/n$ actually lies in this $(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots$. This shows that an open set in the product topology is an open set in the metric topology.

Now we have to do the other way round. Consider any ball in the metric topology. How many t_k does the condition $\hat{D}(c, t) < \epsilon$ constrain? Once $1/k$ gets smaller than ϵ, the value of t_k can be anything. Hence, the open ball in the metric topology is an open set in the product topology. This shows that the metric topology and the product topology are the same.

14.2 Outline of the proof of Urysohn's lemma

So we are left with constructing the embedding.

Lemma 14.2 (Urysohn's lemma)**.** *Let X be a normal space and A, B be disjoint closed subsets. Then there is a continuous function $f : X \to [0, 1]$ with $f|_A = 0$ and $f|_B = 1$.*

Proof. Rather than constructing the function, we construct level sets instead. That is the great idea.

Start with closed sets A and B. Then we can find disjoint open $\mathcal{O}_A$ and $\mathcal{O}_B$ which contain A and B. Then I look at the closed sets B and $X \setminus \mathcal{O}_B$. Using normality, we can choose an open set U_1 which contains B but does not intersect $X \setminus \mathcal{O}$. Then U_1 contains B, $U_0 = \mathcal{O}_A$ contains A, and the closure of U_1 and U_0 are disjoint. This is going to be the sublemma.

Lemma 14.3. *If X is normal and A, B are disjoint and closed, there are $A \subset \mathcal{O}_A$ and $B \subset \mathcal{O}_B$ with $\bar{\mathcal{O}}_A \cap \bar{\mathcal{O}}_B = \emptyset$.*

The first step is to choose open U_0 which contains A and open U_1 which contains B, whose closures are disjoint. Next, we label the rationals by $\{r_2, r_3, \dots\}$. Then for each r_k, we construct a boundary using the normality inductively. $\square$

15 October 7, 2015

Recall that we were proving that a regular space with a countable basis is metrizable. We were trying to embed

$$X \xhookrightarrow{F} \prod_{n=1}^{\infty}[0,1] \subset \prod_{n=1}^{\infty}\mathbb{R}.$$

What we needed to prove was Urysohn's Lemma, and then using the countable basis, we can construct the embedding.

15.1 Proof of Urysohn's Lemma

The basic idea is to prove it by induction. If you give a number, I give a set in which the function attains a value less than that number.

Let A and B be disjoint sets. There is a open set U_0 around A and U_1 around B which have disjoint closure. U_0 will be a space where $f = 0$ and U_1 will be a space where $f = 1$. Now we need to specify other values. Because the real numbers are uncountable, we choose a dense countable set, for instance the set of rational numbers. Let $r_2, r_3, \ldots$ be a labeling of all the rational numbers between 0 and 1. Now for r_2 we will have $0 < r_2 < 1$. Then we have a open set containing the closure of U_0 and whose closure is disjoint from U_1. Let this be U_2. For the next rational, suppose that $0 < r_3 < r_2$. Then we construct U_3 as the open set containing the closure of U_0 and whose closure is in U_2.

Generally, the rule is that for rationals $r_1, \ldots, r_k$, we construct $U_1, \ldots, U_k$ inductively such that for any $p, q \in \{1, \ldots, k\}$, if $r_p < r_q$ then $\bar{U}_p \subset U_q$. If there is a new rational r_{k+1}, we can construct U_{k+1} well by using the normality of the space. Now we have all the level sets at rational numbers $\{U_r\}_{r \in \mathbb{Q} \cap [0,1]}$, which is nested, in the sense that if $r < r'$, then $\bar{U}_r \subset U_{r'}$.

We define the function

$$f(x) = \inf\{r : x \in U_r\}.$$

Why is this function continuous? Consider the inverse image of a basis element (p, q). Then

$$f^{-1}((p,q)) = U_q \cap (X \setminus \bar{U}_q)$$

is open. So f is continuous.

15.2 Dedekind cuts: a digression

This reminds me of Dedekind cuts. (This is a digression.) Dedekind has this great idea of how to define the real numbers. A Dedekind cut is a partition $\mathbb{Q} = A \cup B$ such that

- A does not contain its least upper bound.
- If $x \in A$ and $y < x$ then $y \in A$.
- if $x \in B$ and $y > x$ then $y \in B$.

Then the set of all Dedekind cuts is the rational numbers.

15.3 Proof of Urysohn's metrization theorem (1)

Let $\{G_n\}_{n=1}^\infty$ be our countable basis. Define

$$\Theta = \{(n, m) \in \{1, 2, \dots\} \times \{1, 2, \dots\} : \bar{G}_m \subset \bar{G}_n\}.$$

If $\{G_n\}$ is a basis, then so is $\{G_{n,m} = G_n \setminus \bar{G}_m\}_{(n,m)\in\Theta}$. Why is this? Just use the lemma about finding open sets with disjoint closure. Let p be a point inside a basis element G_n. Then applying the lemma, we have open sets U containing p and $\mathcal{O}$ containing $X \setminus G_n$. Then a basis element in the open set $\mathcal{O} \cap G_n$ does not contain p. Applying a bunch of times, we can actually choose it so that the closure of $\mathcal{O}$ does not contain p but is inside G_n.

Now for any such $(n, m) \in \Theta$, consider the map $f_{n,m} : X \to [0, 1]$ such that $f_{n,m}(\bar{G}_m) = 0$ and $f_{n,m}(X \setminus G_n) = 1$. Then define

$$F = \prod_{(n,m)\in\Theta} f_{n,m} : X \to \prod_{(n,m)\in\Theta} [0, 1].$$

Then F is continuous, and it is an embedding.

16 October 9, 2015

Today we will finish this point set topology and move to algebraic topology.

16.1 Proof of Urysohn's metrication theorem (2)

The Urysohn's theorem was that if X is a regular space with countable base, it is metrizable. For a countable collection $f_1, f_2, \dots$ such that for any $p \neq q$ there is a k such that $f_k(p) \neq f_k(q)$, we define a metric

$$\mathcal{D}(p, q) = \sup_{k=1,2,\dots} \left(\frac{1}{k} \min(|f_k(p) - f_k(q)|, 1) \right).$$

The term $\frac{1}{k}$ guarantees that the topology is finer than the metric topology.

So now if we have functions $f_k : X \to [0, 1]$, we can construct a function $X \to \prod_{k=1}^{\infty}[0, 1]$ by sending

$$x \mapsto (f_1(x), f_2(x), \dots).$$

But how do we choose the functions? We use a countable base $\{G_n\}_{n=1}^{\infty}$. Now this is a review of last class.

We let

$$\Theta = \{(n, m) : \bar{G}_m \subset G_n\}$$

and for each $(n, m) \in \Theta$, we defined a function $f_{n,m} : X \to [0, 1]$ such that it is 0 on $X \setminus G_n$ and 1 on $\bar{G}_m$. Then these functions distinguishes any two distinct points.

Now the function which maps X into the product topology is continuous, and it is a bijection. But we still have to show that it is an open map. Because the $G_{n,m} = G_n \setminus \bar{G}_m$ is a basis element, it is sufficient to show that the image of the complement if this set is closed. However, I think there is a flaw in the proof I prepared. I will upload the modified proof in the website.

16.2 Manifolds

Definition 16.1. X is an n-dimensional **manifold** if

a) it has a countable basis,

b) it is Hausdorff,

c) if $p \in X$, there is an open set $\mathcal{O}_p \subset X$ with a homeomorphism $f_p : \mathcal{O}_p \to$ open set in $\mathbb{R}^n$.

If $\mathcal{O}_r$ and $\mathcal{O}_q$ are neighborhoods of r and q, and they overlap, there is a homeomorphism between $f_r(\mathcal{O}_r \cap \mathcal{O}_q)$ and $f_q(\mathcal{O}_r \cap \mathcal{O}_q)$ in the Euclidean space. Then we can view a manifold as space obtained by gluing different open sets in the Euclidean space.

Example 16.2. *The circle S^1 is a manifold, because for any point, we can project it to some axis. Likewise, any smooth surface in $\mathbb{R}^3$ is a manifold, because we can project the neighborhood to a tangent plane.*

Theorem 16.3. *Suppose that X is a compact n-dimensional manifold. Then there is an embedding $F : X \to \mathbb{R}^N$ for some finite N.*

Proof. Consider any $p \in X$ and there should be a open $\mathcal{O}_p$ containing p with an homeomorphism $f_p : \mathcal{O}_p \to \mathbb{R}^n$. Because the image is an open set, we can choose a $r_p > 0$ such that the ball with radius r_p centered at p is inside the image of f_p. Then

$$Q_p^{1/2} = f_p^{-1}(1/2r_p \text{ radius ball})$$

and

$$Q_p = f_p^{-1}(r_p \text{ radius ball})$$

are both open sets in $\mathcal{O}_p$.

Define a function $h_p : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} h_p = 1 & r \leq \frac{1}{2}r_p \\ h_p = 2(1 - \frac{r}{r_p}) & \frac{1}{2}r_p \leq r \leq r_p \\ h_p = 0 & \text{outside} \end{cases}.$$

This function can be viewed as a continuous function which is 1 inside $Q_p^{1/2}$ and 0 outside Q_p.

Now the set $\{Q_p^{1/2}\}_{p \in X}$ is an open cover, so since X is compact, there is a finite subcover $\{Q_{p_1}^{1/2}, \ldots, Q_{p_m}^{1/2}\}$. Define

$$\psi_{p_k} = \frac{h_{p_k}(x)}{\sum_{i=1}^m h_{p_i}(x)}.$$

Then these ψ_{p_k} are continuous maps, and they form a partition of unity, i.e., $\sum_{i=1}^m \psi_{p_k}(x) = 1$ for any x.

Once I have a partition of unity, I am in business. Define $F : X \to \prod_m \mathbb{R}^{n+1}$, where m is the number of sets $\{Q_{p_1}^{1/2}, \ldots, Q_{p_m}^{1/2}\}$. Then we can set $F(x)$ in the kth factor of $\mathbb{R}^{n+1}$ be

$$(\psi_{p_k}(x) f_{p_k}(x), 1 - \psi_{p_k}(x))$$

in $\mathbb{R}^n \times \mathbb{R}$. You can easily check that it is an embedding, because it is locally embedding. $\square$

17 October 14, 2015

Algebraic topology is the toolbox for studying topological spaces. Topology is really visual to me, so I like to visualize things. Let me demonstrate one of the motivating examples in topology.

There was a famous magician for freeing himself from knots. If I tie my ankle and the leg of this projector, can I theoretically free myself form the projector, when we are allowed to stretch the rope but not cut it?

17.1 Homotopy

Definition 17.1. Let Z and X be topological spaces, and $f : Z \to X$ and $f' : Z \to X$ be continuous maps. Then a continuous map

$$F : Z \times [0, 1] \to X$$

such that $F(z, 0) = f(z)$ and $F(z, 1) = f'(z)$ is called an **homotopy** from f to f', and if there exists a homotopy, the two maps f and f' is called to be **homotopic**.

Example 17.2. *For any Z, consider the map $f_0 : Z \to \mathbb{R}^n$ such that $f_0(z) = 0$ for all z. Consider any other map $f : Z \to \mathbb{R}^n$. Then F defined as*

$$F(z, t) = tf(z)$$

is a homotopy from f_0 to f.

Homotopy is an equivalence relation. We will denote $f \sim f'$ if there is a homotopy from f to f'.

1. Of course it is reflexive because we can take $F(z, t) = f(z)$.
2. If $f \sim f'$ and $f' \sim f''$ then $f \sim f''$. Because an homotopy is a path between maps, we can just put the two paths together. Let $F(z, 0) = f$, $F(z, 1) = f'$, and $G(z, 0) = f'$, $G(z, 1) = f''$. Then we walk two paths twice faster and define

$$H(z, t) = \begin{cases} F(z, 2t) & t \in [0, 1/2] \\ G(z, 2t - 1) & t \in [1/2, 1] \end{cases}$$

3. To show $f' \sim f$ from $f \sim f'$, you can just walk backwards.

Equivalence class under homotopy equivalence of maps from Z to X is denoted by $[Z, X]$, and is called **homotopy classes**.

Lemma 17.3. *If $h : X \to X'$ is a map, then h induces a map $h_* : [Z, X] \to [Z, X']$. If h is a homeomorphism then $h_\star$ is bijective.*

We can draw the following diagram.

$$\begin{tikzcd} X \arrow[r, "h"] & X' \\ Z \times [0,1] \arrow[u, "F"] \arrow[ur, dashed, "h \circ F"'] & \end{tikzcd}$$

If $[f]$ is the homotopy class of $f : X \to Z$, then $h_*[f]$ is defined as the homotopy class of $h \circ f$, which is a map $X' \to Z$.

This is very useful tool. If $[Z, X]$ for some Z cannot be bijective with $[Z, X']$ then X cannot be homeomorphic to X'.

Example 17.4. *Suppose that $Z = \{*\}$ is a point. What does it mean for two maps $f_0(*) = x_0$ and $f_1(*) = x_1$ to be homotopic? The homotopy is just $F : * \times [0, 1] \to X$, which is just a path between x_0 and x_1. So $[*, X]$ is just the set of path components. Path components is just a baby version of homotopy.*

The more common ones people take for Z are spheres; $S^1, S^2, \ldots, S^n$. The sphere is not too complicated that you can't compute, but it is also not so simple that the homotopy is trivial.

There is also a notion called **pointed homotopy**. We choose specific points $z_0 \in Z$ and $x_0 \in X$, and confine our view to maps $f : Z \to X$ such that $f(z_0) = x_0$. Then a pointed homotopy is is a continuous map F such that $F(z, 0) = f$, $F(z, 1) = f'$ and $F(z_0, t) = x_0$ for all t.

The analogous lemma also holds for pointed homotopy.

Lemma 17.5. *Pointed homotopy is an equivalence relation. Also, a map $h : X \to X'$ such that $x_0 \mapsto x'_0$ induces a map*

$$h_* : [(Z, z_0), (X, x_0)] \to [(Z, z_0), (X', x'_0)].$$

If h is a homeomorphism then h_* is a bijection.

Let $Z = S^n \subset \mathbb{R}^{n+1}$ and let $z_0 = (1, 0, \ldots, 0)$. We denote

$$\pi_n(X, x_0) = [(S^n, (1, 0, \ldots, 0)), (X, x_0)].$$

This actually turns out to be a group!

17.2 Groups: a reminder

A **group** G is a set with a multiplication $m : G \times G \to G$ such that (1) it is associative: $m(a, m(b, c)) = m(m(a, b), c)$, (2) there is an identity i such that $m(i, a) = a = m(a, i)$ for all $a \in G$, (3) given a, there is an inverse a^{-1} obeying $m(a, a^{-1}) = m(a^{-1}, a) = i$.

Example 17.6. *The set of integers $\mathbb{Z}$ is a group. It is defined by $m(a, b) = a+b$ and $i = 0$ and $a^{-1} = -a$. This is an additive group, and is abelian*

Example 17.7. *The $\mathbb{Z}/p = \{1, e^{2\pi i/p}, \ldots, e^{2\pi i(p-1)/p}\}$ with $m(\eta, \eta') = \eta\eta'$ and $i = 1$ and $\eta^{-1} = \bar{\eta}$ is a group. This is multiplicative, and is also a group.*

18 October 16, 2015

If I have a space Z and a space X, and two maps $f, f' : Z \to X$, there is a notion of homotopy. This is a map $F : Z \times [0,1] \to X$ such that $F(z,0) = f(z)$ and $F(z,1) = f'(z)$. Last time I said that homotopy is an equivalence relation. This is the first observation and the second observation is that if I have a map $h : X \to X'$, there is a map $h_* : [Z,X] \to [Z,X']$. The new map h_* is a bijection if h is a homeomorphism. Also there was a pointed homotopy $(Z, z_0) \to (X, x_0)$, which is a homotopy F such that $F(z_0, t) = x_0$ for all t.

18.1 Fundamental group of a space

When $Z = S^n \subset \mathbb{R}^{n+1}$ with the base point $z_0 = (1, 0, \dots, 0)$, the homotopy classes of based maps is denoted by $\pi_n(X, x_0)$. It turns out that $\pi_n(X, x_0)$ is a group, and it is called the **nth homotopy group**. If $n = 1$, the group π_1 is called the **fundamental group**.

The rest of the lecture, I will try to explain what the group is. We first consider only S^1. The good thing about the circle is that we can parameterize it by the angle. Then map from S^1 to X is the same as a path that begins and ends at the same point.

We were able to concatenate paths. Suppose I have a path $\gamma : [0,1] \to X$ and $\gamma' : [0,1] \to X$, and $\gamma(0) = x_0, \gamma(1) = x_1$ and $\gamma'(0) = x_1, \gamma'(1) = x_2$. Then we can add them by walking along the path twice as fast. Likewise, if we have two paths with the same base point x_0, we can walk two paths and get another path which starts and ends at x_0.

So we give the group structure by

$$[\gamma][\gamma'] = [\gamma * \gamma'],$$

where $*$ means the concatenation. But there are a several things we have to show. First, we need to show that $[\gamma * \gamma']$ does not depend on the representatives γ and γ' of their equivalence class. Then we need to show that this multiplication satisfies the group axioms.

First let us prove that multiplication is well defined. Let $\gamma, \gamma_1 \in \alpha$ be two paths in a homotopy class α and $\gamma' \in \beta$ be any path. Then we need to show that $\gamma * \gamma'$ is homotopic to $\gamma_1 * \gamma'$. Let $F : [0,1] \times [0,1] \to X$ be a homotopy between γ and γ_1 such that $F(z,0) = \gamma(z)$, $F(z,1) = \gamma_1(z)$, and $F(0,t) = x_0 = F(1,t)$ for each t. We can define

$$G(z,t) = \begin{cases} F(2z, t) & z \in [0, 1/2] \\ \gamma'(2z-1) & z \in [1/2, 1] \end{cases}.$$

This gives an homotopy between a $\gamma * \gamma'$ and $\gamma_1 * \gamma'$ and establishes a multiplication.

Now we show that it is a group. What is the identity i? It is going to be the homotopy class of the constant map $S^1 \to x_0$.

Lemma 18.1 (Reparametrization lemma)**.** *If $\gamma : [0,1] \to X$ is a path and $h : [0,1] \to [0,1]$ is any continuous map such that $h(0) = 0$ and $h(1) = 1$, then $\gamma_h(z) = \gamma(h(z))$ and γ are homotopic.*

Proof. The map $F : [0,1] \times [0,1] \to X$ defined by

$$F(z,t) = \gamma(tz + (1-t)h(z))$$

is a homotopy, because $F(z,0) = \gamma_h(z)$ and $F(z,1) = \gamma(z)$. □

Now the concatenation of i and any path γ is

$$(i * \gamma)(z) = \begin{cases} \gamma(2z) & z \in [0,1/2] \\ x_0 & z \in [1/2,1] \end{cases}$$

and this is just a reparametrization of the path γ. Likewise, the other concatenation $\gamma * i$ is homotopic to γ. This shows that for any homotopy class α, we have $\alpha \cdot i = i \cdot \alpha = \alpha$.

What about the inverse? The inverse of the path γ will be $\gamma^{-1}(z) = \gamma(1-z)$. The concatenation $\gamma * \gamma^{-1}$ will be homotopic to the constant map i, because at time t we can set the path $F(z,t)$ to be the path which walks the $1-t$ of the path γ, and then turn around and walks back. Then $F(z,0)$ will be just $\gamma * \gamma^{-1}$ and $F(z,1)$ will be the consent map. This shows that the homotopy class of γ^{-1} will be the inverse of the homotopy class of γ.

Likewise, we can show that for any three paths $\gamma, \gamma', \gamma''$, the concatenation $\gamma*(\gamma'*\gamma'')$ is homotopic to the $(\gamma*\gamma')*\gamma''$, because it is just a reparametrization. Thus, multiplication is associative.

18.2 Some examples

We're there at last. The fundamental group is a group. Our next job is to calculate it.

Example 18.2. *The fundamental group $\pi_1(\mathbb{R}^n, 0) = 1$, because we can always shrink a map. The group $\pi_1(S^1, x_0) \cong \mathbb{Z}$.*

Example 18.3. *Let $SO(3)$ be the group of rotations of the Euclidean space $\mathbb{R}^3$. Then the fundamental group $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$.*

19 October 19, 2015

One of the things we'll do today is compute the fundamental group $\pi_1(S^1)$. This will be the homotopy classes of based maps from $S^1 \to S^1$ with some base point. Because we can parameterize the points of a circle by $e^{i\theta}$, we can instead consider the group of loops in S^1.

I'll be using a few lemmas implicitly.

Lemma 19.1 (Reparametrization)**.** *Let $f : [0,1] \to X$ be a path, and $h : [0,1] \to [0,1]$ be a continuous functions such that $h(0) = 0$ and $h(1) = 1$. Then $f(h(z))$ is path homotopic to $f(z)$.*

Lemma 19.2 (Concatenation)**.** *(a) Let $f : [0,1] \to X$ and $g : [0,1] \to X$ be two paths from x_0 to x_1 and x_1 to x_2. Suppose that f' is homotopic to f and g' is homotopic to g. Then $f' * g'$ is homotopic to $f * g$.*
(b) It does not matter up to homotopy how you put in the parentheses in

$$f_1 * f_2 * \cdots * f_n.$$

19.1 Fundamental group without the base point

Also, I should have mentioned this, but we have the following lemma.

Lemma 19.3. *If X is path connected, then $\pi_1(X, x_1)$ is isomorphic to $\pi_1(X, x_0)$.*

Proof. Suppose we have a homotopy class $\alpha \in \pi_1(X, x_0)$. Because X is path connected, we can pick a path from x_1 to x_0. Now if we have a path $f : [0,1] \to X$ in α, which will be based on x_0, there is a fairly straightforward way of getting a loop; you follow γ, and then f, and then go back around. If we write it down, it will be

$$\gamma^{-1} * f * \gamma.$$

The concatenation lemma shows that if f' is homotopic to f, then $\gamma^{-1} * f' * \gamma$ is homotopic to $\gamma^{-1} * f * \gamma$. So this gives a map from the homotopy classes $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$. It also preserves multiplication, because

$$(\gamma^{-1} * f * \gamma) * (\gamma^{-1} * g * \gamma) = \gamma^{-1} * f * (\gamma * \gamma^{-1}) * g * \gamma = \gamma^{-1} * (f * g) * \gamma.$$

□

19.2 $\pi_1(S^1)$

Now we get to $\pi_1(S^1)$. I've been working on this all weekend, because I didn't like the proof in the book. It introduced a lot of machinery, but after all, it's just a circle! I think I have an alternative proof, but I need you guys to check my proof.

Theorem 19.4.

$$\pi_1(S^1) \cong \mathbb{Z}.$$

Proof. Let f be any map $f : S^1 \to S^1$. Consider the set $f^{-1}(S^1 \setminus \{1\})$. Since it is clearly open, it should be a countable union of disjoint open sets. Let

$$f^{-1}(S^1 \setminus \{1\}) = \dot{\bigcup_{n=1}} (b_n, a_n).$$

Because at points b_n and a_n the value of f should be 1, in the interval b_n, a_n, the path either leaves 1 clockwise or counterclockwise, and enters 1 either clockwise or counterclockwise. If it comes out and in above the x axis, or both below the x axis, and does all the junk between, then it is just homotopic to not moving at all. So the things we need to worry about is intervals for which the path exist 1 in one direction and enters it in the other direction.

Let N_+ be the number of intervals that wind the circle counterclockwise, and N_- be the number of intervals that winds clockwise. Note that because the map f should be continuous, there cannot be infinitely many these things. That is, N_+ and N_- is finite. Now I give the homomorphism $\pi_1(S^1) \to \mathbb{Z}$ as

$$f \mapsto N_+ - N_-.$$

We should get rid of the junk in the middle of the counterclockwise intervals. If we parametrize the value of f by the angle, we have a map $h : [0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$. Then we see that f in the interval is just homotopic to the simple path walking along the circle in constant speed. Therefore, the path f is actually homotopic to the concatenation of N_+ counterclockwise paths and N_- clockwise paths, and we can cancel out each one of the paths. $\square$

20 October 23, 2015[1]

We showed that $\pi_1(S^1) \simeq \mathbb{Z}$. We shall prove that $\pi_1(S^n) \simeq (1)$ for $n > 1$.

20.1 $\pi_1(S^n)$

Let $(1, 0, 0, \dots, 0)$ be the base point, and consider

$$D = \{(x_1, \dots, x_{n+1}) : |1 - x_1| \leq r\} \cap S^n,$$

where $r = 1/1000$. Then the boundary will be

$$\partial D = \{x_1 = 1 - r, |(x_2, \dots, x_{n+1})| = \sqrt{2r - r^2}\}.$$

Like when we proved that $\pi_1(S^1) \simeq \mathbb{Z}$, we consider the image $f^{-1}(S^n \setminus D)$ for some closed path f. This will consist of countably many open intervals. Let $\mathcal{U}$ be the set of those open intervals. Then on any intervals $(a, b) \in \mathcal{U}$, the function (or path) f sends (a, b) to $S^n \setminus D$ and $[a, b]$ to $S^n \setminus (D \setminus \partial D)$.

Now for any path f from $f(a)$ to $f(b)$ lying in $S^n \setminus D$, you can push the path all around the sphere to make it inside D. (I will post a exact formula in the notes.) Then all the paths lie in the D. Because D is homeomorphic to B^n, every path is homotopic to the point. Therefore, there is only one element in $\pi_1(S^n)$.

Lemma 20.1. *Let $A \subset X$ be a closed set, and let $A \subset U \subset X$ be a open set containing A. Let $f : [a, b] \to X$ such that $f(a), f(b) \in A$. Then $f^{-1}(X \setminus A)$ is a countable collection of disjoint intervals. Let the set of open intervals be $\mathcal{U}$. Then there are only finitely many intervals (a, b) from $\mathcal{U}$ such that there is a point $z \in (a, b)$ such that $f(z) \in X \setminus U$.*

20.2 Retractions on the plane

Why is there a spot on the head where the hair sticks out in all directions? We will show that it is not some evolutionary thing, but a underlying principle in topology.

Definition 20.2. Let X be a space and let $A \subset X$. A continuous map $\mathfrak{r} : X \to A$ such that $\mathfrak{r}(a) = a$ on A is called a **retraction**. If there is a reaction, A is aid to be a **retract** of X.

Example 20.3. *The origin is a retract of $\{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 \leq 1\}$ because we can let $\mathfrak{r}(x, y) = (0, 0)$. The sphere S^{n-1} is a retract of $\mathbb{R}^n \setminus 0$ because $\vec{x} \mapsto \vec{x}/|\vec{x}|$ is a retraction.*

Lemma 20.4. *If A is a retract of X then $j : A \to X$ induces a injective homomorphism*

$$j_* : \pi_1(A, a_0) \to \pi_1(X, a_0).$$

[1] I did not attend class on October 21, 2015.

Proof. Let $f : [0,1] \to A$ be a loop. Suppose that there is a homotopy $F : [0,1] \times [0,1] \to X$ in X such that $F(z,0) = f(z)$ and $F(z,1) = p_0$. Then $\mathfrak{r}(F(z,t))$ is a homotopy from f to a constant map in A. This shows that j_* is injective. $\square$

Example 20.5. *Let D be the disk $x^2 + y^2 \leq 1$, and let A be the boundary. Because $\pi_1(A) \simeq \mathbb{Z}$ and $\pi_1(D) = (1)$, the boundary A is not a retract of D.*

Theorem 20.6. *The identity map from $i : S^1 \to S^1$ is not homotopic to the map $S^1 \to$ point $\in S^1$.*

Proof. Suppose that there is a point $1 \in S^1$, and a homotopy $F(\vec{v},t) : S^1 \times [0,1] \to S^1$ such that

$$F(\vec{v},0) = 1, \quad F(\vec{v},1) = \vec{v}.$$

We can define a retraction $\mathfrak{r} : B^2 \to S^1$ by

$$\mathfrak{r}(t\vec{v}) = F(\vec{v},t).$$

This is continuous, because if t gets small, $F(\vec{v},t)$ gets closer to 1. But we already proved that there is no retraction. Therefore we get a contradiction. $\square$

21 October 26, 2015

I want to deviate from the book and talk about why $\pi_n(S^n)$ is not zero.

21.1 The hair whorl lemma

A retraction of X onto subspace A was a map $\mathfrak{r} : X \to A$ such that the restriction $\mathfrak{r}|_A$ is the identity. If A is a retract of X and $j : A \to X$ is the inclusion map, then there is a canonical inclusion map $j_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ which is injective. Because $\pi_1(S^1) \simeq \mathbb{Z}$ and $\pi_1(S^1) \simeq 0$, the circle S^1 is not a retract of the disk D^2. This implies that the identity map from S^1 to S^1 is not homotopic to the constant map (even allowing the base point moving).

Theorem 21.1. *Let $v : D^2 \to \mathbb{R}^2$ be a vector field, and suppose that $v \neq 0$ anywhere. Then there is a point on ∂D where it points out and one where it point in.*

Proof. Because $v \neq 0$, we actually have a map $v : D \to \mathbb{R} \setminus \{0\}$. Then we get a $q : D \to S^1$ defined by

$$x \mapsto \frac{v(x)}{|v(x)|}.$$

The q on the boundary $q|_{\partial D}$ is null homotopic, i.e., homotopic to a constant map, because there is a homotopy

$$F(\hat{u}, t) = q(t\hat{u}).$$

Suppose that $q(\hat{u}) \neq \hat{u}$ for any $\hat{u} \in S^1$. (This means that the vector field never points out.) Then there is homotopy $G : S^1 \times [0, 1] \to S^1$ defined by

$$G(\hat{u}, t) = \frac{(1-t)q(\hat{u}) - t\hat{u}}{|(1-t)q(\hat{u}) - t\hat{u}|}.$$

This contradicts that the identity map is not null homotopic. □

This is known as the "Hair whorl lemma". This also shows that there must be an eye in a hurricane.

21.2 Deformation retract

Definition 21.2. Let A be a subspace of X. A **deformation retract** is a map $F : X \times [0, 1] \to X$ such that $F(x, 0) = x$ and $F(x, 1) \in A$ for any $x \in X$, and also $F(a, t) = a$ for all $a \in A, 0 \leq t \leq 1$.

If A is a deformation retract of X, then $\pi_1(X, a_0) \simeq \pi_1(A, a_0)$. This is because just you can retract any path of X into A.

Example 21.3. *The space $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1}. The function*

$$F(x, t) = \frac{x}{(1-t) + t|x|}$$

is a deformation retract.

This actually shows that $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \pi_1(S^1) = \mathbb{Z}$. Also, $\pi_1(\mathbb{R}^n \setminus \{0\}) = \pi_1(S^{n-1}) = (1)$ for all $n \geq 3$. Hence $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Example 21.4. *Consider a torus $T \simeq S^1 \times S^1$. Then there is a retract $T \to S^1$ as the projection, but there is no deformation retract from T into a circle, because the fundamental group of T is $\mathbb{Z}^2$ and the fundamental group of S^1 is $\mathbb{Z}$.*

We want to show that $\pi_n(S^n) \neq (1)$. In particular, we shall show that the identity i is not based homotopic to a constant map. The proof will go something like this. Assume for $n \geq 2$ that the identity map on S^{n-1} is not homotopic to a constant map. Then we prove that the identity on S^n is not homotopic to a constant. If we assume the induction hypothesis, then the n-dimensional ball does not retract onto the boundary. If there were a retract $\mathfrak{r} : B^3 \to S^2$, then let $Q(\hat{u}, t) = t\hat{u}$. The homotopy $F(\hat{u}, t) = \mathfrak{r}(Q(\hat{u}, t))$ will be a homotopy from the identity map to a constant map.

22 October 28, 2015

I will prove today that the identity map S^n to S^n is not homotopic to the constant map, or at least convince you. This implies that there is no retraction $B^{n+1} \to S^n$. Also, I won't prove this, but if $m < n$, the homotopy group $\pi_m(S^n)$ is trivial. It follows from this fact that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

What will be $\pi_{n+k}(S^n)$ for $k > 0$? It turns out to be a really hard problem. In many cases, it is nonzero, and it is hard to compute. For instance, $\pi_3(S^2) = \mathbb{Z}$ and $\pi_4(S^2) = \mathbb{Z}/2\mathbb{Z}$.

22.1 $\pi_n(S^n)$

Last time, we started by using induction. Suppose that the identity map on S^{n-1} is not homotopic to S^{n-1}. Then there is no retract of B^n onto S^{n-1}.

Now consider S^n lying inside $\mathbb{R}^{n+1}$. Consider the fixed point $x_0 = (1, 0, \dots, 0)$ on the sphere S^n. Consider the set of unit vectors whose angle from x_0 is σ. Then the set of such points is the same as the S^{n-1}. Also, we can consider

$$B^n = r \cdot \vec{\mu} \mapsto (x_0 = \cos\theta, \sin\theta\vec{\mu}) \quad \text{where } \theta(r) = (1-r)\pi + r\sigma.$$

Then this embeds B^n onto S^n so that it misses only a small open disk enclosed in the $(n-1)$-sphere $\theta = \sigma$.

Now we want to push the map into some disk $\theta \le \pi/4$. For a homotopy F, we define

$$\begin{cases} \Phi_1(x,t) = x & \theta(x) \le \sigma \\ \Phi_1(x,t) = F(x,t) & \theta(x) \ge 2\sigma \\ \Phi_1(x,t) = F(x, (\theta/\sigma - 1)t). & \sigma \le \theta \le 2\sigma \end{cases}$$

We want to find a continuous function $\tau : S^n \to [0,1]$ such that $\Phi_1(x, \tau(x))$ has θ between σ and $\pi/4$ for every $\theta(x) \ge \sigma$. One possible candidate is stopping x on the last crossing of $\pi/4$. That is, we define $\tau_-(x)$ as the greatest lower bound of t such that $\Phi_1(x,t)$ has $\theta < \pi/4$. Is this continuous? Unfortunately, it is not. But if $(x_i) \to x$ converges then $\lim_{i\to\infty} \tau_-(x_i) \le \tau(x)$. But it might be strictly smaller. Likewise if we define $\tau_+(x)$ as the last time $\Phi_1(x,t)$ comes into the closed set $\theta \le \sigma$, then we get $\lim_{i\to\infty} \tau_+(x_i) \ge \tau(x)$.

Consider the interval $[\tau_-(x), \tau_+(x)]$. Let B_x be the ball centered at x such that for any $x' \in B_x$, the point $\frac{1}{2}(\tau_-(x) + \tau_+(x))$ is inside $[\tau_-(x', \tau_+(x')]$. Now we have a open cover

$$B_L = \bigcup_x B_x$$

where B_L is the ball $\theta \ge \sigma$. Then there is a finite subcover $B_{x_1}, \dots, B_{x_N}$. We can consider the partition of unity $\varphi_1, \dots, \varphi_N$, such that $\varphi_k(x) = 0$ outside of B_k for each k and $\sum_k \varphi_k = 1$. Then we can let

$$\tau(x) = \sum_k ((\tau_-(x_k) + \tau_+(x_k))/2) \cdot \varphi_k(x).$$

This is a finite sum of continuous functions and hence continuous.

23 October 30, 2015

We are going to talk about covering spaces.

23.1 Covering maps

Definition 23.1. Let $p : E \to B$ be a continuous and surjective map. An open set $U \subset B$ is called to be **evenly covered** by p if

$$p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$$

where $p : V_\alpha \to U$ is a homeomorphism. The map p is called a **covering map** if for any $x \in B$ there is a open neighborhood of x which is evenly covered by p.

Let us make some observations. If $p : E \to B$ is a covering map, then $p^{-1}(b) = \sum_{\alpha \in A} e_\alpha$ has a discrete topology. Also, p is open; p takes open sets to open sets. This is because p is a "local homeomorphism," i.e., given $e \in E$ there is a neighborhood V of e such that $p : V \to p(V)$ is a homeomorphism.

Example 23.2. *Let Λ be any set with discrete topology. Let B be any space, and let $E = B \times \Lambda$. We can consider the projection map $p : E \to B$ which maps $(b, \alpha) \mapsto b$. This is a trivial covering space.*

Example 23.3. *Consider the map $\mathbb{R} \to S^1$ which maps $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. This is a covering map. For instance, consider a small open neighborhood of $(1, 0)$. The inverse image of this neighborhood will consist of copies of small open intervals around $0, 1, 2, \ldots$ and $-1, -2, \ldots$.*

Example 23.4. *Consider the space of lines in $\mathbb{R}^3$ passing through the origin. This space is called the $\mathbb{R}P^2$. We can give it a topology by the map $\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$. Let $p : S^2 \to \mathbb{R}P^2$ be the map $x \mapsto$ (line through 0 and x). This is a covering map. Likewise, we have a covering map $S^n \to \mathbb{R}P^n$. These are all 2-to-1 covering map.*

Example 23.5. *There is a 2-to-1 covering of a double torus by a triple torus. It can be constructed by cutting one handle of the torus and gluing nontrivially. Likewise, there is a n-to-1 covering of a double torus by a $(n+1)$-torus*

Example 23.6. *Consider the figure eight. This is two S^1s glued by a point. There are two loops a and b. If we interpret walking along a counterclockwise as going up, and walking along b counterclockwise as going right, we get a covering space from the grid graph on $\mathbb{Z}^2$ to the figure eight. Actually, there is a more complicated covering space, which is called the Caley graph.*

23.2 Path lifting lemma

Lemma 23.7 (Path lifting lemma)**.** *Suppose we have a covering map $p : E \to B$ and a point $e_0 \in E$. Let $b_0 = p(e_0)$. Given $f : [0, 1] \to B$ with $f(0) = b_0$, there is a unique "lift" $\hat{f} : [0, 1] \to E$ such that $\hat{f}(0) = e_0$ and $p(\hat{f}(t)) = f(t)$.*

Proof. We know that $f([0,1])$ is compact. Also, for each $t \in [0,1]$ there is an open U_t such that $f(t) \in U_t$ and $p^{-1}(U_t) = \bigcup_{\alpha \in A} V_\alpha$ where $p : V_\alpha \to U_t$ is a homeomorphism. Because U_t is open, there is a small interval $f : (t-\epsilon_t, t+\epsilon_t) \to U_t$. Because these intervals $\{(t-\epsilon_t, t+\epsilon_t)\}$ cover $[0,1]$ there is a finite cover $0 \leq t_1 < \cdots < t_n \leq 1$ such that $(t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$ covers $[0,1]$.

Consider the first interval $[0, t_1 + \epsilon)$. The path lines in U_1 and there is a stack of sets homeomorphic to U_1 over U_1. Then the path f should lift into the stack where the base point lies. Likewise, the original path in $(t_2 - \epsilon_{t_2}, t_2 + \epsilon_{t_2})$ should stick into U_2 and we can lift it uniquely to E. Doing this finitely many times, we get a unique lifting of the path f. $\square$

24 November 2, 2015

Let $p : E \to B$ be a map. An open set $U \subset B$ is evenly covered when $p^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$ for disjoint set where $p : V_\alpha \to U$ is a homeomorphism. E is a covering space of B if every $b \in B$ has an evenly covered neighborhood. We have a lifting lemma. If $f : [0,1] \to B$ and $f(0) = b_0$ then there is a unique lift $\hat{f} : [0,1] \to E$ of the path such that $\hat{f}(0) = e_0$ and $p(\hat{f}) = f$. There is also a lift of homotopies.

24.1 Homotopy lifting lemma

Lemma 24.1. *Let $F : [0,1] \times [0,1] \to B$ be a homotopy, and let $F(0,0) = b_0$. There therein a unique lift of F to $\hat{F} : [0,1] \times [0,1] \to E$ such that $\hat{F}(0,0) = e_0$ and $p(\hat{F}) = F$.*

Proof. We break up the square into a grid of smaller squares. If the squares are small enough, then the image of one small square in B should be evenly covered by p. Now for each square, there is a whole bunch of ways to lift it into E. The only choice we have is where to lift it into.

We label the squares in the grid by labeling the squares in the first row by $1, 2, \dots, k$ in order, and then second row by $(k+1), \dots, 2k$, and proceeding in a similar way. Then because before lifting a square there is already a base point determined, we see that there is a unique way of lifting. $\square$

The (possibly) annoying thing about lifting of paths is that two paths which starts and ends at the same points can be lifted to two paths which starts at the same point but ends at different points. However, if they are homotopic, then the endpoints should be the same.

Theorem 24.2. *If f is homotopic to g then $\hat{f}(1) = \hat{g}(1)$.*

Proof. There is a homotopy $F : I \times I \to B$ such that

$$F(0,t) = b_0, \quad F(1,t) = b_1, \quad F(x,0) = f(x), \quad F(x,1) = g(x).$$

We can lift the homotopy F to a homotopy $\hat{F}$ such that $\hat{F}(0,0) = e_0$. If t is fixed, then $x \to \hat{F}(x,t)$ is a lifting of $f_t(x)$. how $F(1,t) = b_1$ for all t, and because $p^{-1}(b_1)$ has a discrete topology, $F(1,t)$ should be some constant point. Then $\hat{f}(1) = \hat{g}(1) = \hat{F}(1,0) = \hat{F}(1,1)$. $\square$

Let us look at loops. If $f : [0,1] \to B$ is some loop with $f(0) = f(1) = b_0$, then the lift of f should begin at some point e_0 and end at some point e_1 with $e_0, e_1 \in p^{-1}(b_0)$. This induces a map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

If E is path connected, the map ϕ is surjective.

This is relatively easy to show. Let $e_1 \in p^{-1}(b_0)$ be any point in E. Because E is path connected, there is a path γ from e_0 to e_1. Project this downstairs to get $p(\gamma)$. This is a loop f in B which starts and ends at b_0. By the uniqueness of lifting of paths, the lift of f back in E should be just γ. Then ϕ should associate γ with e_1.

The next question is when is it bijective? In fact, the set $p^{-1}(b_0)$ is bijective with the set of right cosets $\pi_1(B, b_0)/p_*(\pi_1(E, e_0))$. That is, we have the following exact sequence.

$$(1) \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\phi} p^{-1}(b_0) \longrightarrow (1)$$

We will try to prove this on Wednesday.

25 November 4, 2015

Today I want to continue exploring covering spaces. Let me remind you that the covering space consists of data (B, E, p). The $p : E \to B$ is a surjective map, and every point in B has an evenly covered open neighborhood. By the path and homotopy lifting lemma, we obtained a map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ which is surjective if E is path connected. But what can we say about injectivity?

25.1 Covering map and the fundamental group

The covering map $p : E \to B$ induces a group homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$.

Lemma 25.1. *p_* is injective.*

Proof. Let $\gamma, \gamma' : [0, 1] \to E$, and let $F : [0, 1] \times [0, 1] \to B$ be a homotopy from $p \circ \gamma$ to $p \circ \gamma'$. Let's take the lift of this homotopy and see what it does.

Let $\hat{F} : [0, 1] \times [0, 1] \to E$ be the lift of F. We only need to check that the sides are what we want. Using unique path lifting, we can just follow $\hat{F}$ along the edges so that they are in fact what we want. $\square$

What are the homotopy classes $[f] \in \pi_1(B, b_0)$ such that $\phi([f]) = e_0$? These are the paths whose lift is a loop which is actually in E. Then there would be a loop $[\hat{f}] \in \pi_1(E, e_0)$ such that $p_*([f]) = f$. Conversely, if $[\hat{f}]$ is in $\pi_1(E, e_0)$ then ϕ maps the projection $[f]$ to just e_0. What we have proved is the following: $\phi[f] = e_0$ if and only if $[f] = p_*[\gamma]$ for $[\gamma] \in \pi_1(E, e_0)$. Or alternatively, we have the following exact sequence (of sets):

$$(1) \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\phi} p^{-1}(b_0) \longrightarrow (1)$$

25.2 Algebraic interlude

Let K and G be a group. To say that $\psi : K \to G$ is a group homomorphism is to say that $\psi(k_1 k_2) = \psi(k_1)\psi(k_2)$. This implies that $\psi(1) = 1$ and $\psi(a^{-1}) = \psi(a)^{-1}$ and $\psi(K) \subset G$ is always a subgroup.

Example 25.2. *Let $G = \mathbb{Z}$ and $K = \mathbb{Z}$ and $\psi_n : K \to G$ sends $\ell \mapsto n \cdot \ell$.*

If $H \subset G$ is a subset, we can think of the right coset space G/H. This is the equivalence classes induced by $g \sim g'$ when $g' = g \cdot h$ for some $h \in H$. One can verify that this is an equivalence relation.

Theorem 25.3. *We have the exact sequence of groups*

$$1 \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \longrightarrow \pi_1(B, b_0)/(p_* \pi_1(E, e_0)) \longrightarrow 1$$

26 November 6, 2015

Most of today's lecture will be about the existence of a covering space. What I showed last time was that if $p : E \to B$ is a covering map then the induced map $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective, and that there is a bijection $\pi_1(B, b_0)/\pi_1(E, e_0) \leftrightarrow p^{-1}(b_0)$.

26.1 Tree covering space of the figure eight loop

Consider the figure eight with the loops a and b. There is a covering space of this which looks like a tree. For each point, there is a vertical branch and a horizontal branch. If you walk clockwise on the a loop, then you walk right in the covering space. If you walk clockwise on the b loop, you walk up. The branches never intersect. This is not possible to draw in the $\mathbb{R}^2$ plane, but you can think it in the space $\mathbb{R}^3$ and push around the branches so they don't intersect. This kind of thing is called a tree.

In the fundamental group, $aba^{-1}b^{-1}$ is not the identity, because in the covering space, the path doesn't bring you to the origin. This shows that the fundamental group is not commutative. In fact, any word $a^{n_1}b^{m_1}a^{n_2}b^{m_2}\cdots a^{n_k}b^{m_k}$ for $m_i \neq 0$ and $n_i \neq 0$ cannot be reduced into simple terms in the fundamental group. This kind of group is called the free group on 2 generators. I will probably talk more about this in the following weeks.

26.2 Constructing a covering space (1)

Definition 26.1. A space X is called **locally path connected** if given any p and open set $p \in U$ there is some $p \in V \subset U$ which is path connected.

Example 26.2. *Consider the topologists sine curve with an extra segment connecting the curve and the y-axis. Then this space is path connected, but not locally path connected.*

Definition 26.3. A space X is **semi-locally simply connected** if for any point b there is an open set $b \in U$ such that $\pi_1(U, b) \to \pi_1(B, b)$ sends everything to i.

Note that this is slightly weaker than "locally simply connected," because the loop can go outside U and then contract to a point.

Example 26.4. *Consider the infinite earring, which is*

$$\bigcup_{n=1}^{\infty} \{(x, y) : (x - \tfrac{1}{n})^2 + y^2 = (\tfrac{1}{n})^2\}.$$

This is not semi-locally simply connected. But if you cover every circle with a cup, then it is semi-locally simply connected.

To construct the covering space, we require our space B to be path connected, locally path connected, and semi-locally simply connected.

Theorem 26.5. *Let B be a path connected, locally path connected, and semi-locally simply connected space. Suppose that $H \subset \pi_1(B, b_0)$ is a subgroup. Then there is a covering space data $p : E \to B$ such that $p_*(\pi_1(E, e_0)) = H$.*

We first construct E as a set. Then we say what p is. Then we give E a topology, and then we verify that p is a covering space, and that $p_*(\pi_1(E, e_0)) = H$.

Proof (part 1). Let

$$\Omega_0 = \{\text{Paths } f : [0,1] \to B \text{ such that } f(0) = b_0\}.$$

We are going to consider the equivalence classes of Ω_0. The equivalence relation is defined by $f \sim f'$ if (i) $f(1) = f'(1)$ and (ii) $f^{-1} * f'$ has its homotopy class in H. We claim that this is an equivalence relation. First $f \sim f$ because $f^-1 * f$ is homotopic to b_0. If $f \sim f'$ then $f^{-1} * f'$is a class in H and then because H is a group $(f^{-1} * f')^{-1} = (f')^{-1} * f$ is in H. Lastly, if $f \sim f'$ and $f' \sim f''$, then we have $[f^{-1} * f'], [f'^{-1} * f''] \in H$. If we multiply these two elements, we get

$$[f^{-1} * f' * f'^{-1} * f''] = [f^{-1} * f''] \in H.$$

Then we see that $f \sim f''$ and hence this is an equivalence relation.

Now define E is the equivalence classes of Ω_0 quotiented by the equivalence relation. $\square$

27 November 9, 2015

We will continue our talk about constructing covering spaces.

27.1 Constructing a covering space (2)

Theorem 27.1. *Let B be a path connected, locally path connected, and semi-locally simply connected space. Let $H \subset \pi_1(B, b_0)$ be a subgroup of the fundamental group. Then there is a covering space data $p : E \to B$ such that*

$$p_*(\pi_1(E, e_0)) = H.$$

Proof. Let

$$\Omega_0 = \text{All paths in } B \text{ that start at } b_0.$$

We give an equivalence relation so that $f \sim f'$ if and only if (i) $f(1) = f'(1)$, and (ii) $f^{-1} * f'$ is a loop at b_0 whose homotopy class is in H. Let

$$E = \text{Set of equivalence classes.}$$

For a path f denote its class by $f^\sharp$. We define a covering map $p : E \to B$ by sending $f^\sharp \mapsto f(1)$.

