Math 230a - Differential geometry

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This course was taught by Tristan Collins. We met on Mondays, Wednesdays, and Fridays from 1:00pm to 2:00pm in Science Center 507. We used the textbook *Differential Geometry: Bundles, