Math 137 - Algebraic Geometry

Taught by Joseph Harris
Notes by Dongryul Kim

Spring 2016

This course was taught by Joseph Harris. We met three times a week, on Mondays, Wednesdays, and Fridays, from 10:00 to 11:00 in Science Center 507. We used the textbook *Algebraic Geometry: A First Course* by Joseph Harris. There were 31 students enrolled, and the grading was based solely on the problem sets. The course assistants for the course were Aaron Landesman and Hannah Larson.

Contents

1 January 25, 2016

There will be weekly recitations, which will be scheduled after we have a better sense of who is actually going to take the course. For this week, it will be on Wednesday and Friday at 3 pm. There will be weekly problem sets, and the grading will be largely based on these problem sets. They will be due Fridays. I will not be assuming any prior knowledge, but I will use languages such as open/closed sets or holomorphic functions, because they are extremely useful.

1.1 What is algebraic geometry

The goal of algebraic geometry is to relate the algebra of polynomials to the geometry of their zero locus.

In the 1810's, Poncelet introduced two fundamental changes. The first one is to work in $\mathbb{C}$ instead of $\mathbb{R}$, and the second one is to work in $\mathbb{P}^n$ rather than $\mathbb{C}^n$. When we work in $\mathbb{R}$, a polynomial can have no zeros. But when we work in $\mathbb{C}$, we always have a constant number of zeros for polynomials of constant degree. But this is not the end of the story. Given a pair of two lines, the intersection is sometimes a single point, while it is sometimes empty, when the lines are parallel to each other. So we add points at infinity so that every two distinct lines meet at one point, or in other words, we work in the projective space. There is a trade-off: we lose geometry intuition. It is much harder to visualize curves in $\mathbb{C}^2$, but they are much well-behaved.

In the first half of the 20th century, There was an algebraicization of algebraic geometry by Zariski and Weil. This was possible because the algebraic language which was not in the 19th century was developed at this point.

In the second half of the 20th century, there was another revolution in algebraic geometry. Concepts such as sheaves and schemes were introduced by Grothendieck, Serre, Mumford, Artin, etc., and the new framework turned out to be extremely powerful. This is what we call modern algebraic geometry.

1.2 Affine and projective spaces

Definition 1.1. Let K be a algebraically closed field. We define the **affine space** of dimension n be the set

$$\mathbb{A}^n = K^n = \{(x_1, \dots, x_n) : x_i \in K\}.$$

Definition 1.2. An **affine variety** $X \subset \mathbb{A}^n$v is a subset describable as the common zero locus of a collection of polynomials $\{f_\alpha(x_1, \dots, x_n)\}$. We denote

$$\mathcal{V}(f_1, \dots, f_k) = \{x \in \mathbb{A}^n : f_\alpha(x) = 0 \text{ for all } \alpha\}.$$

Note that we can assume that there are an infinite number of polynomials, but the variety is determined by the ideal generated by the polynomials, and every ideal in a polynomial is finitely generated. So allowing infinitely many polynomials doesn't really add anything.

Definition 1.3. The **projective space** of dimension n is the set

$$\begin{aligned}\mathbb{P}^n &= \{\text{one-dimensional linear subspaces of } K^{n+1}\}\\ &= (K^{n+1} \setminus \{0\})/K^*.\end{aligned}$$

A point in the projective space is denoted with square brackets like $[z_0, \dots, z_n]$; this is the set of scalar multiples of the vector $(z_0, \dots, z_n)$.

We can think $\mathbb{A}^n$ as sitting inside $\mathbb{P}^n$. There is an isomorphism

$$\mathbb{A}^n \cong \mathcal{U} = \{[z] : z_0 \neq 0\} \subset \mathbb{P}^n$$

which sends

$$(z_1, \dots, z_n) \mapsto [1, z_1, \dots, z_n], \qquad [z_0, \dots, z_n] \mapsto (\tfrac{z_1}{z_0}, \dots, \tfrac{z_n}{z_0}).$$

We can visualize this by thinking $\mathbb{P}^2$ as a union of an affine plane $\mathbb{A}^2$ and a projective line $\mathbb{P}^1$. In fact, it doesn't matter which projective line $\mathbb{P}^1$ we choose. This enables us to identify hyperbolas, ellipses, and parabolas in $\mathbb{R}^2$; an hyperbola is a conic that meets the line of infinity at two points, an ellipse is one that meets at no points, and a parabola is one that is tangent to the line.

2 January 27, 2016

Fun facts from commutative algebra:

1) every ideal in $K[x_1, \dots, x_n]$ is finitely generated.
2) $K[x_1, \dots, x_n]$ is a unique factorization domain.

Right now, we take these statement as just true. I can prove it, but the proof does not illuminate anything about the geometry of varieties. In fact, people almost assumed them when developing algebraic geometry decades before the first proof was formulated.

2.1 Review on affine and projective spaces

As we have defined last time,

Definition 2.1. The affine space is

$$\mathbb{A}^n = \{(x_1, \dots, x_n) : x_i \in K\}$$

and an affine variety is something that looks like

$$Z = V(f_1, \dots, f_k) = \{x \in \mathbb{A}^n : f_\alpha(x) = 0 \text{ for all } \alpha\}.$$

Although I have defined this for an arbitrary algebraically closed field K, I invite you to replace it with $\mathbb{C}$. In fact, we will later see that any property we can prove on $\mathbb{C}$ holds also for any algebraically closed field of characteristic zero.

Definition 2.2. The projective space is

$$\begin{aligned} \mathbb{P}^n &= \{[X_0, \dots, X_n] : x \neq 0\}/K^* \\ &= \{\text{one-dimensional subspaces of } K^{n+1}\}. \end{aligned}$$

For any vector space V over K, we also define

$$\mathbb{P}V = (V \setminus \{0\})/K^*.$$

In the context of projective spaces, I will use square brackets and capital coordinates.

The open set

$$\mathcal{U} = \{[X] \in \mathbb{P}^n : X_0 \neq 0\} \cong \mathbb{A}^n$$

and more generally, if $L : V \to K$ is any homogenous linear polynomial, then

$$\mathcal{U}_L = \{[X] \in \mathbb{P}^n : L(X) \neq 0\} \cong \mathbb{A}^n.$$

Example 2.3. Over $\mathbb{C}$, we have $\mathbb{A}^1 = \mathbb{C}$ and $\mathbb{P}^1 = \{[X_0, X_1]\}/\mathbb{C}^*$. Because $\mathcal{U} = \{[X] : X_0 \neq 0\} \cong \mathbb{A}^1$, we see that $\mathbb{P}^1$ is simply the one-point compactification of $\mathbb{C}$. This is called the Riemann sphere. This becomes much difficult when we try to visualize $\mathbb{P}^2$.

2.2 Projective varieties

We defined the projective space as

$$\mathbb{P}^n = \{[X] \in \mathbb{K}^{n+1} : X \neq 0\}/K^*.$$

These X_α are called homogeneous coordinates on $\mathbb{P}^n$, but remember that these are *not* functions on $\mathbb{P}^n$. But since X_i/X_j are well-defined, for any homogenous polynomial $F(X_0, \dots, X_n)$, its zero locus is well-defined.

Definition 2.4. A **projective variety** is a subset of $\mathbb{P}^n$ describable as the common zero locus of a collection of homogenous polynomials

$$Z = V(\{F_\alpha\}) = \{X \in \mathbb{P}^n : F_\alpha(X) = 0 \text{ for all } \alpha\}.$$

Note that for $\mathcal{U} = \{X \in \mathbb{P}^n : X_0 \neq 0\}$, the intersection $Z \cap \mathcal{U}$ is the affine variety $V(\{f_\alpha\})$ where $f_\alpha = F(1, x_1, \dots, x_n)$. So given a projective variety, when you erase a hyperplane you get an affine variety.

The converse is also true. Given any polynomial $f(x_1, \dots, x_n)$ of degree d defined by

$$f(x) = \sum_{i_1+\cdots+i_n \leq d} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

we can homogenize it to

$$F(X) = \sum a_{i_1,\dots,i_n} X_0^{d-\sum a_i} X_1^{i_1} \cdots X_n^{i_n} = X_0^d \cdot f(\tfrac{X_1}{X_0}, \dots, \tfrac{X_n}{X_0}).$$

Example 2.5. Given an inclusion $K^{k+1} \cong W \subset V \cong K^{n+1}$ of affine spaces, it induces a natural embedding $\mathbb{P}W \hookrightarrow \mathbb{P}V$. We call the image a **linear subspace** of $\mathbb{P}V$. If $k = n - 1$, the image is called a **hyperplane**, and if $k = 1$ it is called a **line**.

Proposition 2.6. *If $\Gamma \subset \mathbb{P}^n$ is finite, then Γ is a projective variety.*

Proof. Let $\Gamma = \{p_1, \dots, p_d\}$. If we can show that for any point $q \notin \Gamma$ there is a polynomial that vanish on Γ but does not vanish on q, we can just take the set of those polynomials. For each i, take a linear homogeneous polynomial L_i such that $L_i(p_i) = 0$ but $L_i(q) \neq 0$. Then we take

$$F = \prod L_i$$

and we get the polynomial we want. □

In fact, this proves that any finite set of d points can be described as a common zero locus of polynomials of degree at most d. We can ask whether how much efficiently we can do this job, and in general, it is an open question.

If F is a homogeneous polynomial, then we call $V(F)$ a hypersurface.

3 January 29, 2016

We've been talking about affine and projective varieties. But the examples we have seen are mundane; we talked about linear subspaces, finite subsets, and hypersurfaces. We will not look at the twisted cubic curve, and generalize it to rational normal curves.

Before we go in, let me introduce a useful notion. For a vector space $V \cong K^{n+1}$, we defined the projective space $\mathbb{P}V \cong \mathbb{P}^n$. The group $GL_{n+1}K$ acts on $\mathbb{P}V$, and the stabilizer will simply be $K^* \subset GL_{n+1}$. So if we define

$$PGL_{n+1} = PGL(V) = GL_{n+1}/K^*,$$

then this PGL_{n+1} acts on $\mathbb{P}V$ faithfully. Let us say that $X, X' \subset \mathbb{P}^n$ are **projectively equivalent** if $A(X) = X'$ for some $A \in PGL_{n+1}$.

3.1 Twisted cubic

Consider the map $\mathbb{A}^1 \to \mathbb{A}^3$ defined by

$$t \mapsto (t, t^2, t^3).$$

Since there is a natural injection from the affine space to the projective space, we can push out this map into the map

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r] \arrow[d, hook] & \mathbb{A}^3 \arrow[d, hook] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^3 \end{tikzcd}$$

that sends

$$[X_0, X_1] \mapsto [X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3].$$

The image is called a **twisted cubic**. In general, a twisted cubic is any set in $\mathbb{P}^3$ that is projectively equivalent to this specific twisted cubic. Alternatively, we can define a twisted cubic as the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ that maps

$$[X] \mapsto [F_0(X), F_1(X), F_2(X), F_3(X)]$$

where F_0, F_1, F_2, F_3 form a basis for the vector space of homogenous cubic polynomials on $\mathbb{P}^1$.

Proposition 3.1. *The twisted cubic is a projective variety.*

Proof. The zero locus of the polynomials

$$Z_0 Z_2 - Z_1^2,\ Z_0 Z_3 - Z_1 Z_2,\ Z_1 Z_3 - Z_2^2$$

is the twisted cubic. $\square$

In fact, these three quadratic polynomials space the space of homogenous quadratic polynomials in Z_0, Z_1, Z_2, Z_3 vanishing on the twisted cubic. How would we prove that? Consider the linear map

$$\{\text{hom. quad. poly. in } Z_0, Z_1, Z_2, Z_3\} \to \{\text{hom. septic poly. in } X_0, X_1\}$$

defined by substituting Z_j with $X_0^{3-j} X_1^j$. We see that the first vector space has dimension 10, while the second has dimension 7. The polynomials we want has to be a basis for the kernel of the map, and thus we need three of quadratic polynomials. In fact, if we select any two arbitrary polynomials and look at the common zero locus, it will be a union of the twisted curve and a line.

We now generalize this to a rational normal curve.

Definition 3.2. A **rational normal curve** is defined to be a variety that is projectively equivalent to the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ that sends

$$[X_0, X_1] \mapsto [X_0^n, X_0^{n-1} X_1, \dots, X_1^n],$$

or in other word,s the image of a map

$$X \mapsto [F_0(X), F_1(X), \dots, F_n(X)].$$

3.2 Zariski topology

I'll come back to this later, but we're desperate in need of definitions.

Definition 3.3. Let X be any variety. The **Zariski topology** on X is the topology whose closed subset are subvarieties of X.

Warning: this is not a "nice topology." This is simply a language, and it has nothing to do with the standard topology. We define it simply because we want to talk about open and closed sets.

Example 3.4. For instance, take $X = \mathbb{A}^1$. Then the open sets are complements of finite sets. This is really bad: any bijection from X to X is continuous.

3.3 Regular function

If we want to define regular functions, we at least want to include polynomials. Also, we would want to take ratios.

Definition 3.5. Let $X \subset \mathbb{A}^n$ be a affine variety, and let $\mathcal{U} \subset X$ be an open set. A **regular function** on $\mathcal{U}$ is a function $\mathcal{U} \to K$ such that for any $p \in \mathcal{U}$, there exists a neighborhood $V \subset \mathcal{U}$ of p such that

$$f(x) = \frac{g(x)}{h(x)}$$

for any $x \in V$, where g and h are polynomials on $\mathbb{A}^n$ and $h(p) \neq 0$.

4 February 1, 2016

Today, the main thing we are going to do is to go through some more definitions. You have to describe what the objects are, and also what the morphisms are.

4.1 Regular maps on affine varieties

Definition 4.1. A **quasi-projective variety** is an open subset of a projective variety.

Observe that these include all affine variety via the inclusion $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$. Roughly you can think of it as requiring some equations to be zero and some to be nonzero. So these are the object we are going to deal with.

We now define the maps between them.

Definition 4.2. For an affine variety $X \subset \mathbb{A}^n$, we define

$$I(X) = \{\text{all polynomials } f \in K[z_1, \dots, z_n] : f \equiv 0 \text{ on } X\}.$$

Definition 4.3. We define the **affine coordinate ring** to be

$$A(X) = K[z_1, \dots, z_n]/I(X).$$

Definition 4.4. A **regular function** on $X \subset \mathbb{A}^n$ is a function locally expressible as $f = g/h$, where g and h are polynomials with $h \neq 0$.

At the end of the day, we will see that the regular functions are simply the polynomials in the affine coordinate ring.

Theorem 4.5 (Nullstellensatz)**.** *The ring of regular functions on an affine variety $X \subset \mathbb{A}^n$ is $A(X)$.*

By the way, I should have said this earlier, but the Zariski topology is a topology. If we have two varieties, we can simply multiply the polynomials and get the union of the two varieties and get a new variety. If we have many varieties, we can take the union of the polynomials and get their intersection.

Definition 4.6. A **regular map** $\phi : X \to Y \subset \mathbb{A}^n$ is a map given by

$$p \mapsto (f_1(p), \dots, f_n(p))$$

where $f_1, \dots, f_n$ are regular function on X.

We can see that the composition of two regular maps are also regular from the fact that the composition of two polynomials are also a polynomial. This also gives the notion of isomorphisms.

We observe that once we have the Nullstellensatz, we can show that $A(X)$ is an invariant of X up to isomorphism. In fact, $A(X)$ determines X up to isomorphism.

This gives rise to the following basic correspondence.

$$\{\text{affine varieties}\} \longleftrightarrow \left\{\begin{matrix}\text{finitely generated nilpotent free rings} \\ \text{over an algebraically closed field}\end{matrix}\right\}$$

When Grothendieck came along, he got rid of all the adjectives on the right side and asked, 'what geometric objects corresponds to rings?' He answered this question by introducing the language of schemes.

4.2 Regular maps on projective varieties

How can we define regular functions on a projective variety? Let $X \subset \mathbb{P}^n$. We may define it as a function that is regular on any affine subvariety of X.

Definition 4.7. A **regular function** on X is a function f such that $f|_{X_i}$ is regular, where $X_i = X \cap \{Z_i \neq 0\}$.

But this involves an arbitrary covering of X. That is why I prefer the following equivalent definition.

Definition 4.8. A **regular function** on $X \subset \mathbb{P}^n$ is a function f locally expressible as a ratio $G(Z)/H(Z)$, where G and H are homogenous polynomials of same degree and $H(Z) \neq 0$.

The basic idea is that although a polynomial does not give a well-defined function on a projective variety, the ratio of homogenous polynomials of same degree gives a function.

Definition 4.9. A map $\phi : X \to Y \subset \mathbb{P}^m$ is called **regular** if ϕ is given locally by regular functions, i.e., if for each $\mathcal{U}_i = \{Z_i \neq 0\} \cong \mathbb{A}^m \subset \mathbb{P}^m$ such that $\phi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \cong \mathbb{A}^m$ is given by an m-tuple $(f_1, \ldots, f_m)$ of regular functions.

It is a pain to write this down and also to check that the definition is independent of coordinates. Equivalently, we have

Definition 4.10. A map $\phi : X \to Y \subset \mathbb{P}^m$ is called **regular** if ϕ is given locally by an $(m+1)$-tuple of homogenous polynomials of same degree

$$p \mapsto [F_0(p), \ldots, F_m(p)].$$

In the affine case, we had a correspondence between affine varieties X and coordinate rings $A(X)$. However, for projective varieties, we cannot do the same thing. For a projective variety $X \subset \mathbb{P}^n$, we can think of the ideal $I(X)$ of homogenous polynomials vanishing on X. But then, the homogenous coordinate ring $S(X) = K[Z_0, \ldots, Z_n]/I(X)$ is not a function. In fact, it is even not an invariant of X under isomorphisms.

Example 4.11. Consider the map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ given by

$$[X_0, X_1] \mapsto [X_0^2, X_0X_1, X_1^2].$$

This is an isomorphism between $\mathbb{P}^1$ and the image curve $C = V(W_0W_2 - W_1^2)$, but one can check that $S(\mathbb{P}^1) \ncong S(C)$.

5 February 3, 2016

5.1 More on regular maps

Recall the definition of regular functions and maps on a variety. For $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$, a regular map $\phi : X \to Y$ can be given locally by an $(n+1)$-tuple of homogeneous polynomials on $\mathbb{P}^m$ with no common zeroes. That is, we would like to describe this map as

$$p \mapsto [F_0(p), \dots, F_n(p)].$$

The condition "locally" is needed because sometimes only one choice of F_is might not work.

Example 5.1. Consider $C = V(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$. We look at the map $C \to \mathbb{P}^1$ given by

$$[X, Y, Z] \mapsto [X, Y - Z],$$

which is the stereographic projection. This is well-defined except for the point $p = [0, 1, 1]$. This can be resolved by observing that

$$[X, Y - Z] \sim [X(Y + Z), Y^2 - Z^2] \sim [X(Y + Z), -X^2] \sim [Y + Z, -X].$$

Then the map

$$[X, Y, Z] \mapsto [Y + Z, -X]$$

agrees with the previous map except for the two points $[0, 1, 1]$ and $[0, 1, -1]$, where they were not well-defined.

5.2 Veronese maps

The next thing I want to introduce is the notion of **Veronese maps**. We first choose n, and a degree d. We look at the map $\nu_{d,n} : \mathbb{P}^n \to \mathbb{P}^N$ given by

$$[X] \mapsto [\dots, X^I, \dots]$$

where I ranges over all homogenous indices of size d. The image is called a Veronese variety. We see that $N = \binom{n+d}{n} - 1$. Note that the case $n = 1$ is exactly the rational normal curve.

Example 5.2. The map $\nu_{2,2}$ is given by

$$[X, Y, Z] \mapsto [X^2, Y^2, Z^2, XY, XZ, YZ].$$

The image is called the **Veronese surface**, which is a misnomer because we could have chosen any d.

Proposition 5.3. *The image of $\nu_{d,n}$ is a subvariety of $\mathbb{P}^N$. In fact, it is the zero locus of*

$$\{X^I X^J - X^K X^L : I + J = K + L\}.$$

Let us just look at the case of $n = d = 2$. The proposition is saying that image of $\nu_{2,2}$ is the zero locus of

$$\begin{aligned} W_3^2 &= W_0 W_1, \quad W_3 W_4 = W_0 W_5, \\ W_4^2 &= W_0 W_2, \quad W_3 W_5 = W_1 W_4, \\ W_5^2 &= W_1 W_2, \quad W_4 W_5 = W_2 W_3. \end{aligned}$$

There are two questions. First, are these all the quadratic polynomials? The answer for this is yes, because we have the surjective map

$$\{\text{hom. quad. poly. in } W_0, \dots, W_5\}^{21} \twoheadrightarrow \{\text{hom. quartic poly. in } X, Y, Z\}^{15}.$$

The kernel would have dimension 6, and the polynomials we wrote above are linearly independent. In fact, these quadratics generate the homogenous ideal $I(S)$, where S is the image. The second question is whether the zero locus is exactly S. The answer is again S, and this we will show later.

5.3 Segre maps

We define the **Segre map** $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ be the map defined by

$$([X], [Y]) \mapsto [\dots, X_i Y_j, \dots].$$

We denote the image of $\sigma_{m,n}$ by $\Sigma_{m,n} \subset \mathbb{P}^{(m+1)(n+1)-1}$.

Proposition 5.4. *The image $\Sigma_{m,n}$ is a variety. Again, it is the common zero locus of*

$$\{W_{i,j} W_{k,l} - W_{k,j} W_{i,l} : 0 \le i, k \le m, 0 \le j, l \le n\}.$$

Note that the domain $\mathbb{P}^m \times \mathbb{P}^n$ is *not* a projective space. However, we can view it as a variety through the Segre map.

6 February 5, 2016

Today we are going to finish our discussion on Veronese and Segre maps, and on Monday we are going to start chapter 3.

6.1 Veronese maps and varieties

The Veronese map $\nu_{n,d} : \mathbb{P}^n \to S \subset \mathbb{P}^N$ is given by

$$[X] \mapsto [\dots, X^I, \dots]$$

where I ranges over all monomials of degree d in $[X_0, \dots, X_n]$. The image S is called a Veronese variety. More generally, a Veronese map can be given as

$$[X] \mapsto [F_0(X), \dots, F_N(X)]$$

where $\{F_0, \dots, F_N\}$ is a basis for the vector space of homogeneous polynomials of degree d on $\mathbb{P}^n$.

We make two observations:

i) The map ν is an embedding, i.e., $S \cong \mathbb{P}^n$, i.e., there is a regular map $\phi : S \to \mathbb{P}^n$ such that $\phi \circ \nu = \mathrm{id}$. For instance, when $n = 1$ the Veronese map ν is given by

$$[X_0, X_1] \mapsto [X_0^d, X_0^{d-1}X_1, \dots, X_0X_1^{d-1}X_1^d]$$

and its inverse ϕ is given by

$$[Z_0, \dots, Z_d] \mapsto [Z_0, Z_1]$$

away from $[0, 0, \dots, 0, 1]$ and

$$[Z_0, \dots, Z_{d-1}] \mapsto [Z_{d-1}, Z_d]$$

away from $[1, 0, \dots, 0]$.

ii) If $Z \subset \mathbb{P}^n$ is any variety then $\nu(Z) \subset S \subset \mathbb{P}^N$ is a sub variety of $\mathbb{P}^N$. Given a polynomial on $\mathbb{P}^N$, we can always pull it back (substitute) to get a polynomial on $\mathbb{P}^n$. Thus we get a pullback map

$$\nu^* : \{\text{hom. poly. of deg. } m \text{ on } \mathbb{P}^N\} \twoheadrightarrow \{\text{hom. poly. of deg. } md \text{ on } \mathbb{P}^n\}.$$

If $Z \subset \mathbb{P}^n$ is any variety, then from some m, the variety Z is the zero locus of polynomials G_α of degree md. (This is because we can multiply monomials to make the polynomials have degree as large as we want.) Then we can take the polynomials whose pullbacks are G_α.

Let us look at an example. We look at the map $\nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ given by

$$[X, Y, Z] \mapsto [X^2, Y^2, Z^2, XY, XZ, YZ].$$

Let $Z = V(X^3 + Y^3 + Z^3)$ and we claim that $\nu(Z)$ is a variety. Because

$$\begin{aligned} Z &= V(X^3 + Y^3 + Z^3) \\ &= V(X^4 + Y^3X + Z^3X, X^3Y + Y^4 + Z^3Y, X^3Z + Y^3Z + Z^4) \end{aligned}$$

we can take the 6 quadratics that cut out S, and throw in the polynomials whose pullback is one of the three polynomials. Then we get 9 polynomials, and their common zero locus will be exactly $\nu(Z)$.

6.2 Segre maps and varieties

The Segre map $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1} = \mathbb{P}^N$ is given by

$$([X], [Y]) \mapsto [\ldots, X_iY_j, \ldots].$$

The image is a variety in $\mathbb{P}^N$ and it is called the Segre variety.

Example 6.1. The Segre map $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by

$$([X_0, X_1], [Y_0, Y_1]) \mapsto [X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1],$$

and the image $S = V(W_0W_3 - W_1W_2)$ is a quadric hypersurface. We also note that the fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ are mapped to lines.

Definition 6.2. A **subvariety** of $\mathbb{P}^m \times \mathbb{P}^n$ is a subvariety of the image $\Sigma \subset \mathbb{P}^N$.

What does a subvariety look like? We again have pullback map

$$\begin{aligned} \sigma^* : &\{\text{hom. poly. of deg. } d \text{ on } \mathbb{P}^N\} \\ &\qquad \to \{\text{bihomogeneous poly. of bidegree } (d,d) \text{ on } \mathbb{P}^m \times \mathbb{P}^n\}. \end{aligned}$$

So a subvareity of $\mathbb{P}^m \times \mathbb{P}^n$ is the zero locus of a collection of bihomogenous polynomials of bidegree (d, d) on $\mathbb{P}^m \times \mathbb{P}^n$. We will show next time that it can be described simply as the zero locus of any collection of bihomogeneous polynomials.

7 February 8, 2016

7.1 More on Segre maps

I want to finish the discussion on Segre maps. A Segre map is a map $\mathbb{P}^m \times \mathbb{P}^n \to \Sigma \subset \mathbb{P}^{(m+1)(n+1)-1}$ given by

$$([X],[Y]) \mapsto [\dots, X_i Y_j = Z_{ij}, \dots].$$

The image is

$$\Sigma = V(\dots, Z_{ij}Z_{kl} - Z_{il}Z_{kj}, \dots).$$

Example 7.1. If $m = n = 1$, then the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is

$$([X_0, X_1],[Y_0,Y_1]) \mapsto [X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1]$$

with images $\Sigma = V(Z_0Z_3 - Z_1Z_2)$.

When I say "the variety $\mathbb{P}^m \times \mathbb{P}^n$," actually I am actually referring to the variety Σ. Nowadays we have the notion of abstract varieties, but for now, we have to just look at Σ instead of the product space.

The subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ is by definition, a subvariety $\Sigma \subset \mathbb{P}^N$. This will be the zero locus of bihomogeneous polynomials.

Example 7.2. Let us look at the twisted cubic $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$t \mapsto [1, t, t^2, t^3].$$

Because $Z_0Z_3 - Z_1Z_2$ vanishes on the curve, we see that the cubic actually lies on the the image of the Segre map.

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[rr] \arrow[dr, hook] & & \mathbb{P}^3 \\
& \mathbb{P}^1 \times \mathbb{P}^1 = \Sigma \arrow[ur, hook] &
\end{tikzcd}$$

In $\mathbb{P}^3$, the twisted cubic is the zero locus of $Z_0Z_3 - Z_1Z_2$, $Z_0Z_2 - Z_1^2$, and $Z_1Z_3 - Z_2^2$. If we pull back the second and third polynomials, we get

$$\begin{aligned}
X_0X_1Y_0^2 - X_0^2Y_1^2 &= X_0(X_1Y_0^2 - X_0Y_1^2), \\
X_0X_1Y_1^2 - X_1^2Y_0^2 &= -X_1(X_1Y_0^2 - X_0Y_1^2).
\end{aligned}$$

This means that the twisted cubic is $V(X_1Y_0^2 - X_0Y_1^2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

We present two useful observations. If $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$, then its product

$$X \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$$

is again a variety. Moreover, if $f : X \to Y$ is a regular map, then the graph

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subset X \times Y$$

is a subvariety.

7.2 Cones

Consider a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$, and a point $p \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$. Suppose that a some variety X lying in $\mathbb{P}^{n-1}$.

Definition 7.3. The **cone** $\bar{X} = \overline{p, X}$ over X with vertex p is defined as

$$\overline{p, X} = \bigcup_{q \in X} \overline{p, q}.$$

Proposition 7.4. *The cone $\bar{X}$ is a variety.*

Proof. Applying a linear transformation, we can make $\mathbb{P}^{n-1} = V(Z_n)$ and $p = [0, \dots, 0, 1]$. Now if $X = V(\dots, F_\alpha(Z_0, \dots, Z_{n-1}), \dots)$ then

$$\bar{X} = V(\dots, F_\alpha, \dots)$$

where the last variable in Z_n is neglected. ☐

Similarly, suppose that $\mathbb{P}^k, \mathbb{P}^l \subset \mathbb{P}^n$ are **complementary**, or in other words, $\mathbb{P}^k, \mathbb{P}^l$ are disjoint and span the whole space $\mathbb{P}^n$. In this setting, we can again define the cone $\overline{X, \mathbb{P}^l}$ over $X \subset \mathbb{P}^k$ with vertex $\mathbb{P}^l$ as

$$\overline{X, \mathbb{P}^l} = \bigcup_{q \in X} \overline{q, \mathbb{P}^l}.$$

7.3 Quadrics

Recall that $\mathrm{char}(K) \neq 2$. It is actually zero, but it is crucial that it not be 2.

Definition 7.5. A **quadric** $X \subset \mathbb{P}^n = \mathbb{P}V$ is the zero locus of a single homogeneous quadratic polynomial Q.

Higher degree polynomials are hard to describe by linear algebra, but quadratic polynomials are in the scope. In fact, we have a correspondence

$$\begin{aligned} &\{\text{homogeneous quadratic polynomials } Q \text{ on } V\} \\ &\qquad\qquad \longleftrightarrow \{\text{symmetric bilinear forms } Q_0 : V \times V \to K\}, \end{aligned}$$

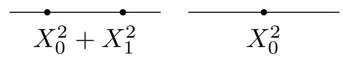
given by $Q(v) = Q_0(v, v)$ and $Q_0(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$.

In our linear algebra course, we learned that any symmetric bilinear form can be diagonalized, i.e., there exists coordinates on V such that

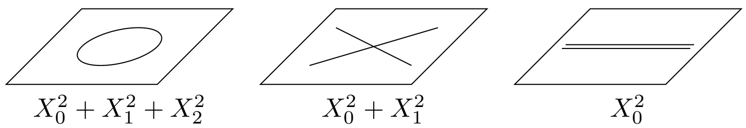
$$Q(X_0, \dots, X_n) = \sum_{i=0}^{k} X_i^2 = X_0^2 + \dots + X_k^2.$$

We define the **rank** of Q as $\mathrm{rank}(Q) = k + 1$.

Example 7.6. We first look at $\mathbb{P}^1$. There can be quadrics of rank 1 and 2.



For $\mathbb{P}^2$, there can be quadrics of rank $1, 2$, and 3.



For $\mathbb{P}^3$, there can be quadrics of rank $1, 2, 3$, and 4.

8 February 10, 2016

Let's get started because we've got some fun stuff today. Recall that a quadric $Q \subset \mathbb{P}^n$ is the zero locus of a single homogeneous quadratic polynomial $Q(X)$. A quadratic is characterized by its rank. A rank 4 quadric in $\mathbb{P}^3$ is a hyperboloid, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, a rank 3 quadric is a conic cone, a rank 2 quadric is two planes, and a rank 1 quadric is a double plane.

Let us now look at rank 4 quadrics over $\mathbb{R}$. There are three cases; if the polynomial is $X^2+Y^2+Z^2+W^2$ then it is the empty set; if it is $X^2+Y^2+Z^2-W^2$ then it is a sphere; if it is $X^2+Y^2-Z^2-W^2$ then it is a hyperbola of one sheet. We can even try to classify the rank 4 quadrics over $\mathbb{R}$ in $\mathbb{A}^3$. Depending on what plane we choose to be the infinity, the shape of the quadric changes. The point I want to make is that our lives are much easier when we work in the projective space, and over the complex numbers.

8.1 Projection map

Let $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a hyperplane and let $p \in \mathbb{P}^n \setminus H$ be a point. We can define a map $\pi_p : \mathbb{P}^n \setminus \{p\} \to H$ given by

$$g \mapsto \overline{qp} \cap H.$$

If we choose homogeneous coordinates Z on $\mathbb{P}^n$ so that $H = V(Z_n)$ and $p = [0, \ldots, 0, 1]$, then we can describe the map in terms of coordinate as

$$\pi_p : [Z_0, \ldots, Z_n] \mapsto [Z_0, \ldots, Z_{n-1}].$$

Proposition 8.1. *If $X \subset \mathbb{P}^n$ is a projective variety and $p \notin X$, then $\bar{X} = \pi_p(X) \subset \mathbb{P}^{n-1}$ is a projective variety.*

Before we prove this, we look at a general result about polynomials.

8.2 Resultant

Let K be a field, and let

$$f(X) = a_m X^m + \cdots + a_0, \quad g(X) = b_n X^n + \cdots + b_0$$

be polynomials in $K[X]$. When do f and g have a common zero? Equivalently, we can ask when do

$$F(X,Y) = a_m X^m + a_{m-1} X^{m-1} Y + \cdots + a_0 Y^m, \quad G(X,Y) = b_n X^n + \cdots + b_0 Y^n$$

have a common zero in $\mathbb{P}^1$? The second formulation is more concrete, because we consider the two polynomials to share a zero even if $a_m = b_n = 0$.

To restate the question, let $\mathbb{P}^m$ be the space of polynomials of degree m on $\mathbb{P}^1$. Consider a subset $\Sigma \subset \mathbb{P}^m \times \mathbb{P}^n$ defined as

$$\Sigma = \{(f, g) : f \text{ and } g \text{ have a common zero}\}.$$

We claim that Σ is a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$. The problem is to find the defining equations of $\Sigma \subset \mathbb{P}^m \times \mathbb{P}^n$.

Here is what I propose to do. To answer this, let V_m be the vector space of polynomials of degree at most m. Consider the map $\varphi : V_{n-1} \oplus V_{m-1} \to V_{m+n-1}$ given by

$$(A, B) \mapsto fA + gB.$$

If f and g both vanish at a point p, then at element of $\operatorname{im} \varphi$ also vanishes at p. This means that φ cannot be an isomorphism. Conversely, if φ is not a isomorphism, then it has nontrivial kernel and $fA + gB \equiv 0$ for some A and B. Then gB has to vanish at all roots, and thus counting the degree, we see that g has to vanish at least one root, which will be the common root of f and g. Hence φ is an isomorphism if and only if f and g have no common zero.

Now, write out the matrix of the map φ in terms of bases given by powers of x. The matrix will be

$$\begin{pmatrix}
a_0 & a_1 & a_2 & \cdot & \cdot & a_m & 0 & \cdot & \cdot & \cdot & 0 \\
0 & a_0 & a_1 & \cdot & \cdot & \cdot & a_m & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & 0 & a_0 & a_1 & \cdot & \cdot & \cdot & a_m \\
b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_n & 0 & \cdot & \cdot & 0 \\
0 & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_n & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & 0 & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_n
\end{pmatrix}$$

The conclusion is that f and g have a common zero if and only if the determinant of the matrix is zero. The determinant is called the **resultant**.

Of course, in practice, we would use the Euclidean algorithm to see whether f and g have a common zero, but we would not know the set is a zero locus of a polynomial.

We look at one generalization of this fact. If F and G are homogeneous polynomials in $Z_0, \ldots, Z_r$, then we can view F and G as polynomials in Z_r with coefficients in $K[Z_0, \ldots, Z_{r-1}]$. Then we can form the same matrix and the determinant is called the **resultant** with repeat to Z_r and is denoted $\operatorname{Res}_{Z_r}(F, G)$.

9 February 12, 2016

Today we are going to finish chapter 3. I want to first recall the basic definitions.

Definition 9.1. For two polynomials in one variables

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_n, \quad g(x) = b_n x^n + \cdots + b_0,$$

we define the **resultant** $R(f, g)$ as the determinant of the $(m + n) \times (m + n)$ matrix.

The polynomials f and g have a common zero if and only if $R(f, g) = 0$. More generally, if $f, g \in K[x_1, \dots, x_r] = K[x_1, \dots, x_{r-1}][x_r]$, we can look at the coefficients of f and g, which will be in $K[x_1, \dots, x_{r-1}]$ and define $R_{x_r}(f, g)$.

9.1 Image of a projection

Let $p \in \mathbb{P}^\alpha$, and let $H \cong \mathbb{P}^{\alpha-1} \subset \mathbb{P}^\alpha$ be a hyperplane. One condition is that $p \notin H$. Define the projection map $\pi_p : \mathbb{P}^\alpha \setminus \{p\} \to \mathbb{P}^{\alpha-1}$ as

$$q \mapsto \overline{qp} \cap H.$$

Without loss of generality, we can let $p = [0, \dots, 0, 1]$and $H = V(Z_\alpha)$ so that

$$\pi_p : [Z_0, \dots, Z_\alpha] \mapsto [Z_0, \dots, Z_{\alpha-1}].$$

Theorem 9.2. *Let $X \subset \mathbb{P}^\alpha$ be a projective variety and let $p \notin X$. Then $\pi_p(X) = \overline{X} \subset \mathbb{P}^{\alpha-1}$ is a projective variety.*

Proof. It amounts to verifying that

$$\overline{X} = V\big(\{\mathrm{Res}_{Z_\alpha}(F, G)\}_{F,G \in I(X)}\big).$$

For any $t \in \mathbb{P}^{\alpha-1}$, we see that

$$\begin{aligned} r \in \pi_p(X) &\Leftrightarrow L = \overline{p, r} \cap X \neq \emptyset \\ &\Leftrightarrow \text{any pair } F, G \in I(X) \text{ have a common zero on } L \\ &\Leftrightarrow \mathrm{Res}(F, G) \text{ vanishes on } r \text{ for any pair } F, G \in I(X). \end{aligned}$$

□

Corollary 9.3. *Let $p \in \mathbb{P}^n$ and $X \subset \mathbb{P}^n$ be a projective variety, and let $p \notin X$. Set*

$$\overline{p, X} = \bigcup_{q \in X} \overline{p, q}$$

be the cone. Then $\overline{p, X}$ is again a variety.

Proof. We already know that the claim is true of X is contained in a hyperplane. So we first choose an hyperplane $H \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and observe that

$$\overline{p, X} = \overline{p, \pi_p(X)}.$$

□

Similarly, let Y be a projective variety and $X \subset Y \times \mathbb{P}^1$ be a projective variety. Consider the projection map $\pi : Y \times \mathbb{P}^1 \to Y$.

Proposition 9.4. *The image $\pi(X) \subset Y$ is a projective variety.*

Proof. Take $(F, G) \in I(X)$ and take the resultant with respect to the $\mathbb{P}^1$ factor. $\square$

More generally, let $X \subset Y \times \mathbb{P}^n$ be a projective variety.