Example 27.2. *If $B = S^1$ then $\pi_1 \simeq \mathbb{Z}$. Suppose, for example, that $H = 3\mathbb{Z}$. The paths in E that are projected onto the base point b_0 in B by the map p are the elements of the fundamental group. But because going three times is same as the constant map, we see that there are exactly three elements in $p^{-1}(b_0)$. Likewise, for any other point, the inverse image consists of three points. If you move the three points around, you will get the 3-fold covering space.*

We need now to give a topology on E. We give the basis of the topology. For each open $U \subset B$ and $f^\sharp \in E$ such that $f(1) \in U$, define

$$\begin{aligned} B(U, f^\sharp) = \{&\text{Equivalence classes of paths } \alpha * f \text{ with} \\ &\alpha \text{ starting at } f(1) \text{ and entirely in } U\}. \end{aligned}$$

We give the topology in E so that $\{B(U, f^\sharp)\}$ is a basis.

For any point $b \in B$, consider a neighborhood U of b which is both path connected, and $\pi_1(U, b) \to (1)$. Let f be a path in B which starts at b_0 and ends at b. The map p takes a path $\alpha * f \in B(U, f^\sharp)$ and maps it to $\alpha(1)$. This shows that p restricted on $B(U, f^\sharp)$ is a surjection onto U. Moreover, it is one-to-one, because for any α, β inside U with $\alpha(0) = \beta(0) = b$, we have, by semi-locally connectedness,

$$[f^{-1} * \beta^{-1} * \alpha * f] = [f^{-1} * f] = [i].$$

This shows that $p|_{B(U, f^\sharp)}$ is a bijection between the set and U, and more work shows that it is a homeomorphism.

Now if we choose a different path f' from b_0 to b, this (possibly) gives a different set $B(U, f'^\sharp)$. These all are homeomorphic to U by the restriction of

p, and thus the inverse image of U will consist of these stacks homeomorphic to U.

The inverse image $p^{-1}(b_0)$ is the set of left cosets $\pi_1(B, b_0)/H$, by construction. This is because $[f] = [f']$ if and only if $[f]^{-1} * [f'] \in H$, or $[f'] \in [f] * H$. $\square$

28 November 11, 2015

What I want to do today is finish the discussion on covering spaces. Then I will do a new topic.

28.1 Constructing a covering space (3)

Theorem 28.1. *Let $b \in B$ and let H be a subgroup of $\pi_1(B, e_0)$. Then there is a covering space E with a map $p : E \to B$ such that $p_*(\pi_1(E, e_0)) = H$.*

Proof. On $\Omega = \{f : [0,1] \to B, f(0) = b_0\}$, we gave a equivalence relation and considered the equivalence classes. We should give a topology now.

A basis $\mathcal{B}$ for a topology is a set of subspaces such that for any $B_1, B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ for which $p \in B_3 \subset B_1 \cap B_2$. We define

$$B(U, f^\sharp) = \{\alpha * f * h : [h] \text{ is in } H,\ \alpha \text{ starts at } f(1) \text{ and stays in } U\}.$$

We have to check the intersection property. For instance, by definition we have

$$B(U, f^\sharp) \cap B(V, f^\sharp) = B(U \cap V, f^\sharp).$$

But this is not only the possible form of intersection. Suppose that $h^\sharp \in B(U, f^\sharp) \cap B(V, g^\sharp)$. We claim that $B(U \cap V, h^\sharp)$ works.

Because $h^\sharp$ is in both basis elements, we see that $h(1) \in U \cap V$. Also, there is a representation of h as

$$h = \alpha * f * \tau = \beta * g * \sigma$$

where $\alpha \subset U$, $\beta \subset V$, and $\tau, \sigma \in H$. By the thing we have shown above, we see that

$$B(U \cap V, h^\sharp) = B(U, h^\sharp) \cap B(V, h^\sharp).$$

Lemma 28.2. *If $h \in B(U, f^\sharp)$, then $B(U, h^\sharp) = B(U, f^\sharp)$.*

Proof. Because $h \in B(U, f^\sharp)$, there is a representation

$$h = \alpha * f * \sigma$$

for some $[\sigma] \in H$. For any $\psi \in B(U, h^\sharp)$, there is a representation

$$\psi = \beta * h * \tau$$

for some $[\tau] \in H$. Then we have

$$\begin{aligned} \psi &\sim \beta * (\alpha * f * \sigma) * \tau \\ &\sim (\beta * \alpha) * f * (\sigma * \tau). \end{aligned}$$

□

Applying the lemma, we see that

$$B(U \cap V, h^\sharp) = B(U, f^\sharp) \cap B(V, g^\sharp).$$

So this is indeed a basis. $\square$

This is very beautiful in theory, but if you can't guess the covering space, you don't know the fundamental group. So it is completely useless in computing the fundamental group.

28.2 Motivation for Seifert-van Kampen theorem

Let U, V, and $U \cap V$ be path connected open sets. What will a typical loop in $U \cup V$ based on a point in $U \cap V$ look like? It will play around in U for some time, then pass $U \cap V$ and then play around in V for some time, then go back and again to U and then come back to $U \cap V$ and play around inside the intersection and then go to the base point. Then by making a few "jogs," we can write this path in some form of $\gamma_3 * \gamma_2 * \gamma_1$ where $\gamma_1, \gamma_3 \in \pi_1(U)$ and $\gamma_2 \in \pi_1(V)$.

Now if V is simply connected, then everything can be reduced into a loop in U. This suggests that we might be able to figure out $\pi_1(U \cup V)$ in terms of $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$.

28.3 Generators of a group

A group is a set G with a binary operation $G \times G \to G$ which satisfies certain certain properties. We can present a group using generators. A set $\{g_1, \ldots, g_N\}$ is called a set of **generators** if every $g \in G$ can be expressed using the generators by

$$g = g_{\alpha_1}^{n_1} g_{\alpha_2}^{n_2} \cdots g_{\alpha_\mu}^{n_\mu},$$

where $g_{\alpha_j} \in \{g_1, \ldots, g_N\}$ and $n_j \in \{1, -1\}$ for any j.

Example 28.3. *The additive group $\mathbb{Z}$ is generated by 1. The multiplicative group in the complex numbers generated by $e^{2\pi i/p}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The group P_n of permutations of $\{1, 2, \ldots, n\}$, which has $n!$ elements, is generated by the transposition (12) and $(123 \cdots n)$.*

An element in G is said to be **torsion** if $x^p = 1$ for some p.

An **abelian group** is a group such that $gh = hg$ for any h and g.

If G is an abelian group, and $G_1, G_2 \subset G$ are subgroups, and every $g \in G$ can be written as $g = g_1 + g_2$ for $g_1 \in G_1$ and $g_2 \in G_2$, we write $G = G_1 + G_2$. If this decomposition is unique, we say that G is a **direct sum** and write

$$G = G_1 \oplus G_2.$$

29 November 13, 2015

Let U, V and $U \cap V$ be path connected sets, and let x_0 be a base point inside $U \cap V$. Then you can know what $\pi_1(U \cup V, x_0)$ by only knowing $\pi_1(U, x_0)$, $\pi_1(V, x_0)$, and $\pi_1(U \cap V, x_0)$. This is the idea of the Seifert-van Kampen theorem.

Example 29.1. *You can cut a torus into a union of two bands, and a rectangle. The union of two bands is homotopic to the figure eight, and we know that the fundamental group of this is just the set of words, which is the free group* $\mathbb{Z} * \mathbb{Z}$.

29.1 Freely generated abelian group

Because the free group of general groups is a bit complicated, let us talk about abelian groups first. Let us look at $G_1 = 2\mathbb{Z}$ and $G_2 = 3\mathbb{Z}$. The group $G = \mathbb{Z}$ is generated by G_1 and G_2, because $n = 3n - 2n$, but there is a certain amount of redundancy, because $6 = 2 + 2 + 2 = 3 + 3$.

Definition 29.2. A group G is **freely generated** by $\{G_\alpha\}$ when for every $g \in G$, there is a unique representation

$$g = g_{\alpha_1} + g_{\alpha_2} + \cdots + g_{\alpha_n}.$$

If G is generated by $\{G_\alpha\}$ we write

$$G = \sum_\alpha G_\alpha.$$

If G is freely generated by $\{G_\alpha\}$ we write

$$G = \bigoplus_\alpha G_\alpha.$$

We now want to construct G that is freely generated by $\{G_\alpha\}_{\alpha \in G}$. If J is finite, we can just consider the product

$$\prod_{\alpha \in J} G_\alpha = G_1 \times G_2 \times \cdots \times G_n$$

which is the set of tuples $(g_1, \dots, g_n)$ endowed with the binary operation inherited from each of G_k. There is a natural embedding $G_k \hookrightarrow G$, and one can prove that G is indeed freely generated by its images.

If J is infinite, then there is a problem because $\prod_\alpha G_\alpha$ is not generated by $\{G_\alpha\}$. So we instead consider the subgroup of $\prod_\alpha G_\alpha$ consisting of elements g such that $(g)_\alpha = 0_\alpha$ except for only finitely many α. One can check that this indeed is freely generated by $\{G_\alpha\}$.

For an element $g \in G$, denote

$$G_g = \{\cdots, -2g, -g, 0, g, 2g, \cdots\} = \langle g \rangle.$$

Then we see that G is generated by $\{G_{g_\alpha}\}$ if and only if G is generated by $\{g_\alpha\}$. If G_g is finite, we say that g is **torsion**. If G_g is infinite, this is said to be infinite cyclic, and is isomorphic to $\mathbb{Z}$.

Definition 29.3. A group G is said to be freely generated by $\{a_\alpha\}_{\alpha\in J} \subset G$ if

$$G = \oplus_{k=1}^{n} \langle a_k \rangle.$$

In this case, $\{a_1, \ldots, a_n\}$ is said to be a basis for G.

If J is finite and has n elements, then

$$G \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

If G is a free abelian group, then $G/2G$ can be regarded as a vector space over $\mathbb{Z}/2\mathbb{Z}$. Because the dimension of a vector space is independent of the choice of the basis, we see that the size of the basis for G is also independent of the choice of the basis. This is a slick proof that we have some notion of "dimension."

29.2 Freely generated group

Now let us look at the non-abelian groups. A group G is generated by $\{G_\alpha\}$ if any element in G can be written as a word of elements in some G_α.

Example 29.4. *Consider the permutation group of* $\{1, 2, 3\}$*. Let* $a = (12)(3)$ *and let* $b = (1)(23)$*. Then* $a^2 = 1$ *and* $b^2 = 1$ *and we let* $G_1 = \{1, a\}$ *and* $G_2 = \{1, b\}$*. Then* G_1 *and* G_2 *generate* G*, because*

$$G = \{1, a, b, ab, ba, bab\}.$$

But it is not "freely generated" because $aba = bab$*.*

Definition 29.5. A group G is freely generated by $\{G_\alpha\}_{\alpha\in J}$ if (i) $G_\alpha \cap G_\beta = \{i\}$ for any $\alpha \neq \beta$, and (ii) $\cdots\cdots$.

30 November 16, 2015

We're still setting up the stage for the Seifert-van Kampen theorem.

30.1 Free product of groups

In the abelian case, given a set $\{G_\alpha\}_{\alpha \in J}$ of abelian groups, we were finding an abelian G with a injective map $i_\alpha : G_\alpha \to G$ such that G is the direct sum of $\{i_\alpha(G_\alpha)\}$.

Now let us look at the non-abelian case. We want to find a group G with an injection $i_\alpha : G_\alpha \to G$ so that G is the free product of $\{i_\alpha(G_\alpha)\}$. This means that for each $g \in G$ there is a unique representation

$$g = g_1 g_2 \cdots g_n$$

with $g_k \in G_{\alpha_k}$ and $g_{k+1} \notin G_{\alpha_k}$. We call this representation a reduced word.

Although this representation is sometimes useful, it is not easy to deal with. Suppose that

$$g = g_1 \cdots g_n, \quad g' = g'_1 \cdots g'_m$$

are reduced words. Then the product is

$$gg' = g_1 \cdots g_n g'_1 \cdots g'_m$$

and if g_n and g'_1 are in the same G_α then it is not a reduced word.[2]

[2] At this point there was an unconfirmed bomb threat and the Science Center was evacuated.

31 November 18, 2015

I had a collection of groups $\{G_\alpha\}_{\alpha \in J}$. The free product of them is a group G along with monomorphism $i_\alpha : G_\alpha \to G$ such that G is generated by $\{i_\alpha(G_\alpha)\}$. It should be freely generated so every $g \in G$ has a unique reduced word

$$g = g_1 g_2 \cdots g_k$$

so that $g_k \in G_\alpha$ implies $g_{k+1} \notin G_\alpha$. In other words, the identity I_G is represented by a unique empty word.

31.1 Construction of free product

We can construct G by considering the set of words

$$W = \{g_1 g_2 \cdots g_k\} = \{\text{set of reduced words}\}.$$

You can give an obvious multiplication on W. If $w = g_1 \cdots g_n$ and $w' = h_1 \cdots h_m$ are two reduced words, we define

$$ww' = g_1 \cdots g_n h_1 \cdots h_m.$$

If g_n and h_1 are from the same group, we can reduce the two things to one $(g_n h_1)$. If it is the identity, you can just cancel it out. Associativity is tedious.

Now this group is called the free product of $\{G_\alpha\}_{\alpha \in J}$, and is denoted $G_1 * G_2 * \cdots * G_n$. By definition, $(G_1 * G_2) * G_3 = G_1 * (G_2 * G_3)$. Let us get back to the fundamental group. The fundamental group of the wedge sum of two S^1 is the free product $\mathbb{Z} * \mathbb{Z}$.

Example 31.1. *What is the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$? It turns out it is the isometry group of $\mathbb{Z}$, where the metric on $\mathbb{Z}$ is given by $d(m, n) = |m - n|$. This is called the infinite dihedral group. Every isometry θ should be of the form $\theta(n) = \theta(0) + n$, or of the form $\theta(n) = \theta(0) - n$. Let $a(n) = -n$, and let $b(n) = 1 - n$. Then $a^2 = b^2 = 1$, and everything is freely generated by a and b, because ab is a translation.*

Example 31.2. *The space $\mathbb{RP}^2$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$. The space gluing two $\mathbb{RP}^2$ at a single point, denoted by $\mathbb{RP}^2 \vee \mathbb{RP}^2$ has fundamental group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. The simply connected covering space will look like an infinite chain of spheres kissing at one point.*

31.2 Free group and the universal property

Let $\{a_\alpha\}_{\alpha\in J}$ be a subset of a group G. Then G is called a **free group** with basis $\{a_\alpha\}$ if G is freely generated by the groups G_α, where

$$G_\alpha = \{\cdots, a_\alpha^{-2}, a_\alpha^{-1}, 1, a_\alpha, a_\alpha^2\}.$$

Then every $g \in G$ has a unique representation as

$$g = a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$$

where $a_i \in \{a_\alpha\}$, $a_i \neq a_{i+1}$, and $n_i \in \mathbb{Z} \setminus \{0\}$. For example, the fundamental group of the figure eight is a free group.

Like the free product, we can construct the free group using formal expressions. First for $\alpha \in J$, we let

$$G_\alpha = \{\cdots, a_\alpha^{-2}, a_\alpha^{-1}, 1, a_\alpha, a_\alpha^2\}.$$

This is a set, but we can give multiplication rules by $a_\alpha^p a_\alpha^q = a_\alpha^{p+q}$ and $a_\alpha^0 = 1$. Then this becomes a group isomorphic to $\mathbb{Z}$. Then we can consider the set of all finite words, and then give a multiplication rule on that.

Lemma 31.3 (Free product lemma)**.** *Let G be a group, and $\{G_\alpha\}$ be subgroups. Suppose that G is the free product of $\{G_\alpha\}$. Then for any group H and any family of homomorphisms $\phi_\alpha : G_\alpha \to H$, there is a unique $\phi : G \to H$ such that $\phi|_{G_\alpha} = \phi_\alpha$.*

Proof. What is going to be the homomorphism? Any g can be written as $g_1 \cdots g_k$ where $g_l \in G_{\alpha_l}$, and then there is one thing g can map to. If we actually define

$$\phi(g) = \phi(g_1)\cdots\phi(g_k) = \phi_{\alpha_1}(g_1)\cdots\phi_{\alpha_k}(g_k),$$

you can show that ϕ indeed is a homomorphism. □

Likewise, if G is a free group with basis $\{a_\alpha\}_{\alpha\in J}$, specifying $\phi(a_\alpha)$ specifies the whole homomorphism $\phi : G \to H$.

31.3 Normal subgroup

Definition 31.4. A subgroup $H \subset G$ is **normal** if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

If H is normal in G, then we have $G/H = H\backslash G$. In other words, the right cosets and the left cosets are the same. This is because $gHg^{-1} = H$ and hence $gH = Hg$. Also, G/H becomes a group. If we denote $[g]$ denote the coset of G containing g, then we can define $[g_1][g_2] = [g_1g_2]$. This does not depend on the choice of a representative, because $g_1hg_2h' = g_1g_2(g_2^{-1}hg_2)h' \in [g_1g_2]$. Then there is a natural homomorphism $\phi : G \to G/H$.

In general, let $\phi : G \to K$ be a group homomorphism. We define the **kernel** of ϕ to be $\ker\phi = \phi^{-1}(i_K)$. Then $\ker\phi$ is always a normal subgroup of G, because $\phi(g_1) = i_K$ and $\phi(g_2) = i_K$ implies $\phi(g_1g_2) = \phi(g_1)\phi(g_2) = i_K$, and $\phi(g_1) = i_K$ implies

$$\phi(g_2g_1g_2^{-1}) = \phi(g_2)i_k\phi(g_2^{-1}) = \phi(g_2g_2^{-1}) = i_k.$$

There is also a notion of the smallest normal subgroup containing K. Define

$$N_K = \langle\{gkg^{-1} : k \in K, g \in G\}\rangle$$

be the group generated by the elements in that set. Then N_K is a normal subgroup. Actually K does not have to be a subgroup.

31.4 Commutator subgroup and abelianization of a group

Define the **commutator** $[x, y]$ to be

$$[x, y] = xyx^{-1}y^{-1}.$$

This is 1 if and only if x and y commute. We can define the **commutator subgroup** of G to be

$$[G, G] = \langle\{[x, y] : x, y \in G\}\rangle.$$

This is a normal subgroup, because

$$gxyx^{-1}y^{-1}g^{-1} = (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}].$$

Also, $G/[G, G]$ is always abelian. This is because

$$[g_1][g_2] = [g_1g_2] = [[g_1, g_2]g_1g_2] = [g_2g_1] = [g_2][g_1].$$

The group $G/[G, G]$ is called the **abelianization** of G.

This tells us an interesting fact about free groups. If G is free group with basis $\{a_\alpha\}_{\alpha\in J}$, we have

$$G/[G, G] = \bigoplus_{\alpha\in J} \mathbb{Z}a_\alpha.$$

So if G is free with basis $\{a_1, \dots, a_n\}$ and also with basis $\{b_1, \dots, b_m\}$, then $m = n$.

Let G be a group generated by $\{a_\alpha\}_{\alpha\in J}$ and F be the free group generated by $\{\hat{a}_\alpha\}_{\alpha\in J}$. Then there is a surjective homomorphism $\phi : F \to G$ defined by $\hat{a}_\alpha \mapsto a_\alpha$. We get an exact sequence

$$1 \longrightarrow \ker\phi \longrightarrow F \xrightarrow{\phi} G1 \longrightarrow 1$$

The kernel $\ker\phi$ is called the group of relations. If a_α freely generate, the map ϕ is a isomorphism. If a_α do not freely generate, then there is a nontrivial representation of the identity

$$i = a_1a_2\cdots a_n.$$

Then the word $\hat{a}_1\hat{a}_2\cdots\hat{a}_n$ is in $\ker\phi$.