Proposition 9.5. *$\pi(X) \subset Y$ is again a projective variety.*

Proof. We use induction on n. Choose $p \in \mathbb{P}^n$, $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ with $p \notin H$. We let

$$\pi = \mathrm{id} \times \pi_p : Y \times (\mathbb{P}^n \setminus \{p\}) \to Y \times \mathbb{P}^{n-1}.$$

Then we see that $\pi(X) \subset Y \times \mathbb{P}^{n-1}$ is a projective variety.

But there is a glitch here. What if $Y \times \{p\}$ is not disjoint from X? Let $V = \{(q \in Y : (q, p) \in X\}$. Then this V is a closed subset, i.e., a subvariety of Y. Now $X \setminus (V \times \{p\})$ is disjoint from $Y \times \{p\}$ and hence we can apply the same argument to show that its projection $\pi_p(X \setminus (V \times \{p\}))$ is a projective variety. Then

$$\pi(X) = \pi_p(X \setminus (V \times \{p\})) \cup V$$

is a variety. $\square$

9.2 Image of a regular map

Finally, we have the following major proposition.

Proposition 9.6. *Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be projective varieties, and let $f : X \to Y$ be a regular map. Then $f(X)$ is a projective variety in Y.*

Proof. Locally in X, the map f may be given by

$$[Z_0, \ldots, Z_m] \mapsto [F_0(Z), \ldots, F_n(Z)].$$

Then the graph of the map $f : X \to Y \subset X \times Y$ is the zero locus of bihomogeneous polynomials $W_i F_j(Z) - W_j F_i(Z)$, in other words, $\Gamma_f \subset X \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$ is a projective variety. Then we can simply project it and see that $f(X) = \pi(\Gamma_f) \subset Y$ is a projective variety. $\square$

If you want to compute things more efficiently, there is a thing called Gröbner basis. The idea is to choose the "right" basis so that we don't have to look at all pairs of polynomials.

10 February 17, 2016

When we look at the set of manifolds isomorphic to S^1, it is too big to be really given a topology or whatever. But in algebraic geometry, the set of conics on a plane is determined by six coefficients, up to a scalar.

10.1 Family of varieties

Definition 10.1. A *family of varieties* in $\mathbb{P}^n$ parametrized by a variety B is a closed subvariety of $B \times \mathbb{P}^n$. The **members** of the family $\mathfrak{X} \subset B \times \mathbb{P}^n$ is the fibers $X_b = \pi^{-1}(b) \subset \mathbb{P}^n$ where $\pi : \mathfrak{X} \to B$ is the projection map.

Example 10.2. By a conic curve $C \subset \mathbb{P}^2$, we mean the zero locus of a single homogeneous quadratic polynomial on $\mathbb{P}^2$. The set of conics is $\mathbb{P}V \cong \mathbb{P}^5$ where V is the vector space of homogeneous quadratic polynomial on $\mathbb{P}^2$. These form a family

$$\mathcal{C} = V(aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ) \subset \mathbb{P}^5_{[a,\dots,f]} \times \mathbb{P}^2_{[X,Y,Z]}.$$

In the example, we can always replace the word "conic" by "zero locus of degree d polynomial" and then get the family

$$\mathcal{C} = V\Big(\sum_{i+j+k=d} a_{ijk} X^i Y^j Z^k\Big) \subset \mathbb{P}^N_{[a_{ijk}]} \times \mathbb{P}^2_{[X,Y,Z]}.$$

More generally we can look at the hypersurfaces $X \subset \mathbb{P}$ of degree d, i.e., the zero locus of a homogeneous polynomial $F(X_0, \dots, X_n)$ of degree d. They form a family

$$\mathfrak{X} = V(\sum a_I X^I) \subset \mathbb{P}^N_a \times \mathbb{P}^n_X.$$

This is called the **universal hypersurface** of degree d in $\mathbb{P}^n$. It is called universal because it actually has the universal property. For any family of hypersurfaces, it can be induced by the universal hypersurface.

Let us look at the special case $d = 1$. We have

$$\mathcal{H} = V(a_0 X_0 + \cdots + a_n X_n) \subset \mathbb{P}^n_a \times \mathbb{P}^n_X.$$

If $\mathbb{P}^n_X = \mathbb{P}V$, then we can think $\mathbb{P}^n_a = \mathbb{P}V^* = \mathbb{P}^{n*}$. It is useful to thing these as different objects.

Given a $\mathfrak{X} \subset B \times \mathbb{P}^n$, there is a natural projection map $\pi : \mathfrak{X} \to B$. We can ask, does there exists a **section**, i.e., a map $\sigma : B \to \mathfrak{X}$ such that $\pi \circ \sigma = \mathrm{id}$? For instance, let us look at the case $n = 1$ and $d = 2$. The question is equivalent to whether there exists polynomials $X(a, b, c)$ and $Y(a, b, c)$ such that

$$aX^2 + bXY + cY^2 \equiv 0.$$

This is clearly false, because we know that

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and this is not a polynomial.

Let me ask a now a more appealing question, which is for $n = 2$ and $d = 2$. Does there exist polynomial functions X, Y, Z with variables $a, \ldots, f$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0?$$

The answer is no. But this is not an obvious question and is instructive. If you are interested, try to solve it using polynomial manipulations.

Let us look at some other examples.

- The families of k-planes in $\mathbb{P}^n$ is called the **Grassmannian**

$$G(k+1, n+1) = \{\Lambda \cong \mathbb{P}^k \subset \mathbb{P}^n\}.$$

- Let us look at the set of twisted cubics in $\mathbb{P}^3$. We can describe the twisted cubic as the zero locus of three quadratics, and so we can think of the space of twisted cubics as lying inside the Grassmannian. The problem is that three quadrics don't generally cut out a twisted cubic, and so we need extra work.

These are hard problems, and there are whole theories to address these issues.

11 February 19, 2016

Let me recall the definition of a family of varieties.

Definition 11.1. A **family** of projective variety parametrized by a variety B is closed subset $\mathfrak{X} \hookrightarrow B \times \mathbb{P}^n$. The members of this family are the fibers $X_b = \pi^{-1}(b)$ of the projection $\pi : \mathfrak{X} \to B$.

Now I am going to introduce a ubiquitous terminology.

11.1 General member of a family

Definition 11.2. Given a family $\mathfrak{X} \subset B \times \mathbb{P}^n$, we say that a **general member** of the family has property P if

$$\{b \in B : X_b \text{ has property } P\} \subset B$$

contains an open dense subset of B.

Example 11.3. Let us look at the statement "a general triple of points (p_1, p_2, p_3) in $\mathbb{P}^2$ are not collinear." This implicitly uses the parameter space $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. That is, we are using the family

$$\mathfrak{X} = \big\{((p, q, r), s) : s \in \{p, q, r\}\big\} \subset (\mathbb{P}^2)^3 \times \mathbb{P}^2.$$

This is indeed a family, because $\mathfrak{X} = \pi_{14}^{-1}(\Delta) \cup \pi_{24}^{-1}(\Delta) \cup \pi_{34}^{-1}(\Delta)$ where π_{ij} is the projection $(\mathbb{P}^2)^4 \to \mathbb{P}^2 \times \mathbb{P}^2$ with the ith and jth indices.

Now in this particular family, what our statement really means is that the locus

$$\{(p, q, r) \in (\mathbb{P}^2)^3 : p, q, r \text{ are collinear}\}$$

is contained in a closed subset of $(\mathbb{P}^2)^3$.

Now after ten minutes we know that this means. How do we prove this? A triple of points

$$([X_0, X_1, X_2], [Y_0, Y_1, Y_2], [Z_0, Z_1, Z_2])$$

are collinear if and only if the determinant is zero. So we have

$$\{\text{collinear triples}\} = V\left(\begin{vmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ Z_0 & Z_1 & Z_2 \end{vmatrix}\right).$$

Note that the determinant is a tri-homogeneous polynomial. So this is indeed a closed set.

Last time we saw that the plane conics form a family

$$V(aX^2 + bY^2 + cZ^2 + dXY + eXY + fYZ) \subset \mathbb{P}^5_{[a,\ldots,f]} \times \mathbb{P}^2_{[X,Y,Z]}.$$

A conic, as we have seen, is characterized by its rank.

Proposition 11.4. *A general conic has rank* 3*.*

Proof. We observe that the parameter space can be viewed as the projectivization of the space of symmetric 3×3 matrices. This matrix has rank 3 if and only if the conic has rank 3. Therefore we see that the set of conics rank at most 2 is contained in the zero locus in the determinant. $\square$

Proposition 11.5. *Let $n > 1$. A general polynomial $F(X_0, \dots, X_n)$ that is homogeneous of degree d is irreducible.*

Proof. Let $\mathbb{P}^{N_d}$ be the space of polynomial homogeneous of degree d, so that $N = \binom{d+n}{n} - 1$. The set of reducible polynomials is the union of the image of the multiplication map

$$\mathbb{P}^{N_a} \times \mathbb{P}^{N_b} \to \mathbb{P}^{N_d}$$

for $a + b = d$. The map is regular, and hence the image is a closed subset of $\mathbb{N}^d$. Thus we only need to show that the map is surjective. This can be checked easily. $\square$

11.2 Relation between ideals and varieties

Let $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ and $X \subset \mathbb{A}^n$. Very often we hear people say that a set of polynomials "cut out" a variety. There are two ways to interpret this. The naive interpretation is that X is the common zero locus of $f_1, \dots, f_k$. The other way is to think that $f_1, \dots, f_k$ generate $I(X)$.

Let me phrase this distention in another way. There is some kind of a correspondence.

$$\left\{ \begin{array}{c} \text{closed} \\ \text{subvarities of} \\ X \subset \mathbb{A}^n \end{array} \right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{ \begin{array}{c} \text{ideals} \\ I \subset K[x_1, \dots, x_n] \end{array} \right\}$$

By definition, we have $V \circ I = \text{id}$. But the other way $I \circ V$ is not always the identity, i.e., it is not a one-to-one correspondence. For instance, (x^2) goes to (x). Indeed, if we let $I = (f_1, \dots, f_k)$, then

$$I(V(I)) = \{g \in K[x_1, \dots, x_n] : g(x) = 0 \text{ for any } x \text{ such that } f_1(x) = \dots = f_k(x) = 0\}.$$

Definition 11.6. Let $I \subset R$ be any ideal. We define the **radical** of I as

$$r(I) = \{f \in R : f^m \in I \text{ for some } m\}.$$

This is again an ideal.

Theorem 11.7 (Nullstellensatz)**.** *For any ideal I, $I(V(I)) = \mathfrak{r}(I)$.*

12 February 21, 2016[1]

12.1 Varieties and Ideals

Recall that there is a kind of a correspondence

$$\left\{\begin{array}{c}\text{closed}\\ \text{subvarities of}\\ X \subset \mathbb{A}^n\end{array}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{array}{c}\text{ideals}\\ I \subset K[x_1, \ldots, x_n]\end{array}\right\}$$

Theorem 12.1 (Nullstellensatz). *For any ideal $I \subset K[x_1, \ldots, x_n]$,*

$$I(J(I)) = \mathfrak{r}(I).$$

We remark that this gives a correspondence between the set of varieties and radical ideals. But Grothendieck on the other hand enlarged the set of varieties to schemes and made a correspondence between schemes and ideals.

Definition 12.2. We say that a collection of polynomials $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$ **cut out X set theoretically** if $V(f_1, \ldots, f_n) = X$. On the other hand, we say that that they **cut out X scheme theoretically** if $(f_1, \ldots, f_k) = I(X)$.

Things get a bit messier in the projective space. Let $S = K[X_0, \ldots, X_n]$, and consider the subsets S_m consisting of homogeneous polynomials of degree m. Then clearly $S = \oplus_{m \geq 0} S_m$.

Definition 12.3. We say that $I \subset S$ is a **homogeneous ideal** if I is generated by homogeneous polynomials.

we note that $I \subset S$ if and only if $I = \oplus_{m \geq 0}(I \cap S_m)$.

We want something like

$$\left\{\begin{array}{c}\text{closed}\\ \text{subvarities of}\\ X \subset \mathbb{P}^n\end{array}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{array}{c}\text{homogeneous}\\ \text{ideals}\\ I \subset K[X_0, \ldots, X_n]\end{array}\right\}.$$

However, there are problems. For instance, (XY) and (X^2Y, XY^2) have the same vanishing locus.

Definition 12.4. For a homogeneous ideal $I \subset S = K[X_0, \ldots, X_n]$, we define the **saturation** of I as

$$\mathrm{Sat}(I) = \{f : \text{there exists } d \text{ such that } fg \in I \text{ for all } g \in S \text{ with } \deg g \geq d\}.$$

We observe that if I is a homogeneous ideal, then $\mathrm{Sat}(I)$ is a homogeneous ideal.

[1] I did not attend class. The following is adapted from Aaron Landesman's notes.

Definition 12.5. Let $f_1, \dots, f_k \in S$ be homogeneous polynomials. We say that $f_1, \dots, f_k$

(1) **cut out** $X \subset \mathbb{P}^n$ **set theoretically** if $X = V(f_1, \dots, f_k)$.

(2) **cut out** $X \subset \mathbb{P}^n$ **scheme theoretically** if $\mathrm{Sat}(f_1, \dots, f_k) = I(X)$.

(3) **generate the homogeneous ideal of** $X \subset \mathbb{P}^n$ if $(f_1, \dots, f_k) = I(X)$.

The polynomials $f_1, \dots, f_k$ generating the homogenous ideal of X implies cutting out X scheme theoretically, and cutting out X scheme theoretically implies cutting out X set theoretically.

12.2 Irreducibility

Consider a hypersurface $X = V(f)$ with $f \in K[x_1, \dots, x_n]$. We can factorize f into $f = \prod_i f_i^{m_i}$ for f_i irreducible, and then we have $V(f) = \bigcup_i V(f_i)$. This suggests that there might be a way to break a variety into irreducible parts.

Definition 12.6. A topological space is called **irreducible** if X cannot be written as a union of proper closed subset. A topological space is called **reducible** if it is not irreducible.

For instance, if $f, g \in S$ and neither f nor g divides the other, then $X = V(fg) = V(f) \cup V(g)$ is reducible. More generally, X is irreducible if and only if $I(X)$ is prime. We shall prove the following theorem next time.

Theorem 12.7. *Any variety $X \subset \mathbb{A}^n$ is uniquely expressible as a union of finite collection of irreducible varieties. Equivalently, any radical ideal $I \subset K[x_1, \dots, x_n]$ is uniquely expressible as an intersection of n prime ideals.*

This theorem shows why ideals are called ideals. In some cases, like in Dedekind domains, ideals behave more nicely than numbers. People originally called ideals "ideal numbers."

13 February 24, 2016

Today we are going to talk about prime decomposition of radical ideals and Nullstellensatz.

13.1 Noetherian rings

Definition 13.1. Let R be a commutative ring. R is called a **Noetherian ring** if every increasing sequence of ideals

$$\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \mathfrak{a}_3 \supset \cdots$$

terminates.

There is a geometric interpretation of this condition. The polynomial ring $K[x_1, \dots, x_n]$ being Noetherian is equivalent to the fact that any decreasing sequence of subvarieties of $\mathbb{A}^n$

$$\mathbb{A}^n \subset X_1 \subset X_2 \subset X_3 \subset \cdots$$

terminates.

Anyways, back to our discussion, there are equivalent definitions of a Noetherian ring.

- every ideal of R is finitely generated.
- every submodule of a finitely generated R-module is finitely generated.
- any collection of ideals $\mathfrak{a} \subset R$ has maximal elements.

The key fact is the Hilbert basis theorem.

Theorem 13.2 (Hilbert basis theorem)**.** *If R is Noetherian, then $R[x]$ is also Noetherian.*

It follows that every polynomial ring $R[x_1, \dots, x_n]$ is Noetherian. It follows from the definition that if R is Noetherian, then $R/\mathfrak{a}$ is Noetherian. Therefore if R is Noetherian, then every finitely generated R-algebra is Noetherian. Gaitsgory's Math 123 notes are a great reference to this material.

Proposition 13.3. *Let $I \subset K[x_1, \dots, x_n]$ is a radical ideal. Then there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ such that*

$$I = \bigcap_{i=1}^{m} \mathfrak{p}_i.$$

Proof. Consider the set of radical ideals $I \subset K[x_1, \dots, x_n]$ that are *not* expressible as a finite intersection of prime ideals. Suppose that this set is nonempty. From the Noetherian condition, we can look at the maximal element of this collection, and let it be I_0. Clearly, I_0 is not prime, and thus there are $a, b \in K[x_1, \dots, x_n]$ such that $a, b \notin I_0$ but $ab \in I_0$.

Set $I_1 = \mathfrak{r}(I_0, a) \supsetneq I_0$ and $I_2 = \mathfrak{r}(I_0, b) \supsetneq I_0$ be the radicals of some ring. From the maximality assumption of I_0, we have decompositions

$$I_1 = \bigcap \mathfrak{p}_i, \quad I_2 = \bigcap \mathfrak{q}_j.$$

We claim that $I_1 \cap I_2 = I_0$. Consider any $f \in I_1 \cap I_2$. Then there has to be $g_1, g_2 \in I_0$ and positive integers m and n such that

$$f^m = g_1 + h_1 a, \quad f^n = g_2 + h_2 b.$$

Then the multiple of them is

$$f^{m+n} = g_1 g_2 + g_1 h_2 b + g_2 h_1 a + h_1 h_2 ab$$

is in I_0. This shows that $I_1 \cap I_2 = I_0$ and thus I_0 is also expressible as a finite intersection of primes. □

13.2 Proof of the Nullstellensatz

Let us recall the Nullstellensatz.

Theorem 13.4 (Nullstellensatz)**.** *Let $I \subset K[x_1, \dots, x_n]$ be any ideal. Then $I(V(I) = \mathfrak{r}(I)$.*

Theorem 13.5 (Weak Nullstellensatz)**.** *If $I \subset K[x_1, \dots, x_n]$ has no common root, then $I = (1)$.*

We first show that Weak Nullstellensatz implies the Nullstellensatz. This is something called the trick of Rabinowitsch.

Proposition 13.6. *The Weak Nullstellensatz implies the regular Nullstellensatz.*

Proof. Consider any ideal $I \subset K[x_1, \dots, x_n]$. Let $f \in K[x_1, \dots, x_n]$ vanish on $V(I)$. We need to show that $f^m \in I$ for some m.

Consider the variety

$$\Sigma = \{(x_1, \dots, x_n) \in \mathbb{A}^{n+1} = \mathbb{A}^n_x \times \mathbb{A}^1_y : y f(x_1, \dots, x_n) = 1\}.$$

Now because $f \equiv 0$ on $V(I)$, we see that $V(I, yf(x_1, \dots, x_n) - 1) = \emptyset$. Then by Weak Nullstellensatz, we can write

$$1 = g_0(yf(x) - 1) + \sum h_i y^i$$

for some $g_0 \in K[x_1, \dots, x_n, y]$ and $h_i \in I$. In this expression, we substitute $1/f$ for y. Then $1 = \sum_{i=0}^m h_i f^{-i}$ and multiplying by f^m, we get $f^m \in I$. □

We have five minutes left, so let me at least get started in the proof of the Weak Nullstellensatz.

Proposition 13.7. *Any maximal ideal* $\mathfrak{m} \subset K[x_1, \ldots, x_n]$ *is of the form* $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

If we prove this fact, then every $I \subsetneq (1)$ will be contained in some maximal ideal, which is in that form. Then $V(I) \neq \emptyset$.

14 February 26, 2016

Let $X \subset \mathbb{A}^n$ be an affine variety, and let $I(X) \subset K[x_1, \dots, x_n]$. We know that the coordinate ring $A(X) = K[x_1, \dots, x_n]/I(X)$ gives regular functions on X. It might be possible that some regular function is not globally given as a polynomial and only locally given as polynomials. But the Nullstellensatz says that this is not possible. That is, the ring of regular functions on X is simply $A(X)$.

Let $K \subset L$ be a field extension. We say that L/K is **algebraic** if every element of L satisfies a polynomial over K. Otherwise, say that L/K is **transcendental**. We observe that if L/K is transcendental, then L is not a finitely generated K-algebra.

Proposition 14.1. *$K(x)$ is not a finitely generated K-algebra.*

Proof. Suppose that $z_1, \dots, z_k \in K(x)$ generated $K(x)$ as a K-algebra. Write $z_i = p_i(x)/q_i(x)$. Given any irreducible polynomial $f(x) \in K[x]$, we can write

$$\frac{1}{f} = \text{polynomial in } \frac{p_i(x)}{q_i(x)}.$$

When we clear the denominator, we see that f has to divide at least one of the $q_i(x)$. But then it follows that there are only finitely many irreducible polynomial, and using the argument of Euclide, we see that this is not the case. □

Proposition 14.2. *Let K be an algebraically closed field. Every maximal ideal $\mathfrak{m} \subset K[x_1, \dots, x_n]$ is of the form $(x - a_1, \dots, x - a_n)$.*

This, if proven, will imply the Weak Nullstellensatz. We have to use a completely opaque lemma.

Lemma 14.3. *Say R is a Noetherian ring and let S be any ring $R \subset S \subset R[x_1 \dots, x_n]$. If $R[x_1, \dots, x_n]$ is finitely generated as an S-module, then S is finitely generated as an R-algebra.*

Proof. Say that $y_1, \dots, y_k \in R[x_1, \dots, x_n]$ generate $R[x]$ as an S-module. Then we can write

$$x_i = \sum a_{ij} y_j, \quad x_i x_j = \sum b_{ijk} y_k$$

for $a_{ij}, b_{ijk} \in S$.

Let $S_0 \subset S$ be a sub-algebra generated over R by $\{a_{ij}, b_{idk}\}$. So far we have $R \subset S_0 \subset S \subset R[x_1, \dots, x_n]$. We know that S_0 is a finitely generated R-algebra, and thus is Noetherian. Now $R[x_1, \dots, x_n]$ is finitely generated as an S_0-module by $y_1, \dots, y_k$. But then S is finitely generated as an S_0-module, and we see that S is finitely generated as an R-algebra. □

Proof of proposition 14.2. Let $\mathfrak{m}$ be a maximal ideal. Then $K[x_1, \dots, x_n]/\mathfrak{m}$ is a field extension of K. If the field is algebraic over K, then it is simply K because K is algebraically closed. If it is not K, the it means that it is a transcendental extension. This cannot be the case since $K[x_1, \dots, x_n]/\mathfrak{m}$ is finitely generated as a K-algebra. □

14.1 Tensor products

Let V be a vector space of dimension n over K, and let $\mathbb{P}V$ be the set of 1-dimensional subspaces of V. We define the **Grassmannian** as

$$G(k, V) = \{k\text{-dimensional subspaces of } V\}.$$

This is so far a set. So the first order of business of to show that this has a structure of a variety, i.e., give an inclusion $G(k, v) \hookrightarrow \mathbb{P}^N$ such that the image is a projective variety.

Let V and W be vector spaces of K, where K is a field of characteristic zero, not necessarily algebraically closed. We define the **tensor product** $V \otimes W$. There are three equivalent definitions.

(1) Choose bases $v_1, \ldots, v_m$ for V and $w_1, \ldots, w_n$ for W, and let $V \otimes W$ be the K-vector space with basis $\{v_i \otimes w_j\}$.

(2) We look at the vector space $K\langle\{v \otimes w : v \in V, w \in W\}\rangle$ and define $V \otimes W$ as the quotient by the subspace spanned by $\{(\lambda v)\otimes w - \lambda(v \otimes w), v \otimes (\lambda w) - \lambda(v \otimes w), (v_1 + v_2) \otimes w - (v_1 \otimes w + v_2 \otimes w), v \otimes (w_1 + w_2) - (v \otimes w_1 + v \otimes w_2)\}$.

(3) We define $V \otimes W$ as an universal object for bilinear maps.

$$\begin{tikzcd}
V \times W \arrow[r, "\text{bilinear}"] \arrow[d] & \mathcal{U} \\
V \otimes W \arrow[ur, dotted, "\text{linear}"'] &
\end{tikzcd}$$

If you are into it, we can prove they are indeed equivalent definitions.

15 February 29, 2016

Today and Wednesday, we are going to be looking at Grassmannians.

Let me make a remark. We recall problem 5 from the last homework.

Proposition 15.1. *Let $d \geq 1$ and $m \leq \binom{d+2}{2}$. A general collection of m points $p_1, \dots, p_m \in \mathbb{P}^2$ impose independent conditions on the vector space of polynomials of degree d on $\mathbb{P}^2$.*

What does it mean for m points to impose independent conditions? It meant that the evaluation at $p_1, \dots, p_m$

$$\begin{Bmatrix} \text{hom. poly. of} \\ \text{deg. } d \text{ on } \mathbb{P}^2 \end{Bmatrix} \longrightarrow K^m$$

has full codimension. The parameter space is simply $(\mathbb{P}^2)^m$. To prove the proposition, we have to check three things.

- $\mathcal{U} = \{(p_1, \dots, p_m) : p_i \text{ impose independent conditions}\} \subset (\mathbb{P}^2)^m$ is open (or is contained in an open set).
- $\mathcal{U} \neq \emptyset$.
- $(\mathbb{P}^2)^m$ is irreducible.

The last thing is necessary, because an open set might not be dense if $(\mathbb{P}^2)^m$ can be decomposed into multiple varieties.

15.1 Symmetric and exterior powers

We have looked at the tensor product last time. Let us look at the special case $V \otimes V$. In this space, there are the symmetric tensors, spanned by $v \otimes w + w \otimes v$, and the skew-symmetric tensors, spanned by $v \otimes w - w \otimes v$. We define

$$\operatorname{Sym}^2 V = \langle v \otimes w + w \times v \rangle = V \otimes V / \langle v \otimes w - w \otimes v \rangle,$$
$$\textstyle\bigwedge^2 V = \langle v \otimes w - w \otimes v \rangle = V \otimes V / \langle v \otimes w + w \otimes v \rangle.$$

More generally, we define $\operatorname{Sym}^k V$ as the space of symmetric tensors in $V^{\otimes k}$ and $\bigwedge^k V$ as the skew-symmetric tensors. We note that

$$\operatorname{Sym}^2 V^* = \{\text{symm. bilinear forms on } V\},$$
$$\textstyle\bigwedge^2 V^* = \{\text{alt. bilinear forms on } V\}.$$

Let us look again at the Grassmannian.

$$G(k, V) = \{k\text{-dimensional linear subspaces of } V\}.$$

We want a set-inclusion $G(k, V) \hookrightarrow \mathbb{P}^N$ such that the image is the zero locus of homogeneous polynomials in $\mathbb{P}^N$. This is equivalent to finding a parameter space for the k-dimensional subspaces. The idea is, for each $\Lambda \subset V$ choosing a basis $v_1, \dots, v_k$ for λ and mapping

$$[\Lambda] \mapsto [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}\textstyle\bigwedge^k V.$$

This gives a well-defined map.

16 March 2, 2016[2]

16.1 The Grassmannian

For a k-dimensional linear subspace $\Lambda \subset V$, we denote the corresponding point by $[\Lambda] \in G(k, V)$. We defined a map $\iota : G(k, V) \to \mathbb{P}\bigwedge^k V$ by

$$[\Lambda] = [\langle v_1, \dots, v_k \rangle] \mapsto [v_1 \wedge \cdots \wedge v_k].$$

We note that this is well-defined. Also, it is injective, because then V is the set of vectors v such that $v_1 \wedge \cdots \wedge v_k \wedge v = 0$.

We now claim that the image of this map ι is a projective variety in $\mathbb{P}\bigwedge^k V$. We use a lemma.

Lemma 16.1. *Let $\eta \in \bigwedge^k V$ and $v \in V$. Then*

$$\epsilon = v \wedge \phi$$

for some $\phi \in \bigwedge^{k-1} V$ if and only if $\epsilon \wedge v = 0$.

Proof. One direction is clear because $v \wedge v = 0$. For the other direction, express η in terms of a basis $v = e_1, \dots, e_n$. $\square$

This lemma shows that η is totally decomposable if and only if $\dim \ker \phi_\eta = k$, or equivalently, the rank of ϕ_η is at most $n - k$, where $\phi_\eta : V \to \bigwedge^{k+1} V$ is given by

$$\phi_\eta : v \mapsto v \wedge \eta.$$

Now we have a map $\phi : \bigwedge^k V \to \operatorname{Hom}(V, \bigwedge^{k+1} V)$ given by $\eta \mapsto \phi_\eta$. Then the image of ι is the zero locus of the determinate of all $(n-k+1) \times (n-k+1)$ minors of the $n \times \binom{n}{k+1}$ matrices in $\operatorname{Hom}(V, \bigwedge^{k+1} V)$. This shows that $G(k, V)$ can be given a structure of a projective variety. In fact, the Grassmannian is cut out by quadratic polynomials, but it takes a little more work to see this.

16.2 Universal family of k-planes

The Grassmannian is a parameter space for linear subspaces in V, but more interestingly it is also a parameter space for linear spaces in $\mathbb{P}V$. That is,

$$\begin{aligned} G(k, V) &= \{k\text{-dimensional subspaces } \Lambda \subset V\} \\ &= \{(k-1)\text{-dimensional spaces in } \mathbb{P}V\}. \end{aligned}$$

If we want to interpret the Grassmannian as spaces in the projective space, we shall write

$$\mathbb{G}(k-1, n-1) = \mathbb{G}(k-1, \mathbb{P}V) = G(k, V)$$

for an n-dimensional vector space V.

[2] I missed class, and this is adopted from Aaron Landesman's notes.

Definition 16.2. We define the **universal family of k-planes** in $\mathbb{P}^n$

$$\Sigma = \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n.$$

Then we have natural projections:

$$\begin{tikzcd} & \Sigma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{G}(k,n) & & \mathbb{P}^n \end{tikzcd}$$

To show that this is indeed a family, we need to check that Σ is a projective variety.

Lemma 16.3. *The universal family of k-planes Σ is indeed a variety in*

$$\mathbb{G}(k,n) \times \mathbb{P}^n \subset \mathbb{P}(\textstyle\bigwedge^{k+1} k^{n+1}) \times \mathbb{P}^n.$$

Proof. We can realize Σ as the vanishing of wedge products; for

$$\Phi = \{([\eta],[v]) : \eta \wedge v = 0\} \subset \mathbb{P}(\textstyle\bigwedge^{k+1} k^{n+1}) \times \mathbb{P}^n,$$

we have $\Sigma = (\mathbb{G}(k,n) \times \mathbb{P}^n) \cap \Phi$. □

We can construct other interesting varieties using this universal family. For instance, if $\Phi \subset \mathbb{G}(k,n)$ is a projective variety, then

$$X_\Phi = \bigcup_{[\Lambda]\in\Phi} \Lambda \subset \mathbb{P}^n$$

is a projective variety. This is because $X_\Phi = \pi_2(\pi_1^{-1}\Phi)$. Also, if $X \subset \mathbb{P}^n$ is any projective variety,

$$\{[\Lambda] \in \mathbb{G}(k,n) : \Lambda \cap X \neq \emptyset\} \subset \mathbb{G}(k,n) = \pi_1(\pi_2^{-1}X)$$

is a projective variety.

17 March 7, 2016

This week we will talk about rational functions and rational maps. This is not an exact language, but we'll see.

17.1 Rational functions

Let $X \subset \mathbb{A}^n$ be an irreducible variety.

"Definition". A **rational function** on X is a function locally expressible as a ratio f/g, where f and g are regular functions.

I put the definition in quotes, because f/g is not necessarily a function. Bear this example in mind: the map $h(x, y) = y/x : \mathbb{A}^2 \to \mathbb{A}^1$ cannot be extended to the whole plane.

Definition 17.1. Let $I(X)$ be the ideal of X in $K[x_1, \ldots, x_n]$, and let $A(X) = K[x_1, \ldots, x_n]/I(X)$. We define $K(X)$ as the field of fractions of $A(X)$, and call it the **function field** of X. A **rational function** on X is simply an element of $K(X)$.

Note that we use the condition X being irreducible here; because X is irreducible, $A(X)$ is an integral domain and this is why we can take its field of fractions. There is also something I just swept aside. By the Nullstellensatz we can show that anything locally expressible as regular functions can be globally expressible as polynomial. So we can get rid of the "locally expressible."

We can also realize $K(X)$ for $X \subset \mathbb{P}^m$ similarly. We can define it as the function field of any open affine subset. Or, we can look at the fraction of $S(X)$, where $S(X) = K[X_0, \ldots, X_n]/I(X)$. Note that because $S(X)$ is a graded ring, its field of fractions $S(X)$ is a graded ring including those with negative degree. Then we define $K(X)$ be the 0th graded piece of $S(X)$.

Now in presenting these definitions, I have left out a lot of loose ends. Let me now give the right definitions.

Definition 17.2. A **rational function** on X is an equivalence class of pairs (U, f) for $U \subset X$ nonempty open and f regular on U, under the relation

$$(U, f) \sim (V, g) \quad \Leftrightarrow \quad f = g \text{ on } U \cap V.$$

Definition 17.3. A **rational map** $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f) with $f : U \to Y$ regular.

Rational functions are not easily defined even sheaf theoretically. These objects were the most resistant in Grothendieck's rewriting of the subject, because they aren't exactly maps.

We also remark that the definition can be easily extended to reducible varieties. In this case, the functions on irreducible parts do not have to agree at the intersection, because they are closed proper subsets.

Definition 17.4. Say that X and Y are **birational** if there exists rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ that are inverses to one another.

Definition 17.5. Say that X is **rational** if there is a birational map $f : X \to \mathbb{A}^n$, or equivalently $K(X) = K(x_1, \ldots, x_n)$.

Example 17.6. Look at the quadric surface $Q = V(XY - ZW) \subset \mathbb{P}^3$ and for $p = [0, 0, 0, 1]$ consider the projection map $\pi_p : Q \dashrightarrow \mathbb{P}^2$ given by $q \mapsto \overline{pq} \cap \mathbb{P}^2$. If we look at the lines $L = V(X, Z)$ and $M = V(Y, Z)$, we see that both L and Z are mapped to points. But π_P is birational and in fact we can construct its inverse $\varphi : \mathbb{P}^2 \dashrightarrow Q$ as

$$[X, Y, Z] \mapsto [XZ, YZ, Z^2 XY].$$

18 March 9, 2016

There is sometimes a huge different between how people think about things and how they are defined. Of course I am talking about rational maps. How people think about it is a map given by $[f_0, \ldots, f_n]$ for $f_i \in K(x)$. But it i defined by an equivalence class of pairs (U, f) where $U \subset X$ open and dense and $f : U \to Y$ regular.

There is a natural question: in each equivalence class, is there a maximal (U, f_0)? We will not prove this, but the answer is yes. The maximal U is called the **domain** of f, and $X \setminus U$ is called the **indeterminacy locus**.

Example 18.1. Let $p \in X = \mathbb{P}^2$ and $L = V(Z) \cong \mathbb{P}^1 \subset \mathbb{P}^2$. The projection map $\pi_p : \mathbb{P}^2 \dashrightarrow L$ that sends $q \mapsto \overline{pq} \cap L$ is a rational map.

18.1 Rational maps as maps

Given a rational map $X \dashrightarrow \mathbb{P}^2$, we want to define

- composition of rational maps,
- the image of f,
- the preimage $f^{-1}(Z)$ for any subvariety $Z \subset \mathbb{P}^n$,
- and the graph.

They are not hard to define, but we still have to specify what we mean.

Definition 18.2. For a rational map f, take (U, f_0) in the equivalence class. Since $f_0 : U \to \mathbb{P}^n$ is a regular map, we have a well-defined graph $\Gamma_{f_0} \subset U \times \mathbb{P}^n \subset X \times \mathbb{P}^n$. Then we define Γ_f as the closure of Γ_{f_0} in $X \times \mathbb{P}^n$.

Now there is a natural projection:

$$\begin{tikzcd} & \Gamma_f \subset X \times \mathbb{P}^n \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & \mathbb{P}^n \end{tikzcd}$$

Then we can define the **image** of f as $\beta(\Gamma_f)$, and then for any $Z \subset X$, we define

$$f(Z) = \beta(\alpha^{-1}(Z)).$$

Likewise for $W \subset \mathbb{P}^n$, we define its **preimage**

$$f^{-1}(W) = \alpha(\beta^{-1}(Z)).$$

You always have to careful, because the image of a point might not be a point! For instance, in the projection map π_p, the image of p is a whole line.

Definition 18.3. Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be rational maps. Suppose that there exist pairs (U, f_0) and (V, g_0) with $U \subset X$, $V \subset Y$, and $f_0(U) \not\subset Y \backslash V$. Then we define the **composite** $g \circ f$ to be the equivalence class containing the pair

$$(f_0^{-1}(V), g_0 \circ f_0).$$

Definition 18.4. We say that X and Y are **birational** if

- there exist $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ such that $f \circ g$ and $g \circ x$ are defined and are equal to the identity maps.
- there exist dense open subsets $U \subset X$, $V \subset Y$ such that $U \cong V$.
- $K(X) \cong K(Y)$.

We say that X is **rational** if X is birational to $\mathbb{P}^n$ for some n.

Example 18.5. Consider any quadric hypersurface $Q \subset \mathbb{P}^{n+1}$ of rank $n+2$, and pick any point $p \in Q$. Let $\pi_p : Q \dashrightarrow \mathbb{P}^n$ be the projection map. Then this is a rational map.

So, quadric hypersurfaces are rational. What about cubic hypersurfaces? For $n = 1$, any smooth cubic curve in not rational; for example $X^3+Y^3+Z^3 = 0$ is rational. This is proven in the early 19th century, and was a major groundwork for many different fields. For $n = 2$, any smooth cubic surface is rational, and for $n = 3$, any smooth cubic 3-fold is not rational. The $n = 3$ case was proved in 1972 by Clemens-Griffiths by using Hodge theory.

19 March 11, 2016

In the 18th century, people were frustrated, because they were able to calculate

$$\int \frac{dx}{\sqrt{x^2-1}}$$

but not

$$\int \frac{dx}{\sqrt{x^3-1}}.$$

Why can we integrate the first integral? This is because the curve $y^2 = x^2 - 1$ is rational. We can parameterize the curve as $x = (1+t^2)/(1-t^2)$ and $y = 2t/(1-t^2)$, and this makes our integral into

$$\int \frac{dx}{y} = \int R(t)dt$$

for some rational function $R \in \mathbb{C}(t)$. However, the problem with $\sqrt{x^3-1}$ is that the curve $y^2 = x^3 - 1$.

In fact, the surface $V(Y^2 - X^2 + Z^2) \subset \mathbb{P}^2_{\mathbb{C}}$ is topologically a sphere, but $V(Y^2Z - X^3 + Z^3)$ is a torus. There is a fundamental difference between the sphere and the torus; a line integral on a sphere can be well-defined once we have the two endpoints, but a line integral on a torus is not well-defined. So the integral is actually a doubly periodic function on the complex plane. That is why we have to introduce the Weierstrass P function.

19.1 Blow-ups

Consider the map $\pi_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ for $p = [0,0,1]$ given by $[X,Y,Z] \mapsto [X,Y]$. Then we see that the graph Γ of π_p is

$$\Gamma = \{(q,r) \in \mathbb{P}^2 \times L : p, q, r \text{ colinear}\} = \{([X,Y,Z],[A,B]) : BX = AY\}.$$

There is a natural projection $\alpha : \Gamma \to \mathbb{P}^2$, and let us look at the fiber of α over an arbitrary $q \in \mathbb{P}^2$. Then we see that it is

$$\begin{cases} \text{single pt} & \text{if } q \neq p \\ \mathbb{P}^2 & \text{if } q = p. \end{cases}$$

The graph Γ along with the map $\Gamma \to \mathbb{P}^2$ is called the **blow-up** of $\mathbb{P}^2$ at p. The picture is that we take all the lines passing through p and make it disjoint.