Example 31.5. *For instance, in the permutation subgroup P_3, the permutations $a = (12)(3)$ and $b = (1)(23)$ generate P_3. However, it was not freely generated because all $a^2 = b^2 = ababab = 1$ is the identity. This is an example of a relation.*

If G has a finite set of generators, and the kernel of the corresponding ϕ also is generated by a finite set, then we say that G is **finitely presented**. I suspect P_3 can be finitely presented.

32 November 30, 2015[3]

I was hoping to finish talking about Seifert-van Kampen theorem today. Let X be a space and suppose that open sets U and V cover the space X. Assume that U, V, and $U \cap V$ are all path-connected, and let $x_0 \in U \cap V$ be the base point. Then there are homomorphisms

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "j_U"] & \\ \pi_1(U \cap V) \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & \pi_1(X) \\ & \pi_1(V) \arrow[ur, "j_V"'] & \end{tikzcd}$$

and this induces a homomorphism $\pi_1(U) * \pi_1(V) \to \pi_1(X)$. Now if we let N be the smallest normal subgroup containing $\pi_1(U \cap V)$ in $\pi_1(U) * \pi_1(V)$, we get an exact sequence

$$0 \longrightarrow N \longrightarrow \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X) \longrightarrow 0.$$

Then we see that $\pi_1(X) \simeq (\pi_1(U) * \pi_1(V))/N$.

32.1 Application of the Seifert-van Kampen theorem

Now using this, we can calculate the fundamental group of a oriented surface in $\mathbb{R}^3$. Consider a surface of genus 3, i.e., a triple torus. If we let U be a disc on the surface, and let V be the complement, it is not hard to see that V deformation contracts into 6 circles. This shows that $\pi_1(U) = (1)$ and $\pi_1(V) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Then the intersection deformation retracts to a circle, and in terms of the free generators of $\pi_1(V)$, it is

$$aba^{-1}b^{-1}cdc^{-1}d^{-1}efe^{-1}f^{-1}.$$

So denoting the smallest normal subgroup containing this element by N, we see that

$$\pi_1(X) \simeq (\mathbb{Z} * \cdots * \mathbb{Z})/N.$$

If we take the commutator subgroup of this group, we see that

$$\pi_1/[\pi_1, \pi_1] \simeq \bigoplus_{2g} \mathbb{Z}$$

where g is the genus of the surface. This shows that a doughnut is not homeomorphic to a pretzel.

[3]I did not attend class on November 23, 2015. Apparently Prof. Taubes has proved the Seifert-van Kampen theorem during the last lecture.

32.2 Attaching a 2-cell

Suppose that there is a Hausdorff space X, a closed subspace $A \subset X$, and a map $h : D^2 \to X$ such that (i) $h : \partial D^2 \to A$, and (ii) $h : \mathrm{int}(D) \to X \setminus A$ is a homeomorphism.

Theorem 32.1. *The map $\pi_1(A) \to \pi_1(X)$ is surjective and the kernel of the map is the smallest normal subgroup in $\pi_1(A)$ containing the homotopy class $[h(S^1)]$.*

This means that when we attach a disc to a space, the only loops that gets "killed" is the ones on the boundary.

Proof. Let q be the image of the origin of D^2 under h. Then $X \setminus \{q\}$ deformation retracts onto A by the function

$$\Psi(t, z) = \frac{z}{t|z| + 1 - t}.$$

This shows that $\pi_1(X \setminus \{q\}) \simeq \pi_1(A)$.

We now apply the Seifert-van Kampen theorem to

$$X = (X \setminus \{q\}) \cup (X \setminus A).$$

Then because $U \cap V = \mathrm{int}(D^2) \setminus \{0\}$, we see that $\pi_1(U \cap V) \simeq \mathbb{Z}$ and thus the result follows. □

We recall that a finitely presented group is a group with finite generators and m relations. Using the theorem above, given a finitely presented group we can build a space for which the fundamental group is precisely that group. First we take a bouquet of circles $x_1, \dots, x_n$, which are generators. The fundamental group of this space is just $\langle x_1, \dots, x_n \rangle$. Now to "kill" the relations $r_1, \dots, r_m$, we attach the 2-cells to the space. For instance, if $r_k = x_1 x_2$ then we attach the disk D_2 with the boundary $x_1 x_2$. That is, we take the disjoint union and quotient it out by some equivalence relation

$$\mathcal{B}_n \cup D_1 \cup \dots \cup D_m / \sim .$$

There is also an interesting conjecture about this. A balanced representation is a representation where the number of generators is the number of relations. There is a trivial representation for a trivial group, for instance, $\langle x_1, x_2 \mid x_1, x_2 \rangle$. There is also a nontrivial representation for a trivial group, for instance,

$$\langle a, b \mid aba = bab, a^5 = b^4 \rangle.$$

Nielsen moves are the following five moves on the ordered set of relations:

- $r_1, r_2, \dots, r_n \quad \to \quad r_2, r_1, r_3, \dots, r_n$
- $r_1, r_2, \dots, r_n \quad \to \quad r_2, r_3, \dots, r_n, r_1$

- $r_1, r_2, \ldots, r_n \quad \to \quad r_1^{-1}, r_2, \ldots, r_n$
- $r_1, r_2, \ldots, r_n \quad \to \quad r_1 r_2, r_2, \ldots, r_n$
- $r_1, r_2, \ldots, r_n \quad \to \quad g r_1 g^{-1}, r_2, \ldots, r_n$

These moves do not change the represented group.

Conjecture 32.2 (Andrews-Curtis)**.** *Every balanced representation of a trivial group can be changed to a trivial representation by a finite number of Nielsen moves.*

This is related to 4-manifolds; a counterexample of this conjecture would lead to a potential counterexample of the 4-dimensional Poincaré conjecture.

A Category theory

Definition A.1. A **category** is a collection of points, and a collection of arrows between the points such that

- For any object x, there exists a map id_x that sends $x \to x$, and composing leaves other maps the same.
- For any two maps $f : x \to y$ and $g : y \to z$, there exists a composite map $g \circ f : x \to z$.
- Composition is associative.

Example A.2. *The set of one object, with one identity, is a category. The category of sets* **Set** *has sets as objects, and maps as arrows. The category of topologies* **Top** *has topological spaces as objects, and continuous maps as arrows. There is the category of locally small categories* **Cat**. *Also, for any group, there is a one-object category with each element of the group as arrows.*

Definition A.3. A (covariant) **functor** is a map between categories, which sets objects to objects and functions to functions, such that

- $F \cdot \mathrm{id}_x = \mathrm{id}_{Fx}$.
- $Fg \circ Ff = F(g \circ f)$.

Example A.4. *There is the stupid functor, which sends the one-element category to any category. There are the forgetful functor* **Grp** $\to$ **Set** *which forgets the group structure.*

Definition A.5. A **natural transformation** is a map between functors $\eta : F \to G$, thought of as a collection of maps, one for each object, such that the following diagram commutes for all objects $a, b \in C$.

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\eta_a"] & Fb \arrow[d, "\eta_b"] \\ Ga \arrow[r, "Gf"] & Gb \end{tikzcd}$$

Example A.6. *There is also a functor from* **Set** *to* **Grp**, *sending a set to a free group generated by the elements.*

A locally small category is a category for which for any x and y, the set of maps from x to y can be defined. In locally small categories, $C(x, y)$ denotes the set of maps from x to y.

Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor, and $F : \mathbf{Set} \to \mathbf{Grp}$ be the free functor. Then there is a natural isomorphism

$$\mathbf{Grp}(FX, G) \cong \mathbf{Set}(X, UG).$$

This kind of thing is called the adjoint functor.

Example A.7. *There is the functor* $\pi_1 : \mathbf{Top} \to \mathbf{Grp}$ *which sends a topological space to its fundamental group.*

Definition A.8. Two functors $F, G : C \to D$ is said to be **naturally isomorphic** if there exists a natural transformation $\eta : F \to G$ such that η_c is a isomorphism for each $c \in C$.

Definition A.9. Given two categories C, D and two functors $F : C \to D$ and $G : D \to C$, we say there is an **equivalence of categories** if GF is naturally isomorphic to the identity map id_C.

Note that this is much more loose than what we usually call bijection for maps.

Definition A.10. A locally small functor $F : C \to D$ is called

- **full** if $C(a, b)$ is surjective onto $D(Fa, Fb)$.
- **faithful** if $C(a, b)$ is inject to $D(Fa, Fb)$.
- **essentially surjective** if for any $d \in D$, there exists $c \in C$ such that $Fc \cong d$.

Theorem A.11. *Given an equivalence of categories* $F : C \to D$ *and* $G : D \to C$*, we have* F *is full, faithful, and essentially surjective. Further, given a full, faithful, essentially surjective functor* $F : C \to D$*, there exists a functor* $G : D \to C$ *such that the pair* F, G *induces an equivalence of categories.*

Proof. Exercise, because it is ugly. □

Given $c \in C$, we have a functor $C(c, -) : C \to \mathbf{Set}$ which sends

$$d \mapsto C(c, d).$$

Consider the following diagram:

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "F"] & d' \arrow[d, "F"] \\ C(c,d) \arrow[r, "f\circ -"] & C(c,d') \end{tikzcd}$$

We define $f \circ -$ to be the map which sends $g : c \to d$ to $f \circ g : c \to d'$.

Theorem A.12 (Yoneda's lemma)**.** *Let* $F : C \to \mathbf{Set}$ *be a functor, and let* $h_c = C(c, -)$*. Also, let* $\mathrm{Nat}(h_c, F)$ *be the set of natural transformations from* h_c *to* F*. Then we define a map* $\mathrm{Nat}(h_c, F) \to F(c)$ *by* $T \mapsto T_c(\mathrm{id}_c)$*. Then this is a natural isomorphism.*

Proof. You should do it, because you need to get used to doing these things. It is fairly straightforward. □

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