Definition 19.1. Suppose we have closed subvarieties $Z \subset X \subset \mathbb{P}^m$. We can find homogeneous polynomials $F_0, \dots, F_n$ of the same degree d such that $\mathrm{Sat}(F_0, \dots, F_n) = I(Z)$. Consider the rational map $\varphi : X \dashrightarrow \mathbb{P}^n$ given by

$$\varphi = [F_0, \dots, F_n].$$

Then the graph $\Gamma \subset X \times \mathbb{P}^n$ of φ with the projection map $\Gamma \to X$ is called the **blow-up of X along Z**. We note the fiber of $\Gamma \to X$ for any point in $X \setminus Z$ is a single point. We denote this map as $\mathrm{Bl}_Z(X) \to X$.

I recommend looking up the definition in Hartshorne, and it has the undeniable virtue that it does not involve the choice of the polynomials.

Theorem 19.2. *If $\varphi : S \dashrightarrow T$ is any birational isomorphism between smooth surfaces S, T, the φ factors into a sequence of blow-ups at points.*

Let $X, Y \subset \mathbb{P}^n$ be disjoint varieties. The the set

$$J = \bigcup_{p \in X, q \in Y} \overline{pq}$$

is a projective variety, because it is the image of the map $X \times Y \to \mathbb{G}(1, n)$ given by $(p, q) \mapsto \overline{pq}$. Now this works even if X and Y are not disjoint. Then we have a map $s : X \times Y \to \mathbb{G}(1, n)$, and we can define the **joint** of X and Y to be

$$J = \bigcup_{L \in \mathrm{im}(s)} L.$$

In fact, we can take $X = Y$, and in this case we get the **secant variety** $\mathrm{Sec}(X)$.

20 March 21, 2016

Now we have the vocabulary, and we are going to start proving theorems about them.

20.1 Dimension

There are global and local invariant of a variety. The degree, hilbert polynomial and hilbert function are all global attributes of $X \subset \mathbb{P}^n$. The property of smoothness, on the other hand, is a local property.

For the time being, X is going to be an irreducible projective variety. I will tell you later what to do if it is not. So, how do we define the dimension? This never occurred to the 19th and early 20th century mathematicians. Even when there was no excuse, they defined a dimension d manifold as one having ∞^d points! But it was clear what they were trying to convey: it locally has d parameters.

Proposition 20.1. *Let $U \subset X$ be an open and dense subset, where X is a manifold over $\mathbb{C}$. Then U is a complex manifold.*

Then we can define $\dim X$ as the dimension of U as a complex manifold. But this will not do for us, because we do not want to invoke all the complex geometry into this.

First, we all agree that $\dim(\mathbb{P}^k)$ has to be k. Let us look at this definition.

"def". Say X has **dimension** k if there exists a finite (having finite fibers) surjective regular map $f : X \to \mathbb{P}^k$.

One objection is that there might not be such an f. But in fact, it is not hard to construct one f. If $X \subsetneq \mathbb{P}^n$ is projective variety, then we choose a $p \notin X$ and look at the projection $\pi_p : X \to \mathbb{P}^{n-1}$. Then this has finite fiber, and the image is again a projective variety. So we can repeat until the map is surjective. Another objection is that there might be several k such that f exists.

Let me describe this in another way. Given $X \subset \mathbb{P}^n$ we can find (for some k) an $(n-k-1)$-plane $\Lambda \subset \mathbb{P}^n$ such that $\Lambda \cap X = \emptyset$ and the projection map $\pi_\Lambda : X \to \mathbb{P}^k$ regular. Then π_Λ being surjective is equivalent to every $(n-k)$-plane $\Gamma \supset \Lambda$ meeting X.

In other words, we can say $\dim X = k$ if there exists a $(n-k-1)$-plane $\Lambda \subset \mathbb{P}^n$ such that $\Lambda \cap X = \emptyset$ but every $(n-k)$-plane $\Gamma \supset \Lambda$ does meet X. Because the $(n-k-1)$-planes meeting X form a closed subvariety in the Grassmannian, we have the following:

Proposition 20.2. *If $X \subset \mathbb{P}^n$ has dimension k, a general $(n-k-1)$-plane is disjoint from X but every $(n-k)$-plane meets X.*

An algebraic characterization of the dimension is that if $f : X \to \mathbb{P}^k$ is finite and surjective, then this induces a map $f^* : K(\mathbb{P}^k) = K(x_1, \dots, x_k) \to K(X)$ which is a finite algebraic extensions.

Definition 20.3. An irreducible projective variety X has dimension k if the function field $K(X)$ have transcendence degree k over K.

We will focus more on the geometric interpretation of the dimension, but this will be our standard definition.

Also, if $Y \subsetneq X$ is a closed subvariety, then $\dim Y < \dim X$. If $Y = H \cap X$ for $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and $H \not\supset X$ then $\dim Y = \dim X - 1$.

Definition 20.4. If X is a irreducible quasi-projective variety then $\dim X = \dim \overline{X}$. If $X = X_1 \cup \cdots \cup X_m$ for irreducible X_i, then $\dim X = \max(\dim X_i)$. For $p \in X$, the **local dimension** is defined as $\dim_p X = \max\{\dim X_i : X_i \ni p\}$.

Example 20.5. If $X = V(F) \subset \mathbb{P}^n$ and F has degree $d \geq 1$ then $\dim X = n-1$.

21 March 23, 2016

I recommend thinking about this problem:

Problem 21.1. *Let $F(X, Y, Z, W)$ be a general homogeneous polynomial of degree d and $S = V(F) \subset \mathbb{P}^3$ the corresponding surface. Does S contain a line?*

21.1 The basic theorem on the dimension

Let X be a projective irreducible variety, and $f : X \to Y = f(X) \subset \mathbb{P}^n$ be a surjective regular map. Then Y is projective and irreducible. For any $q \in Y$, let $\lambda(q) = \dim f^{-1}(q) \subset X$.

Theorem 21.2. *λ is upper semicontinuous in the Zariski topology, and if $\lambda_0 = \min \lambda(q)$ then $\dim X = \dim Y + \lambda_0$.*

We have encountered an example of a surjective regular map with fibers of varying dimension.

Example 21.3. Let $X = \mathrm{Bl}_p \mathbb{P}^n \to \mathbb{P}^n$. We know that

$$X = \{(q, r) \in \mathbb{P}^n \times \mathbb{P}^{n-1} : p, q, r \text{ collinear}\}.$$

Then the fiber over $q \neq p$ is a single point of dimension 0, and the fiber over p is $\mathbb{P}^{n-1}$, which has dimension $n - 1$.

Corollary 21.4. *Let X be a projective variety, and let $f : X \to Y$ be a surjection to a irreducible projective variety Y. If every fiber of f is irreducible of dimension k, then X is irreducible of dimension $\dim X = \dim Y + k$.*

Proof. Say that $X = \bigcup X_i$, each of X_i being irreducible. Consider the restrictions $f_i = f|_{X_i}$ and $\lambda_i(q) = \dim f_i^{-1}(q) \subset X_i$. Then $k = \lambda(q) = \max(\lambda_i(q))$. Because the dimension is upper semicontinuous, there is a i such that $\lambda_i(q) = k$ on an open subset of Y. Then because Y is irreducible, $\lambda_i(q) \equiv k$. This means that $f_i^{-1}(q) = f^{-1}(q)$ for every q, and there fore $X_i = X$. □

Example 21.5. Consider the Grassmannian $G(k, n)$. Choose $\Gamma \cong K^{n-k} \subset V$ and consider the set

$$\mathcal{U}_\Gamma = \{\Lambda \in G(k, n) : \Lambda \cap \Gamma = 0\}$$

Choose a basis $e_1, \dots, e_n$ for V such that $\Gamma = \langle e_{k+1}, \dots, e_n \rangle$. Then any $\Lambda \in \mathcal{U}_\Gamma$ can be represented as a row space of the matrix of form

$$\left(\begin{array}{c|c} I_k & A \end{array} \right)$$

where A is a $k \times (n - k)$ matrix. This shows that $\mathcal{U}_\Gamma \cong \mathbb{A}^{k(n-k)}$. It follows that $G(k, n)$ is irreducible of dimension $k(n - k)$.

Example 21.6. Let us look at the example of the universal k-plane over $\mathbb{G}(k, n)$

$$\Sigma = \{(\Lambda, p) \in \mathbb{G}(k, n) \times \mathbb{P}^n : p \in \Lambda\} \subset \mathbb{G}(k, n) \times \mathbb{P}^n.$$

Then because the projection $\pi_1 : \Sigma \to \mathbb{G}(k, n)$ are k-planes, we see that $\dim \Sigma = \dim \mathbb{G}(k, n) + k = (k + 1)(n - k) + k$. If we look at the projection $\pi_2 : \Sigma \to \mathbb{P}^n$, then for every $p \in \mathbb{P}^n$ the inverse image is

$$\pi_2^{-1}(p) = \{\Lambda \in \mathbb{G}(k, n) : \Lambda \ni p\} \cong \mathbb{G}(k - 1, n - 1).$$

Example 21.7. Let $X \subset \mathbb{P}^n$ be an irreducible variety of dimension ℓ, and consider the set

$$\mathscr{C}_X = \{\Lambda \in \mathbb{G}(k, n) : \Lambda \cap X \neq \emptyset\}.$$

We know this is a closed subvariety because $\mathcal{C}_X = \pi_1(\pi_2^{-1}(X))$. We use the same setting to calculate the dimension. Because $\pi_2^{-1}(X) \to X$ has fibers $\mathbb{G}(k - 1, n - 1)$, we see that $\pi_2^{-1}(X)$ is an irreducible variety with dimension $\ell + k(n - k)$.

Now we observe that if $k + \ell < n$, a general fiber of $\pi_1 : \pi_2^{-1}(X) \to \mathbb{G}(k, n)$ is finite, i.e., for a general k-plane Λ such that $\Lambda \neq X$, $\Lambda \cap X$ is finite.

I also recommend thinking about this problem.

Problem 21.8. *For any closed subvariety $X \subset \mathbb{P}^n$, Then $\{L \in \mathbb{G}(1, n) : L \subset X\} \subset \mathbb{G}(1, n)$ is a closed subvariety of $\mathbb{G}(1, n)$.*

22 March 25, 2016

Let me put our basic theorem on our board. I still owe you a proof.

Theorem 22.1. *Let X, Y be irreducible projective varieties and let $f : X \to Y$ be a surjective regular map. For $q \in Y$, let $\lambda(q) = \dim f^{-1}(q)$. then λ is upper-semicontinuous and if $\lambda_0 = \min \lambda(q)$ then $\dim X = \dim Y$.*

Last time we saw that $\dim(G(k,n)) = k(n-k)$. But how do we see that it is irreducible? Because PGL_n acts on $G(k,n)$, if we fix $\Lambda_0 \subset K^n$, then we get a map $PGL_n \to G(k,n)$ given by $A \mapsto A(\Lambda_0)$. From this we see that the Grassmannian is irreducible, and it is a nice exercise to calculate the dimension using this.

22.1 Planes inside a variety

The universal k-plane is given by

$$\Phi = \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n.$$

There are projection maps $\pi_1 : \Phi \to \mathbb{G}(k,n)$ and $\pi_2 : \Phi \to \mathbb{P}^n$. It follow from the maps that $\dim \Phi = (k+1)(n-k) + k = k(n-k) + n$.

If $X \subset \mathbb{P}^n$ is irreducible of dimension l, with $k + l \leq n$, then

$$\mathscr{C}_X = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \emptyset\} = \pi_1(\pi_2^{-1}(X)) \subset \mathbb{G}(k,n).$$

Lemma 22.2. *If $k + l \leq n$, then a general k-plane Λ meeting X meets X in finitely many pints.*

I think the is a good point leading into the problem I posed last time.

Problem 22.3. *Let $S \subset \mathbb{P}^3$ be a general surface of degree d. Does S contain a line?*

Let

$$F_k(X) = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset X\} \subset \mathbb{G}(k,n).$$

This is a closed subvariety of $\mathbb{G}(k,n)$, because if we look at the map $\alpha : \pi_2^{-1}(X) \to \mathbb{G}(k,n)$, then $F_k(X) = \{\Lambda : \dim \alpha^{-1}(\Lambda) \geq k\}$.

Solution. Let us first introduce

$$\mathbb{P}^N = \{\text{homogeneous polynomial of degree } d \text{ in } \mathbb{P}^3\}/\text{scalar}.$$

Then we have a family

$$\Psi = \{(S, L) : L \subset S\} \subset \mathbb{P}^N \times \mathbb{G}(1,3)$$

and projection maps

$$\begin{tikzcd} & \Psi \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^N & & \mathbb{G}(1,3). \end{tikzcd}$$

We observe that the fibers of β are $\mathbb{P}^{N-d-1}$. Hence Ψ is irreducible and the dimension is $N-d-1+4=N-d+3$.

An immediate consequence is that if $d \geq 4$, then there are no lines on a general S. If $d=1,2$, then there exist lines, because planes and rank 3 quadrics contain lines. The problem is $d=3$. If we show that there exists a cubic surface $S \subset \mathbb{P}^3$ with finitely many lines, then a general S will contain lines. This is because we can use the upper-semicontinuity theorem. In fact, the surface $X^3+Y^3+Z^3+W^3=0$ contains finitely many lines. I will leave this as a challenge. □

One more problem: what happens in $d=4$? The quartic surfaces that do contain a line form a hypersurface in $\mathbb{P}^{34}$. Then we can ask what the degree of the polynomial that cut out the hypersurface has. This is well beyond the scope of what I would consider appropriate material, but the answer is 320. However, by looking at the family, we change this highly linear problem into a linear condition.

23 March 28, 2016

I owe you one proof of the fundamental theorem on dimension. But I will continue to only use it.

23.1 Secant varieties

Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. Suppose that X is non-degenerate (not contained in a hyperplane) and has dimension k. We want to take the union of all secant lines to X.

We have a rational map $\sigma : X \times X \dashrightarrow \mathbb{G}(1, n)$ given by

$$(p, q) \mapsto \overline{pq}.$$

(When $k = 1$, this map σ will be regular if and only if X is a smooth curve.)

Definition 23.1. We define the **variety of secant lines to** X as

$$\mathscr{S}(X) = \mathrm{Im}(\sigma) \subset \mathbb{G}(1, n)$$

and define the **secant variety** as

$$S(X) = \bigcup_{\ell \in \mathscr{S}} \ell \subset \mathbb{P}^n.$$

To set up, introduce

$$\Sigma = \{(\ell, p) \in \mathscr{S} \times \mathbb{P}^n : p \in \ell\}.$$

There are projection maps $\Sigma \to S(X)$ and $\Sigma \to \mathscr{S}(X)$.

The general fiber of $X \times X \to \mathscr{S}$ has dimension 0, because of the general fiber has dimension > 0, it means that X contains the line going any two points of X, which means that X is a linear subspace. This is impossible, and hence $\dim \mathscr{S} = 2k$. Then because each fiber of $\Sigma \to \mathscr{S}(X)$ is $\mathbb{P}^1$, it follows that $\dim \Sigma = 2k + 1$.

Now what is the dimension of $S(X)$? Clearly $\dim S(X) \leq \dim \Sigma = 2k + 1$ and $\dim S(X) \leq n$, and so $\dim S(X) \leq \min(n, 2k + 1)$. But things get more interesting. Say X is **deficient** if $\dim S(X) < \min(n, 2k + 1)$. To give some facts, no curve is deficient, and there exists a unique deficient surface.

Example 23.2. Consider the Veronese surface $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ given as the image of the map

$$[X, Y, Z] \mapsto [X^2, Y^2, Z^2, XY, XZ, YZ].$$

This map sends any line $L \subset \mathbb{P}^2$ to a plane conic curve $C \subset \Lambda \cong \mathbb{P}^2$. Suppose $r \in S(X)$ is a general point, and let $r \in \overline{\nu(p), \nu(q)}$ for some $p, q \in \mathbb{P}^2$. Then the image of $\nu(p, q)$ is a plane conic curve, and therefore there is a one-dimensional family of lines passing through r that meets X at at least two points. Then the general fibers of $\pi_2 : \Sigma \to S(X)$ has dimension 1, and hence $\dim S(X) = 4$.

There is another way to look at this. We note that

$$X = \left\{ [W] : \operatorname{rank} \begin{pmatrix} W_0 & W_3 & W_4 \\ W_3 & W_1 & W_5 \\ W_4 & W_5 & W_2 \end{pmatrix} \leq 1 \right\}.$$

Then because any two rank 1 matrices add up to a rank at most 2 matrix, we see that $S(X)$ is the space of symmetric rank 2 matrices. Then $S(X)$ is a cubic hypersurface in $\mathbb{P}^5$.

Let us look at 3×3 matrices of rank ≤ 2. How do we know that this is irreducible? We introduces

$$\Phi = \{([v], [A]) \in \mathbb{P}^2 \times \mathbb{P}^8 : v \in \ker A\}.$$

Because the projection $\Phi \to \mathbb{P}^2$ has fibers $\mathbb{P}^5$, the variety Φ is irreducible of dimension 7. Then X also will be irreducible of dimension 7.

Let me end by proposing a problem:

Problem 23.3. *Let $M \cong \mathbb{P}^{mn-1}$ be the projectivization of the space of $m \times n$ matrices, and let $M_k = \{$matrices of rank $\leq k\}$. What is the dimension of M_k?*

24 March 30, 2016

Today we are going to do some more dimension counting.

24.1 More dimension counting

Let

$$M = \{m \times n \text{ matrices}\}/\text{scalars} = \mathbb{P}^{mn-1} = \mathbb{P}(\text{Hom}(V, W)),$$

where V and W are m and n dimensional vector spaces. For any $1 \leq k < \min(m, n)$, the set

$$M_k = \{\varphi : V \to W : \text{rank}(\varphi) \leq k\}$$

is a closed subvariety. We can ask the dimension of M_k and whether it is irreducible.

Let us look at the special case $k = 1$. In this case, M_1 is the Segre variety $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$, which is irreducible and of dimension $m + n - 2$. If $m = n$ and $k = m-1$, then M_{m-1} will be the set of singular matrices, and thus is irreducible of dimension $m^2 - 2$.

Because we suck in dealing with higher degree polynomials, we change the problem into a linear problem. Introduce

$$\begin{tikzcd}[column sep=small] & \Phi = \{(\Lambda, \Phi) \in G(m-k, V) \times M : \varphi(\Lambda) = 0\} \arrow[dl] \arrow[dr] & \\ G(m-k, m) & & M_k \end{tikzcd}$$

The point is that we know exactly the fibers of $\Phi \to G(m-k, m)$. They are isomorphic to $\mathbb{P}(\text{Hom}(V/\Lambda, W)) \cong \mathbb{P}^{nk-1}$. Then Φ is irreducible of dimension $k(m-k) + nk - 1$. Also, a general fiber of the projection map $\Phi \to M_k$ is 0 dimensional, because the fiber of a rank k matrix is a single point. Therefore M_k is irreducible of dimension $k(m+n-k)-1$, or of codimension $(m-k)(n-k)$.

Example 24.1. The first nontrivial example is $m = 2$, $n = 3$, and $k = 1$.

Fix n and d. When is a general homogeneous polynomial of degree d in n variables expressible as the determinant of a $d \times d$ matrix of linear forms? Let us look at $n = 3$ first.

$$\begin{tikzcd} \{d \times d \text{ matrices of linear forms in } X, Y, Z\}/\text{scalar} \cong \mathbb{P}^{3d^2-1} \arrow[d, dashed] \\ \mathbb{P}^{\binom{d+2}{2}-1} \end{tikzcd}$$

What is the general fiber? At least it must contain $PGL_d \times PGL_d$ because multiplying scalar matrices on the left and right does not change the determinant. This shows that the dimension of the space of such polynomials is at most

$3d^2 - 1 - 2(d^2 - 1) = d^2 + 1$. But this is actually pretty much useless, because $\binom{d+2}{2} - 1 \leq d^2 + 1$. Let us look at the case with more variables. If $n = 4$, then the dimension is at most $4d^2 - 1 - 2(d^2 - 1) = 2d^2 + 1$, where the dimension of the space of polynomials is $\binom{d+3}{3} - 1$. Therefore we see that if $n = 4$ and $d \geq 4$, then a general polynomial is not expressible as a determinant.

Here is a challenge problem:

Problem 24.2. *Suppose $S \subset \mathbb{P}^3$ is a quartic surface. Show that S is determinantal if and only if S contains a twisted cubic curve.*

25 April 1, 2016[3]

25.1 Proof of the main theorem

Theorem 25.1. *Let X and Y be irreducible varieties and let $f : X \to Y$ be a surjective map. For $q \in Y$, let $\lambda(q) = \dim f^{-1}(q)$. Then λ is upper-semicontinuous. Further, if $\lambda_0 = \min_{q \in Y} \lambda(q)$, then $\dim X = \dim Y + \lambda_0$.*

To do so, we will first prove the local version of this theorem and use it to prove our theorem.

Definition 25.2. Let $p \in X$ be a point on a vareity. We define the **local dimension** of X at p to be

$$\dim_p(X) = \min\{\dim U : U \text{ is an open set contianing } p\}.$$

Theorem 25.3. *Let X be a quasi-projective variety and $f : X \to \mathbb{P}^n$ be a regular map; let Y be the closure of the image. For any $p \in X$, let $X_p = f^{-1}(f(p))$, and let $\mu(p) = \dim_p(X_p)$ be the local dimension of X_p at p. Then $\mu(p)$ is an upper-semicontinuous function of p, in the Zariski topology on X. Moreover, if $X_0 \subseteq X$ is any irreducible component, $Y_0 \subseteq Y$ the closure of its image and μ_0 the minimum valeu of $\mu(p)$ on X_0, then*

$$\dim(X_0) = \dim(Y_0) + \mu_0.$$

Proof of theorem 25.1. We need to show that $\{q \in Y : \lambda(q) \geq m\} \subset Y$ is closed. But because $\lambda(q) = \mu(p)$ for any $p \in f^{-1}(q)$,

$$\{q \in Y : \lambda(q) \geq m\} = f(\{p \in X : \mu(p) \geq m\}).$$

By theorem 25.3, $\{p \in X : \mu(p) \geq m\}$ is a closed set, and hence the image is also closed because f is regular. The dimension identity follows immediately. $\square$

Proof of theorem 25.3. Because the statement is completely local, we may assume that both X and Y are affine vareities. Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$, and let $f : X \to Y$ be given by

$$(x_1, \ldots, x_m) \mapsto (f_1(x), \ldots, f_n(x)).$$

Because we are not good at dealing with arbitrary regular maps, we use a trick to make f into a projection map. We first identify X with the vareity

$$\{(f_1(x), f_2(x), \ldots, f_n(x), x_1, \ldots, x_m)\} \subset \mathbb{A}^{m+n}.$$

Then $f : X \to Y$ becomes a projection map and factors into

$$\mathbb{A}^{m+n} \to \mathbb{A}^{m+n-1} \to \cdots \to \mathbb{A}^{n+1} \to \mathbb{A}^n.$$

It now suffices to prove the following:

[3] Again, from Aaron Landesman's notes.

Lemma 25.4. *Theorem 25.3 holds for the map π of the form*

$$\begin{tikzcd} \mathbb{A}^l \arrow[r, "\pi"] & \mathbb{A}^{l-1} \\ X \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

Proof. For such a map π, the corresonding function $\pi(p)$ is given by

$$\mu(p) = \begin{cases} 1 & \text{if } \pi^{-1}(\pi(p)) \subseteq X \\ 0 & \text{otherwise.} \end{cases}$$

It suffies to show that $\{p \in X : \mu(p) = 1\}$ is a closed subset of X.

Let $X = V(g_\alpha)$ where

$$g_\alpha(z_1, \ldots, z_l) = \sum a_{\alpha,i}(z_1, \ldots, z_{l-1}) z_l^i.$$

Then the locus on which $\mu(p) = 1$ is precisely $V(a_{\alpha,i})$ and hence is a closed subset of X. □

Therefore theorem 25.3 holds for any map $f : X \to Y$. □

25.2 Parameter space of twisted cubics

Problem 25.5. *Does a general surface $S \subset \mathbb{P}^3$ of degree d contain a line?*

Here is another quesiton along the same lines.

Problem 25.6. *Does a general surface $S \subset \mathbb{P}^3$ of degree d contain any twisted cubics?*

It is difficult to interpret this question because we don't have a paramater space for twisted cubics. We need a variety $\mathscr{H}$ so that hte points of $\mathscr{H}$ correpond bijectively to twisted cubic curves, and we further need a map

$$\begin{tikzcd} \mathscr{C} \arrow[r, hook] \arrow[d] & \mathscr{H} \times \mathbb{P}^3 \arrow[dl] \\ \mathscr{H} & \end{tikzcd}$$

for which the fibers consist of twisted cubic curves. This variety $\mathcal{H}$ is called a **Hilbert scheme**. For today, we will assume that this exists and do dimension counts.

What is the dimension of the space of twisted cubics? We will give two explanations. (We haven't defined parameter spaces yet.)

Lemma 25.7. *The parameter space for twisted cubics is irreudicible and* 12-*dimensional.*

First explanation. Recall that a twisted cubic curve is defined to be a curve projectively equivalent to the map $f : \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$[X_0, X_1] \mapsto [X_0^3, X_0^2 X_1, X_0 X_1^2, X_2^3].$$

Define $C_0 = \operatorname{im} f$. Then we have a map $\phi : PGL_4(K) \to \mathscr{H}$ given by $A \mapsto A(C)$. Because PGL_4 consists of nonsingular matrics modulo scalars, is has dimension 15. Also, because two twisted cubics agree if and only if they are related by precomposing with an automorphism of $\mathbb{P}^1$, the fibers have dimension 3. This implies $\dim \mathscr{H} = 12$. $\square$

Second explanation. We use the fact that for any $p_1, p_2, \dots, p_6 \in \mathbb{P}^3$ in general linear position, there exists a unqiue twisted cubic passing thorugh $p_1, \dots, p_6$. So we have an incidence correspondence

$$\Psi = \{(p_1, \dots, p_6, C) \in U \times \mathscr{H} : p_1, \dots, p_6 \in C\}$$

where U is the open subset of $(\mathbb{P}^3)^6$ consisting of points in general linear position. We have projection maps

$$\begin{tikzcd} & \Psi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathscr{H} & & (\mathbb{P}^3)^6 \end{tikzcd}$$

Because π_2 has zero-dimensional fibers, $\dim \Psi = 18$. Also, because π_1 has fibers isomorphic to an open set C^6 which has dimenison 6, it follows that $\dim \mathscr{H} = 12$. $\square$

Let us go back to our problem about a general surface containing a twisted cubic. Given a $C = \phi(\mathbb{P}^1) \subset \mathbb{P}^3$, it induces a pullback map

$$\{\text{polynomials of deg. } d \text{ on } \mathbb{P}^3\} \to \{\text{polynomials of deg. } 3d \text{ on } \mathbb{P}^1\}.$$

This maps an $(N+1)$-dimensional vector space to a $(3d+1)$-dimensional vector space. Then we have an incidence correspondence

$$\begin{tikzcd} & \Phi \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^N & & \mathscr{H} \end{tikzcd}$$

Beause $\mathscr{H}$ has dimension 12 and the fibers are isomorphic to $\mathbb{P}^{N-3d-1}$. So Φ is irreducible of dimneion $N - 3d - 1 + 12$. Because this is larger than N for $d \geq 4$, a general surface of degree ≥ 4 cannot contain a twisted cubic.

26 April 4, 2016

26.1 Hilbert functions

Given a projective variety $X \subset \mathbb{P}^n$, we want to know how many homogeneous polynomials of a given degree m vanish on X. That is, what is the dimension of the mth graded piece of the ideal ring? Because we don't know what this is, we define this quantity.

Definition 26.1. We define the **Hilbert function** h_X as

$$h_X(m) = \dim A(X)_m = \binom{m+n}{n} - \dim I(X)_m.$$

where $A(X) = S/I(X)$.

Example 26.2. Consider $X = \{p_1, p_2, p_3\} \subset \mathbb{P}^2$. Clearly $h_X(0) = 1$, and

$$h_X(1) = \begin{cases} 3 & \text{if } p_i \text{ are not colinear} \\ 2 & \text{if } p_i \text{ are colinear.} \end{cases}$$

Then $h_X(2) = 3$ for any such X because the evaluation map $S(\mathbb{P}^2)_2 \to K^3$ is surjective. Even if $d \geq 3$, this map is surjective and hence $h_X(d) = 3$.

In general, if $X = \{p_1, \dots, p_d\} \subset \mathbb{P}^n$, by the same logic, $h_X(m) = d$ for $m \geq d-1$. this is becasue $S(\mathbb{P}^n)_m \to K^d$ is surjective. This is because given a point p_k, we can find a polynomial F of degree m such that $F(p_i) = 0$ for $i \neq k$ and $F(p_k) \neq 0$. If $p_1, \dots, p_d$ is general, then the map $S(\mathbb{P}^n)_m \to K^d$ has maximal rank. That is, $h_X(m) = \max\{\binom{m+n}{n}, d\}$.

Example 26.3. Let as consider the twisted cubic $X \subset \mathbb{P}^3$ given as the image of the map $[F_0, F_1, F_2, F_3]$, e.g., $[X_0^3, X_0^2X_1, X_0X_1^2, X_1^3] = [Z_0, Z_1, Z_2, Z_3]$. I want to know how many degree m polynomials vanish on X. We have a map

$$S(\mathbb{P}^3)_m \to S(\mathbb{P}^1)_{3m}$$

and I claim that this map is surjective for any m. Then $h_X(m) = 3m + 1$.

Example 26.4. We can generalize this to rational normal curves $X \subset \mathbb{P}^d$. The map

$$S(\mathbb{P}^d)_m \to S(\mathbb{P}^1)_{dm}$$

is surjective and hence $h_X(m) = dm + 1$.

Example 26.5. If I have a Veronese vareity $\nu_d : \mathbb{P}^n \to X \subset \mathbb{P}^N$, then we again have a surjective pullback map $S(\mathbb{P}^N)_m \to S(\mathbb{P}^n)_{md}$ and thus $h_X(m) = \binom{md+n}{n}$.

You might have noticed that in most of the cases, h_X is in fact a polynomial. Even in the case with three colinear points, h_X eventually becomes a polynomial.

For instance, let $X = V(F) \subset \mathbb{P}^2$ where F is a homogeneous polynomial of degree d without repeated factors. Then clearly $I(X) = (F)$ and there is a short exact sequence

$$0 \longrightarrow S(\mathbb{P}^2)_{m-d} \cong I(X)_m \longrightarrow S(\mathbb{P}^2)_m \longrightarrow A(X)_m \longrightarrow 0.$$

Then for $m \geq d$,

$$h_X(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = dm - \frac{d^2+3d}{2}.$$

Likewise for a hypersurface $X = V(F) \subset \mathbb{P}^n$, we have $I(X) = F$ and hence

$$h_X(m) = \binom{m+n}{n} - \binom{m-d+n}{n}$$

for large enough m. This is a degree $n-1$ polynomial in m.

Next time, we will prove that this is indeed a polynomial for large m.

27 April 6, 2016

Today we will finish Hilbert polynomials.

27.1 Hilbert polynomials

Let $X \subset \mathbb{P}^n$ be a projective variety and define

$$h_X(m) = \dim A(X)_m$$

where $A(X)_m$ is the mth graded piece of $A(X) = S(\mathbb{P}^n)/I(X)$. This is the Hilbert function, and we have observed that in many cases this eventually becomes a polynomial.

Theorem 27.1. *For any projective variety $X \subset \mathbb{P}^n$, there exists an m_0 and a polynomial p_X such that*

$$h_X(m) = p_X(m)$$

for all $m \geq m_0$. Moreover, $\dim X = \deg p_X$.

We define the polynomial p_X the **Hilbert polynomial** of X. For instance, if X is d disjoint points, then $h_X(m) = d$ for all $m > d$ and hence $p_X = m$.

Lemma 27.2. *Let $X \subset \mathbb{P}^n$ be a projective vareity. For a general hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$, the intersection $Y = H \cap X$ satisfies*

$$h_Y(m) = h_X(m) - h_X(m-1)$$

for sufficiently large m.

If we have this lemma, then we would be able to use induction on n to get the theorem 27.1.

Proof of lemma 27.2. We use the following facts:

Lemma 27.3. *Let $H = V(L)$ be a hyperplane where L is a linear form. If H does not contain any irreducible components of X, then L is not a zero divisor in $A(X)$.*

Proof. If L is a zero divisor, then there must be a nonzero polynomial M not in $I(X)$ such that its product with L is in $I(X)$. This means that the union of $H \cap X$ and $V(M) \cap X$ is the whole vareity X but $H \cap X$ does not contain any irreducible componenet of X. Then $V(M)$ must contain X, and this contradicts that M is not $I(X)$. $\square$

Proposition 27.4. *FOr a linear form L for which $V(L)$ does not contain any irreducible componenet of X, the ideal of $X \cap V(L)$ is $I(X \cap V(L)) = \mathrm{Sat}(I(X), L)$.*

We shall prove this later when we know about tangent spaces and transversaility.

Onece we have the two statements, we can construct a short exact sequence

$$0 \longrightarrow A(X)_{m-1} \xrightarrow{\times L} A(X)_m \longrightarrow A(Y)_m \longrightarrow 0$$

for sufficeintly large m. To check exactness, we use the proposition and the fact that $\mathrm{Sat}(I) = \mathrm{Sat}(J)$ implies $I_m = J_m$ for sufficiently large m. (This will be on the homework.) Therefore we get the statement about the dimension. $\square$

27.2 The Hilbert syzygy theroem

Definition 27.5. A **graded module** M over a graded ring $S = \oplus_k S_k$ is a module of the form $M = \oplus_l M_l$ satisfying $S_k M_l \subset M_{l+k}$. A morphism between graded modules M and N over S is a map $\phi : M \to N$ such that $\phi(M_l) \subset N_l$.

Definition 27.6. Givne a graded module M over a graded ring S, we define $M(k)$ to be the "shifted" graded module given by

$$M(k)_l = M_{k+l}.$$

This is isomophic to M as an S-module, but not as a graded S-module. So for example, if $F \in S = S(\mathbb{P}^n)$ is a homogeneous polynomial of degree d, then it induces a map $S(-d) \to S$ given by

$$G \mapsto FG.$$

Theorem 27.7 (Hilbert syzygy theorem)**.** *Let X be a projective variety. Let $I(X) = (F_1, \ldots, F_l)$ with $\deg F\alpha = d_\alpha$. Then there is a surjective (and clearly not injective) homomogphism of graded modules*

$$\bigoplus_\alpha S(-d_\alpha) \xrightarrow{\phi_0} I(X) \longrightarrow 0.$$

Let $M_1 = \ker \phi_0$. Since S is Noetherian, this must also be a finitely generated module. Let $g_1, \ldots, g_m$ be the generators of M_1, where $g_i \in (M_1)_{d_{1,i}}$. Then we can do the same thing and extend it to

$$\bigoplus_\beta S(-d_{1,\beta}) \xrightarrow{\phi_1} \bigoplus_\alpha S(-d_{0,\alpha}) \xrightarrow{\phi_0} S \longrightarrow A(X) \longrightarrow 0.$$

When we repeat this process, it terminates after at most n steps, where $S = K[z_0, \ldots, z_n]$.

We will not prove this theorem in this course, but using it we can get theorem 27.1 quickly.

Alternative proof of theorem 27.1. There is an exact sequence

$$0 \to \bigoplus S(-d_{n,\gamma}) \to \cdots \to \bigoplus S(-d_{1,\beta}) \to S \to A(X) \to 0.$$

Since this is an exact sequence of finite length, for sufficiently large m,

$$\begin{aligned} h_X(m) = \dim A(X)_m &= \sum_i (-1)^i \dim \left(\bigoplus S(-d_{i,j})\right)_m \\ &= \sum_{i,j} (-1)^i \dim S_{m-di,j} = \sum_{i,j} (-1)^i \binom{m+n-d_{i,j}}{n}. \end{aligned}$$

□

As an exercise, try to write out the resolution for the twisted cubic.

28 April 8, 2016

28.1 Free resolution of a twisted cubic

Let me look at one example of a free resolution. Consider the twisted cubic $\mathbb{P}^1 \to C \subset \mathbb{P}^3$ given by

$$[X_0, X_1] \mapsto [X_0^3, X_0^2 X_1, X_0 X_1^2, X_1^3].$$

Then C is cut out be Q_1, Q_2, Q_3, where

$$Q_1 = Z_0 Z_2 - Z_1^2, \quad Q_2 = Z_0 Z_3 - Z_1 Z_2, \quad Q_3 = Z_1 Z_3 - Z_2^2.$$

Then we can start the resolution by letting

$$S(-2)^{\oplus 3} \xrightarrow{(Q_i)} S \longrightarrow A(C) \longrightarrow 0.$$

By dimension counting, we see that $\dim I(C)_2 = 3$ and $\dim I(C)_3 = 10$. When we look at the 3rd graded piece of $S(-2)^{\oplus 3} \to S$, it is the multplication map

$$I(C)_2 \otimes S_1 \to I(C)_3.$$

The first one is 12-dimensional and the second one is 10-dimensional. So the kernel must have dimension 2. The two relations we get is

$$\begin{aligned} Z_2 Q_1 - Z_1 Q_2 + Z_0 Q_3 &\equiv 0 \\ Z_3 Q_1 - Z_2 Q_2 + Z_1 Z_3 &\equiv 0. \end{aligned}$$

And actually these two are all the relations. In other words, finally have a resolution

$$0 \longrightarrow S(-3)^{\oplus 2} \longrightarrow S(-2)^{\oplus 3} \longrightarrow S \longrightarrow A(C) \longrightarrow 0$$

where the $S(-3)^{\oplus 2} \to S(-2)^{\oplus 3}$ is given by

$$\begin{pmatrix} Z_2 & -Z_1 & Z_0 \\ Z_3 & -Z_2 & Z_1 \end{pmatrix}^T.$$

Then we can calculate the Hilbert polynomial.

$$h_C(m) = \dim A(C)_m = \dim S_m - 3 \dim S_{m-2} + 2 \dim S_{m-3} = 3m + 1.$$

28.2 Tangent spaces

Many of the notions that we define algebraically has its origin in simpler categories, such as manifolds. So let us look at a manifold $X \subset \mathbb{R}^n$ and a point $p \in X$. Locally around p, X is the zero locus of C^∞ functions $f_\alpha(x_1, \ldots, x_n)$. Then when we look at the matrix $(\partial f_\alpha / x_i)$, we define the tangent space as $T_p X$, in the manifold case.

Let me just go straight to the definition now.

Definition 28.1. Let $\mathcal{O}_{X,p}$ be the **ring of germs** of regular functions at p. That is, it is the equivalence class of pairs (U, f) where U is an open neighborhood of $p \in X$ and they are equivalent if and only if they agree along an open neighborhood of p. Let $\mathfrak{m}_p$ be the ideal of functions that vanish on p. We define the **Zariski cotangent space** of X at p as $T_p^* = \mathfrak{m}_p/\mathfrak{m}_p^2$, and the **tangent space** as $T_p = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$.

Something to bear in mind is that the cotangent and tangent spaces are always vector spaces.

Definition 28.2. We say that X is **smooth** at p if $\dim T_p X = \dim X$.

Also, one observation is that if $f : X \to Y$ maps $p \mapsto q$, then there is a pullback map $f^* : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, and this clearly maps $\mathfrak{m}_q \to \mathfrak{m}_p$. The conclusion is, there is a map

$$T_q^* Y \to T_p^* X$$

the transpose of this map is

$$T_p X \to T_q Y.$$

This is the derivative df of f. Next week, we are going to look at a bunch of examples.

29 April 11, 2016

Let X be any variety and let $p \in X$. We have defined the Zariski tangnet space.

There is another extrinsic way to define the tangent space. Let $X \subset \mathbb{A}^n$, (just look at the open affine subset) and let $I(X) = (f_1, \dots, f_k)$. Then for the Jacobian matrix

$$M_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix},$$

we can define T_pX as the kernel of M_p, or more simply,

$$T_p = \operatorname{Ann}\langle df_1(p), \dots, df_k(p)\rangle.$$

The intrinsic definition is $T_p = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$.

29.1 Smooth and singular points

Observe that always $\dim_p X \le \dim T_pX$. If equality holds, we say that X is **smooth** at p and if strict inequality holds, we say that X is **singular** at p.

Proposition 29.1. *Let us define*

$$X_{\text{sing}} = \{\textit{singular pts of } X\} \subset X,$$
$$X_{\text{sm}} = X \setminus X_{\text{sing}}.$$

Then X_{sing} is a closed subset and X_{sm} is an open subset.

This will be the homework. You can just reduce into the case of hypersurfaces and to things.

Proposition 29.2. *If X has irreducible decomposition $X = \bigcup X_i$, then*

$$X_{\text{sing}} = \bigcup (X_i)_{\text{sing}} \cup \bigcup_{i \neq j} X_i \cap X_j.$$

Example 29.3. Consider the variety $V(y^2 - x^4) = V(y - x^2) \cup V(y + x^2)$. The tangent space at the origin is *not* a line. It is the whole plane. Likewise, the tangent space of $V(x^2 + y^2 - z^4)$ at the origin is the whole space.

29.2 The Gauss map and tangential varieties

Sometimes, people refer to a different object by the word tangent space. Suppose that $X \subset \mathbb{A}^n$ and let $I(X) = (f_1, \dots, f_k)$. We can look at the naıve "tangent space" that is literally tangent to X at p. We can define the **affine tangent space of X at** $p = (z_1, \dots, z_n)$ as

$$\Big\{(w_1, \dots, w_n) : \sum \frac{\partial f_\alpha}{\partial z_i} = 0 \text{ for all } \alpha\Big\}.$$

Likewise, we can define the **projective tangent space** as

$$\mathbb{T}_p X = \Big\{ [W_0, \ldots, W_n] : \sum \frac{\partial F_\alpha}{\partial Z_i}(p) W_i = 0 \Big\},$$

where $I(X) = (F_1, \ldots, F_k)$ and $p = [Z_0, \ldots, Z_n]$. This is a linear space in the ambient projective space.

Let $X \subset \mathbb{P}^n$ be an smooth irreducible variety of dimension k. Then we can define the **Gauss map**

$$\begin{aligned} \mathscr{G} : X &\to \mathbb{G}(k, n) \\ &\mapsto \mathbb{T}_p X. \end{aligned}$$

Even if X is singular, we get a rational map

$$\mathscr{G} : X \dashrightarrow \mathbb{G}(k, n)$$

because X_{sm} is open in X.

Definition 29.4. We define the **locus of tangent k-planes**

$$\tau(X) = \operatorname{im}(\mathscr{G}) \subset \mathbb{G}(k, n).$$

Likewise, we define **tangential variety** as

$$T(X) = \bigcup_{\Lambda \in \tau(X)} \Lambda \subset \mathbb{P}^n.$$

If we do the same thing we did for secant varieties, i.e, define

$$\Psi = \{(p, q) : p \in \mathbb{P}^n, q \in \mathbb{T}_p X\},$$

then Ψ has dimension $2k$ and hence $\dim TX \leq 2 \dim X$. This leads to a bunch of other questions: when does the equality hold? is the TX itself singular? Try these for the twisted cubic curve in $\mathbb{P}^3$.

Definition 29.5. Let X be an irreducible (smooth) dimension k variety, and let $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ by an hyperplane. Say that H is **tangent** to X at $p \in X$ if $H \supset \mathbb{T}_p X$.

Proposition 29.6. *The set of tangent hyperplanes to X in $\mathbb{P}^{n*}$ is a projective variety.*

Proof. We look at

$$\Phi = \{(p, H) : H \supset \mathbb{T}_p X\} \subset X \times \mathbb{P}^{n*}.$$

Because each fiber of $\Phi \to X$ has fiber $\mathbb{P}^{n-k-1}$, the variety Φ has dimension $n-1$. Then the set of tangent hyperplanes is the image of the projeciton $\Phi \to \mathbb{P}^{n*}$. $\square$

30 April 13, 2016

Recall that for a projective irreducible vareity of dimension k, then the Gauss map $\mathscr{G} : X \dashrightarrow \mathbb{G}(k, n)$ given by

$$p \mapsto \mathbb{T}_p X.$$

This map is regular if and only if X is regular.

30.1 Dual varieties

Let X be a smooth surface. We define its **dual variety** as

$$X^* = \{H \in \mathbb{P}^{n*} : H \supset \mathbb{T}_P X \text{ for some } p \in X\}.$$

If we consider

$$\Phi = \{(p, H) \in X \times \mathbb{P}^{n*} : H \supset \mathbb{T}_P X\}$$

then because each fiber of $\Phi \to X$ is $\mathbb{P}^{n-k-1}$, it follows that Φ is is irreudible of dimension $k + (n - k - 1) = n - 1$. Hence either X^* is a hypersurface, or a general fiber of $\Phi \to \mathbb{P}^{n*}$ has positive dimension.

Now, the key result is:

Theorem 30.1. *For an irreducible nondegenerate variety $X \subset \mathbb{P}^n$,*

$$(X^*)^* = X.$$

Idea of proof. It suffices to show that for any p, $\mathscr{G}_{X^*}(\mathscr{G}_X(p)) = p$. Let us look at the case of plane curves.

For a point q near p, the tangent line $\mathbb{T}_X q$ approaches $\mathbb{T}_X p$, and then the intersection of $\mathbb{T}_X p$ and $\mathbb{T}_X q$ approaches p. Because the dual of this point is the line connecting $\mathscr{G}_X(p)$ and $\mathscr{G}_X(q)$, this line will aproach $\mathbb{T}_{X^*}(\mathscr{G}_X(p))$. It follows that $\mathscr{G}_{X^*}(\mathscr{G}_X(p)) = p$. □

30.2 Nash blow-ups

Question. *Given a variety X, does there always exist a smooth vareity $\tilde{X}$ with a map $\pi : \tilde{X} \to X$ that is generically one-to-one?*

Hironaka in 1960 showed that the answer is yes for characteristic 0. But it is still open in characteristic p.

In the case of plane curves, singular points are isolated, and so the recipe for getting $\tilde{X}$ is to blow up whenever you see a singular point. But if you have a higher dimensional variety, then there are many choices, and Hironaka's theorem does not give a nice algorithm.

Example 30.2. Let us look at the rational cubic plane curve that has a double point, namely $V(Y^2 - X^2(X + 1))$. We can blow-up at the double point and get a resolution.

But we might do something like this.

Definition 30.3. Given a k-dimensional $X \subset \mathbb{P}^n$, we define the **Nash blow-up** of X as the graph of $\mathscr{G}$ projecting to X.

Taking the Nash blow-ups resolve singularities to some extent because, for instance, if there is a curve with double points, then the Gauss map can distinguish them.

Question. *Do iterated Nash blow-ups always resolve singularities?*

If true, this will give an algorithm for finding the resolution.

31 April 15, 2016[4]

31.1 Bertini's theorem

Theorem 31.1 (Bertini). *If $X \subset \mathbb{P}^n$ is a smooth quasi-projective variety, then for a general hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$, the intersection $X \cap H$ is smooth.*

Before proving the theroem, let me list some corollaries.

Corollary 31.2. *If $X \subset \mathbb{P}^n$ is quasi-projective, then for a general hyperplane,*

$$(X \cap H)_{\mathrm{sing}} \subset X_{\mathrm{sing}} \cap H.$$

Proof. Replace X by X_{sm} and apply theorem 31.1. $\square$

Corollary 31.3. *If $X \subset \mathbb{P}^n$ is smooth and quasi-projective, and $Y \subset \mathbb{P}^n$ is a general hypersurface of degree d, then $X \cap Y$ is smooth.*

Proof. We map X into a larger projective space by the Veronese map as

$$X \hookrightarrow \mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^N,$$

where $N = \binom{n+d}{n} - 1$. Then hyperplane sections pull back to intersections with degree d hypersurfaces. $\square$

We now prove Bertini's theorem.

Proof of theroem 31.1. Let $X \subset \mathbb{P}^n$ be of dimension k. We see that $H \cap X$ is singular if and only if $\mathbb{T}_p X \subset H$ for some $p \in X$.

Set up the incidence correspondence

$$\Phi = \{(p, H) \subset \mathbb{P}^n \times \mathbb{P}^{*n} : p \in X, \mathbb{T}_p X \subset H\}.$$

Then the fibers of $\pi_1 : \Phi \to X$ are isomorphic to $\mathbb{P}^{n-k-1}$, and hence Φ has dimension $n-1$. This means that the image of the projection $\pi_2 : \Phi \to \mathbb{P}^{*n}$ has dimension at most $n-1$. This shows that a general hyperplane has smooth intersecton. $\square$

31.2 The Lefschetz principle

There is another proof of the Bertini theorem using a geometric property of varieties over $\mathbb{C}$. If our base field is $\mathbb{C}$, then saying that something is a smoth variety is the same as saying that it is a smooth manifold.

We use the folloing result, which is the weakest form of Sard's theorem.

Theorem 31.4 (Sard's theorem). *Let $f : M \to N$ be a differentible map between smooth manifolds. Then there is a $p \in N$ such that $f^{-1}(p) \subset M$ is a smooth submanifold.*

[4]Again, from Aaron Landesman's notes.

We shall not prove this here, but rather use it to obtain another proof of Bertini's theorem over $\mathbb{C}$.

Alternative proof of theroem 31.1 over $\mathbb{C}$. Consider the set

$$\Omega = \{(H, p) \in \mathbb{P}^{*n} \times \mathbb{P}^n : p \in H \cap X\}.$$

The fibers of $\Omega \to X$ are $\mathbb{P}^{n-1}$, and it follows that Ω is smooth. The fibers of $\Omega \to \mathbb{P}^{*n}$ are the hyperplane sections, and by Sard's theorem, there is at least one p such that the fiber over p is smooth. It follows that a general fiber is smooth. □

Now there is something called the Lefschetz principle.

Lefschetz principle. *Any theorem that is true over* $\mathbb{C}$ *is true over any algebraically closed field of characteristic* 0.

There is a question of what "any theorem" means. This is thought as an assertion on the existence or nonexistence of a solution to a collection of polynomail equations. We are going to use this to prove Bertini's theorem in the general case.

Theorem 31.5 (Proof of theorem 31.1)**.** *Let K be an arbitrary algebraically closed field of characteristic* 0. *By definition, this is defined as the common zero locus of a finite number of polynomials, which will have finitely many coefficients. That is $X = V(\{f_\alpha\})$ where each f_α has coefficeints $\{c_{\alpha,I}\} \subset K$. We know that the field $L = \mathbb{Q}(c_{\alpha,I})$ is embedded in $\mathbb{C}$ and hence the general Bertini's theorem follows from the $\mathbb{C}$ case.*

The first proof is still preferable because it directly shows that the theorem holds over arbitrary fields, but this is a good illustration of the Lefschetz principle.

31.3 Degree of a variety

Let $X = V(f)$ be a hypersurface. Then we define its degree as $\deg X = \deg f$. On the other hand, if X is a set of points, i.e., 0-dimensional, then we define its degree just as the number of points. These two definitions agree at least in $\mathbb{P}^n$, when X is a set of points on a line.

We now want to extend this notion. Let X be k-dimensional and irreducible, and let Γ, Λ, and Ω be general $\mathbb{P}^{n-k-2}$, $\mathbb{P}^{n-k-1}$, and $\mathbb{P}^{n-k}$ embedded in $\mathbb{P}^n$. Then in general $\Omega \cap X$ will be a finite number of points, whereas $\Lambda \cap X = \emptyset$. We can define the degree as the size of $\Omega \cap X$, or we can define it as the degree of

$$\pi_\Gamma|_X : X \to X_0 \subset \mathbb{P}^{k+1},$$

becuase X_0 will be a hypersurface in $\mathbb{P}^{k+1}$.

Definition 31.6. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k.

(1) Let Γ be a general $n-k-2$ plane. We define the **degree** of X to be

$$\deg(\pi_\Gamma(X) \subset \mathbb{P}^{k+1}).$$

(2) Let Ω be a general $n-k$ plane. We define the **degree** of X to be the number of points in $X \cap \Omega$.

(3) Let $\pi_\Lambda : X \to \mathbb{P}^k$ be the projection map, where Λ is a general $n-k-1$ plane. This map induces a map between function fields

$$\pi_\Lambda^* : K(\mathbb{P}^k \to K(X).$$

We define the **degree** as the degree of the field extension $[K(X) : K(\mathbb{P}^k)]$.

We can even give na interpretation of the degree using the Hilbert polynomial, when X is irreducible.

Lemma 31.7. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k. Let $p_X(m) = a_1 m^k + \dots$ be its Hilbert polynomial. Then* $\deg X = k!a_1$.

Proof. Let $\dim X = k$, and $H_1, \dots, H_k$ be k general hyperplanes. If we define

$$X_i = X \cap H_1 \cap \cdots \cap H_i,$$

then we immediately have

$$p_{X_i}(m) = p_{X_{i-1}}(m) - p_{X_{i-1}}(m-1).$$

Because $X_k = X \cap \Omega$ has d points, we see that $p_{X_k} = d$. Hence the leading coefficient of X must be $d/k!$. □

Example 31.8. Recall that the twisted cubic has Hilbert polynomial $h_X(m) = 3m + 1$. Hence it has degree 3.

32 April 18, 2016[5]

Recall that the degree of an irreducible projective variety is defined as

(1) a degree of the image of X under the projection map $\pi_\Gamma : X \to \overline{X} \subset \mathbb{P}^{k-1}$, where $\Gamma \cong \mathbb{P}^{n-k-2}$ is a general space,

(2) a size of a general fiber of the projection map $\pi_\Lambda : X \to \mathbb{P}^k$, where $\Lambda \cong \mathbb{P}^{n-k-1}$ is a general subspace,

(3) the size $\#(\Omega \cap X)$, where $\Omega \cong \mathbb{P}^{n-k}$ is a general subspace,

(4) the leading coefficient of the Hilbert polynomial p_X times $k!$.

In general, if $X = \bigcup_i X_i$, where X_i are the irreducible components, then

$$\deg X = \sum_{\dim X_i = k} \deg X_i.$$

Also, recall the following result, which is in chapter 7 of our book. The proof is similar to the proof of theorem 25.1.

Proposition 32.1. *Let $f : X \to Y$ be a finite surjective map of irreducible projective varieties. Then there exists an open $U \subset Y$ on which the size of $f^{-1}(q)$ is equal to* $\deg[K(X) : K(Y)]$.

Idea of proof. Consider the case in which $Y = \mathbb{A}^k$ and X is affine. The map $f : X \to Y$ can be factored into projections. We focus on the last projection $X \subset \mathbb{A}^{k+1} \to Y = \mathbb{A}^k$. In this case, X will be a hypersurface, and thus we can write $X = V(g)$ where $g = \sum g_\alpha(x_1, \ldots, x_k) x_{k+1}^\alpha$. We can construct a proper open set on which the number of roots in the last variable is equal to g. $\square$

32.1 Bezout's theorem I

Definition 32.2. Let $X, Y \subset \mathbb{P}^n$ be subvarieties of dimension k and l. Let $p \in X \cap Y$. We say that X and Y intersect **transversely** at p if

$$\dim(\mathbb{T}_p X \cap \mathbb{T}_q Y) = k + l - n.$$

If X and Y intersect transversely at each point in $X \cap Y$, then we say that X and Y intersect **transversely**, or simply that $X \cap Y$ is **transverse**.

It can be shown that if X and Y intersect p transversely, then X and Y must both be at least smooth at p.

Definition 32.3. We say that X and Y intersect **generically transversely** if $X \cap Y \neq \emptyset$ and $X \cap Y$ is transverse at a general point of any irreducible component of $X \cap Y$. If X and Y intersect generically transversely, we say that $X \cap Y$ is **generically transverse**.

[5] Again, shamefully from Aaron Landesman's notes.

Note that X and Y can intersect generically transversely only when $k+l \geq n$.

Theorem 32.4 (Bezout's theorem)**.** *Say $X, Y \subset \mathbb{P}^n$ are irreducible projective varieties of dimension k, l with $k + l \geq n$. If $X \cap Y$ is generically transverse, then*

$$\deg X \cap Y = \deg X \cdot \deg Y.$$

We will prove this theorem later in class.

32.2 Degree of the Veronese

Example 32.5. Let $X \subset \mathbb{P}^n$ be any k-dimensional subvariety and $Y \cong \mathbb{P}^{n-k} \subset \mathbb{P}^n$ be a linear subspace. Then Bezout's theroem tells us that

$$\#(X \cap \mathbb{P}^{n-k}) = \deg X$$

whenever $X \cap \mathbb{P}^{n-k}$ is transverse. In particular, for a curve $C \subset \mathbb{P}^2$ of degree d, the intersection of C and a line fails to be d points only when the line is tangent to C or intersects C at a singular point.

Example 32.6. Consider the Veronese variety given as the image of $\nu_d : \mathbb{P}^n \to X \subset \mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$. What is its degree $\deg X$? There are two ways to compute this.

The first way is to use Bezout's theorem. Let $\Lambda \cong \mathbb{P}^{N-n} \subset \mathbb{P}^N$ be a general subspace. Then we have

$$\begin{aligned} \deg X &= \#(X \cap \Lambda) = \#(X \cap H_1 \cap \cdots \cap H_n) \\ &= (\nu^{-1}(H_1) \cap \cdots \cap \nu^{-1}(H_n)). \end{aligned}$$

Because $\nu^{-1}(H_i)$ is a general hypersurface, and they generally are generically transverse to each other, (a stronger version of Bertini's theorem) we see by Bezout's theorem that the number of intersections is d^n.

Alternatively, we can use Hilbert polynomials. Because the map is surjective,

$$\{\text{hom. poly. of deg. } m \text{ in } \mathbb{P}^n\} \longrightarrow \{\text{hom. poly. of deg. } dm \text{ in } \mathbb{P}^n\},$$

the Hilbert polynomial must be

$$h_X(m) = \binom{n + dm}{n} = \frac{d^n}{n!} m^n + \dots .$$

Hence the degree is d^n.

Example 32.7. Like in the previous example, consider the Veronese map $\nu_d : \mathbb{P}^n \to X \subset \mathbb{P}^N$, and this time, look at the image of an irreducible $Z \subset \mathbb{P}^n$ under the map. That is,

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W = \nu(Z) \end{array}$$

What is the degree of W?

Again, there are two ways. First, we see that

$$\deg W = \#(W \cap H_1 \cdots \cap H_k) = \#(Z \cap \nu^{-1}(H_1) \cap \cdots \cap \nu^{-1}(H_k)) = \deg W \cdot d^k.$$

We can also look at the Hilbert functions. In fact, at the level of Hilbert functions, we have the identity $h_W(m) = h_Z(md)$. It follows that $\deg W = d^k \cdot \deg Z$.

33 April 20, 2016

33.1 Degree of the Segre

Consider the map $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ given by

$$([Z_0, \ldots, Z_n], [W_0, \ldots, W_m]) \mapsto [\ldots, Z_i W_j, \ldots].$$

The image is the Segre variety $\Sigma_{m,n}$. What is the Hilbert function of Σ? We have a exact sequence

$$0 \longrightarrow I(\Sigma)_k \longrightarrow \begin{Bmatrix} \text{hom. poly. of deg.} \\ k \text{ on } \mathbb{P}^N \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{bihom. poly. of bideg.} \\ (k,k) \text{ on } \mathbb{P}^n \times \mathbb{P}^m \end{Bmatrix} \longrightarrow 0$$

and thus

$$h_\Sigma(k) = \binom{n+k}{n}\binom{m+k}{m} = \frac{k^{m+n}}{n!m!} + O(k^{m+n-1}).$$

Therefore the degree of the Segre is $\dim \Sigma = \binom{n+m}{n}$.

There is another method of computing the degree in terms of the cohomology ring, and it is in the text, but it is beyond the scope of this course. Moreover, if you want to compute the degree of the image of some variety, then you need a more finer invariant, which is actually the cohomology ring.

33.2 Cones, projections, and joins

For a $X^k \subset \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ and a point $p \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$, we have defined the cone

$$\overline{pX} = \bigcup_{q \in X} \overline{pq}.$$

If we denote by $\Gamma = \mathbb{P}^{n-k} \subset \mathbb{P}^{n+1}$ a general $(n-k)$-plane, we can compute

$$\deg(\overline{pX}) = \#(\overline{pX} \cap \Gamma) = \#(X \cap \pi_p \Gamma) = \deg X.$$

Similarly we can project a variety $X^k \subset \mathbb{P}^{n+1}$ from a point $p \notin X$ and look at the projection $\pi_p(X)^k \subset \mathbb{P}^n$. Then

$$\deg(\pi_p(X)) = \#(\Lambda \cap (\pi_p X)) = \#(p\overline{\Lambda} \cap X) = \deg(X).$$

But the equality between the second and third term hold *only if* π is generically one-to-one. If π is generically multiple-to-one, then

$$\deg(\pi_p(X)) = \deg(X)/(\text{degree of the projection map}).$$

Let $X^k, Y^l \hookrightarrow \mathbb{P}^n$ and assume that $X \cap Y = \emptyset$. The join of X and Y is defined by

$$J = J(X,Y) = \bigcup_{x \in X, y \in Y} \overline{xy}.$$

There is a regular map $X \times Y \to \mathscr{J} \subset \mathbb{G}(1,n)$ given by $(x,y) \mapsto \overline{xy}$, and the join is simply

$$J = \bigcup_{\ell \in \mathscr{J}} \ell \subset \mathbb{P}^n.$$

Because $\dim \mathscr{J} = k + \ell$, *if* a general point of J lies only on finite many lines, *then* $\dim J = k + \ell$. That is, we need an additional assumption. Now that we know what the dimension is, what is the degree $\deg J$?

Before answering this, let us consider a simpler situation. Consider $\mathbb{P}^n = \mathbb{P}V$ and look at

$$\begin{tikzcd} \Lambda_1 = \mathbb{P}^n \arrow[r, hook] & \mathbb{P}(V \oplus V) \cong \mathbb{P}^{2n+1} \\ \Lambda_2 = \mathbb{P}^n \arrow[ur, hook] & \end{tikzcd}$$

then these two linear subspaces are far apart. Now we can embed X and Y in Λ_1 and Λ_2, and get $\tilde{X} = \mathrm{im}(X) \hookrightarrow \Lambda_1$ and $\tilde{Y} = \mathrm{im}(Y) \hookrightarrow \Lambda_2$. Then these two $\tilde{X}$ and $\tilde{Y}$ will also be far apart. But how do we relate $\tilde{X}$ and $\tilde{Y}$ to the original ones? Define $\Gamma = \mathbb{P}(\Delta) \cong \mathbb{P}^n$ be another subspace disjoint from both Λ_1 and Λ_2. Then the projection map π_Γ is defined outside Γ and in particular on X and Y. Then we can project stuff from Λ and get something in $\mathbb{P}^n$ again.

So let us set $\tilde{J} = J(\tilde{X}, \tilde{Y}) \subset \mathbb{P}^{2n+1}$. The homogeneous polynomial ring can be described as

$$S(\mathbb{P}^{2n+1}) = K[Z_0, \dots, Z_{2n+1}] = K[Z_0, \dots, Z_n] \otimes_K K[Z_{n+1}, \dots, Z_{2n+1}].$$

This ring is bi-graded and can be decomposed into

$$S_{i,j} = \{\text{poly. } F(Z_0, \dots, Z_{2n+1}) \text{ bihom. bideg. } (i,j) \text{ in } (Z_0, \dots, Z_n), (Z_{n+1}, \dots, Z_{2n+1})\}.$$

For any $F \in K[Z_0, \dots, Z_{2n+1}]$ of degree m, we can write

$$F = G_0 + \cdots + G_m,$$

where G_i is bihomogeneous of bidegree $(i, m-i)$, and $F \in I(\tilde{J})$ if and only if $G_i \ni I(\tilde{J})$ for all i. It follows that

$$A_m(\tilde{J}) = \oplus_{i+j=m} A_i(\tilde{X}) \otimes A_j(\tilde{Y})$$

after you think about it. So we can compute a Hilbert function as

$$h_{\tilde{J}}(m) = \sum_{i+j=m} h_X(i) \cdot h_Y(y).$$

I don't have enough time, and I'll just write down the binomial identity we will use.

$$\sum_{i+j=m} \binom{a+i}{a} \binom{b+j}{b} = \binom{a+b+m+1}{m}.$$

34 April 22, 2016

There are only three classes left, and I want to wrap up Bezout's theorem. Because $\binom{a+i}{i} = (-1)^i\binom{-a-1}{i}$,

$$\begin{aligned}\sum_{i+j=m}\binom{a+i}{i}\binom{b+j}{j} &= \sum_{i+j=m}\binom{-a-1}{i}\binom{-b-1}{j}(-1)^{i+j} \\ &= (-1)^m\binom{-a-b-2}{m} = \binom{a+b+1+m}{m}.\end{aligned}$$

34.1 Join of varieties and Bezout's theorem

Going back to joins, let $X, Y \subset \mathbb{P}^n$ be of dimension k, l and degree d, e respectively. The trick is to embed $\mathbb{P}^n$ in $\mathbb{P}^{2n+1}$ in two different ways. Let $\Lambda_1, \Lambda_2 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be two disjoint subspaces, namely

$$\Lambda_1 = V(Z_{n+1}, \dots, Z_{2n+1}), \qquad \Lambda_2 = V(Z_0, \dots, Z_n).$$

Also, consider the diagonal $\Gamma = V(Z_0 - Z_{n+1}, \dots, Z_n - Z_{2n+1})$. Then there is a rational projection map $\pi_\Gamma : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^n$ that will send $\Lambda_1, \Lambda_2 \to \mathbb{P}^n$, $\tilde{X}, \tilde{Y} \to X, Y$, and $\tilde{J} = J(\tilde{X}, \tilde{Y}) \to J = J(X, Y)$.

We can decompose the ring of homogeneous polynomial as

$$S(\mathbb{P}^{2n+1})_m = \bigoplus_{i+j=m} S(\Lambda_1)_i \otimes S(\Lambda_2)_j.$$

Then we see that a polynomial vanishes if and only if each component vanishes on $\tilde{X}$ and $\tilde{Y}$ respectively, i.e.,

$$S(\tilde{J})_m = \bigoplus_{i+j=m} \big(S(\tilde{X})_i \otimes S(\tilde{Y})_j\big) \cong \bigoplus_{i+j=m} \big(S(X)_i \otimes S(Y)_j\big).$$

Then the Hilbert function of $\tilde{J}$ can be expressed in terms of the Hilbert functions of X and Y, namely

$$\begin{aligned}h_{\tilde{J}}(m) &= \sum_{i+j=m} h_X(i)h_Y(j) = \sum_{i+j=m}\Big(d\frac{i^k}{k!} + O(i^{k-1})\Big)\Big(e\frac{j^l}{l!} + O(j^{l-1})\Big) \\ &= de\binom{m+k+l+1}{m} + O(m^{k+l}).\end{aligned}$$

Hence we conclude that $\deg \tilde{J} = de$.

So if $\pi_\Gamma : \tilde{J} \to J$ is generically 1-1, i.e., if a general $p \in J$ lies on a unique line, then $\deg J = de$. If this condition is not met, then the degree can be different and even the dimension can be different.

Now observe that this construction can be carried out even if X and Y do intersect. Then we see that $\tilde{J} \cap \Gamma$ is set-theoretically equal to $X \cap Y$. It can also be checked that the generic transversality $X \pitchfork_g Y$ can be translated to $\tilde{J} \pitchfork_g \Gamma$. Because the Γ is then a linear space generically transverse to $\tilde{J}$, we have

$$\deg(X \cap Y) = \deg(\tilde{J} \cap \Gamma) = \deg \tilde{J} = \deg(X)\deg(Y).$$

So we get Bezout's theorem.

34.2 Bezout's theorem II

Example 34.1. Let $C, D \subset \mathbb{P}^2$ be plane curves, and let $C = V(F), D = V(G)$, where F and G have degree d and e. Bezout's theorem says if $C \pitchfork D$, then $\#(C \cap D) = de$ is the number of common solutions to $F = G = 0$. This can be considered as a 2-variable version of the fundamental theorem of algebra.

There is another proof of Bezout's theorem in the plane curve case focusing on that perspective. Let us write

$$F(X,Y,Z) = \sum A_i(X,Y)Z^i, \qquad G(X,Y,Z) = \sum B_j(X,Y)Z^j.$$

These polynomials have a common zero if and only if $R(X,Y) = 0$, where R is the resultant. It is not hard to check that the resultant is homogeneous in X, Y and is of degree de. What about transversality? It is not trivial, but also not impossible to check that $C \pitchfork D$ at p if and only if R has a simple zero at p. This approaches even gives a notion of multiplicity. If we denote by $m_p(C, D)$ the multiplicity at p, then we can say something like $m_p(C, D) = 1$ if and only if $C \pitchfork D$ at p. Also, we would have $\sum m_p(C, D) = de$ whenever $\#(C \cap D) < \infty$.

This motivates the strong form of Bezout's theorem.

Theorem 34.2. *We can assign, to any component Z of a proper intersection $X \cap Y \subset \mathbb{P}^n$ such that $\dim(X \cap Y) = \dim X + \dim Y - n$, a multiplicity $m_Z(X, Y)$ such that*

$$\sum m_Z(X,Y) \deg Z = \deg X \deg Y.$$

People struggled to find this right definition of multiplicity literally for centuries. The right answer came in the 1950s, defined by Serre as the rank of the Tor functor. This is the reason I didn't really tell you.

35 April 25, 2016

Today we are going to go back to what I mentioned in the beginning of this course.

35.1 Real plane curves and Harnack's theorem

Question. *Given a polynomial $f(x, y) \in \mathbb{R}[x, y]$ of degree d, describe the zero locus in $\mathbb{R}$. In particular, how many connected components*[6] *are there?*

In the case $d = 2$, there are three types of curve aside from the emptyset: there is the hyperbola, the parabola, and the ellipse. In the projective plane $\mathbb{P}^2_{\mathbb{R}}$, they are all the same thing with the line of infinity passing, tangent, and not meeting the curve. For $d = 3$, things get more complicated, and there is the elliptic curve with two components and the elliptic curve with one component.

In any case, things are complicated over $\mathbb{R}$ and so we pass from $\mathbb{R}^2$ to $\mathbb{P}^2_{\mathbb{C}}$ to answer our question as we did at the beginning of our course. If we homogenize the polynomial and write it as $F(X, Y, Z) = 0$, then the zero locus will be a compact, connected, oriented 2-manifold. This will be topologically described by its genus g.

Now what is g? The key observation is that this g only depends on the degree d of F. Why is this? Consider the universal family

$$\mathscr{C}_U \subset \mathscr{C} = \{(C, p) : p \in C\} \subset \mathbb{P}^N \times \mathbb{P}^2,$$

where $\mathscr{C}_U$ is the universal family over the open subset $U \subset \mathbb{P}^N$ of smooth curves. Because the complement of U has real codimension 2 (the dimension of $\mathbb{C}$), the open set U must be connected. Now $\mathscr{C}_U$ is a fiber bundle over U, and because each fiber is a smooth manifold, we see that each fiber is homeomorphic.

So what is g in terms of d? For $d = 2$, the answer is $g = 0$, and for $d = 3$, the answer is $g = 1$, as we saw in the case of the integration of special functions. To exactly compute g, we start with a union of d "lines" in $\mathbb{P}^2_{\mathbb{C}}$. Then we add an ϵ factor in to make in smooth. When we add ϵ, what happens is we smooth out all the transverse inversections of each pairs of lines (spehres) and replace it by a cylinder. This is in some sense doing surgery on the spheres.

Let $X_0 = V(\prod \ell_i)$ and let $X_1 = V(\prod \ell_i + \epsilon)$. Because X_0 is the identification of $\binom{d}{2}$ pairs of points from d unions of spheres, we have

$$\chi(X_0) = 2d - \binom{d}{2}.$$

Then we replace the disjoint unions of discs by cylinders, and each process decreases the Euler characteristic by 1, because a contractible space has Euler characteristic 1 and the cylinder has 0. So

$$2 - 2g = \chi(X_1) = 2d - \binom{d}{2} - \binom{d}{2} = -d(d-3), \qquad g = \binom{d-1}{2}.$$

[6] 19th century geometers called this "ovals".

Going back to the question of real plane curves, how many connected components are there in a real curve $C_{\mathbb{R}}$? We now the case of the complex curve, and there is a continuous involution $\tau : C_{\mathbb{C}} \to C_{\mathbb{C}}$ given by

$$[X, Y, Z] \mapsto [\bar{X}, \bar{Y}, \bar{Z}],$$

that is orientation-reversing. The fixed point set will be $C_{\mathbb{R}}$.

The question then becomes what's the quotient $C_{\mathbb{C}}/\tau$? This will be a 2-manifold with boundary $C_{\mathbb{R}}$, and if $C_{\mathbb{R}}$ has δ connected components, then we can complete $C_{\mathbb{C}}/\tau$ to a compact manifold $\overline{C}$ by adding δ discs. We know that $\chi(C_{\mathbb{C}}) = -d(d-3)$ and thus $\chi(C_{\mathbb{C}} \setminus C_{\mathbb{R}}) = -d(d-3)$. When we quotient by the the involution, we get

$$\chi(C_{\mathbb{C}} \setminus C_{\mathbb{R}}/\tau) = -\frac{d(d-3)}{2}$$

and thus

$$2 \geq \chi(\overline{C}) = \frac{-d(d-3)}{2} + \delta.$$

It follows that

$$\delta \leq \binom{d-1}{2} + 1.$$

This is known as Harnack's theorem.

36 April 27, 2016

By the way, as everyone comes, there is something I would like to add about real plane curves. We showed that for a plane curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ of degree d, the number of connected components is at most $\binom{d-1}{2}+1$, and this is sharp. There was a whole problem of constructing such varieties, and it is challenging. For example, if $d=4$, then you take two conics that meet at four points, and add a constant. If you add the constant in the right sign, you will get 4 components. Also, if we look at curves in $\mathbb{A}^2_{\mathbb{R}}$, then we can take out the line at infinity.

36.1 Parameter spaces

We wish to construct parameter spaces for varieties in $\mathbb{P}$. If we have this, it will be easy to get stuff like "parameter spaces for regular maps" from here. So, for now, our goal is given a class $\mathscr{C} = \{X \subset \mathbb{P}^n\}$ of projective varieties, a bijection between $\mathscr{C}$ and the points of a variety $\mathscr{H}$. We also want a incidence correspondence

$$\Phi = \{([X], p) : p \in X\} \subset \mathscr{H} \times \mathbb{P}^n.$$

Example 36.1. Let $\mathscr{C} = \{\text{hypersurfaces of degree } d \text{ in } \mathbb{P}^n\}$. Then we have an inclusion

$$\mathscr{C} \hookrightarrow U \subset \mathbb{P}^N = \{\text{hom. poly. of deg. } d \text{ on } \mathbb{P}^n\}/\text{scalar}.$$

But what about other classes of varieties? For instance let us look at the set of twisted cubics.

Example 36.2. Let $\mathscr{C} = \{\text{twisted cubic } C \subset \mathbb{P}^3\}$. You can try to consider it as the image of a map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. But then, we would have to take the quotient by $PGL_2(K)$ (which is the automorphism group of $\mathbb{P}^1$), and taking quotient by algebraic groups of high dimension is a tricky business that we don't want to involve in.

So alternatively, we try to reduce the class of twisted cubics to some class of hypersurfaces. Let $C \subset \mathbb{P}^3$ be a twisted cubic and consider the incidence variety

$$\Phi = \{(l,p) : p \in l \cap C\} \subset \mathbb{G}(1,3) \times C$$

and projections

$$\begin{tikzcd} & \Phi \arrow[dl] \arrow[dr] & \\ \mathbb{G}(1,3) \supset \Sigma_C & & C. \end{tikzcd}$$

More generally, we can consider

$$\Psi = \{(H_1, H_2, p) : p \in H_1 \cap H_2 \cap C\} \subset \mathbb{P}^{3*} \times \mathbb{P}^{3*} \times C$$

and projection maps

$$\begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ \mathbb{P}^{3*} \times \mathbb{P}^{3*} \supset \Gamma_C & & C. \end{tikzcd}$$

Then Γ_C is a hypersurface in $\mathbb{P}^{3*} \times \mathbb{P}^{3*}$, and it can be checked that Γ_C is the zero locus of a bihomogeneous polynomial of bidegree $(3,3)$ on $\mathbb{P}^{3*} \times \mathbb{P}^{3*}$. Then we can associate to each C a polynomial

$$C \mapsto [F] \in \mathbb{P}(\text{bihom. poly. of bideg. } (3,3) \text{ on } \mathbb{P}^{3*} \times \mathbb{P}^{3*}),$$

where $\Gamma_C = V(F)$. Then we get a injection

$$\{\text{twisted cubics}\} \hookrightarrow \mathbb{P}^{399}.$$

This was Chow's construction of the parameter space, and this was the only way to parameter varieties, before Grothendieck came along.

36.2 Hilbert schemes

Now we have looked at the twisted cubic as the variety cut out by three linearly independent quadratic polynomials. But instead of looking at the basis, we look at the vector space generated by it.

Example 36.3. Again, we look at twisted cubics. Associate to each C a vector space

$$C \mapsto I(C)_2 \subset S(\mathbb{P}^3)_2.$$

Then we immediately get and injection

$$\{\text{twisted cubics}\} \hookrightarrow G(3,10).$$

To see that this is a closed variety, we can look at the incidence correspondence

$$\Phi = \{(\Lambda, p) \; inG(3,10) \times \mathbb{P}^3 : Q(p) = 0 \text{ for all } Q \in \Lambda\} \to G(3,10)$$

The general fiber of the map is eight points, by Bezout's theorem. So we have to consider only points with positive fiber dimension, and this will be a closed variety by our basic theorem.

In general, consider $\mathscr{C} = \{X \subset \mathbb{P}^n \text{ with Hilb. poly. } p\}$. For each $X \in \mathscr{C}$, we associate

$$X \mapsto (I(X)_m \subset S(\mathbb{P}^n)_m) \in G\Big(\binom{m+n}{n} - h_X(m), \binom{m+n}{n}\Big).$$

We need two things. We need to show that given a p, there exists a m_0 such that for any $m \geq m_0$ and a subvariety $X \subset \mathbb{P}^n$ with Hilbert polynomial p, the Hilbert function is $h_X(m) = p_X(m)$. Only after then, we can construct the Hilbert scheme. This is the basic lemma proved by Matsusaka in the 1950s. If you get to learn more about schemes, you will appreciate how well this Hilbert scheme behaves.

Index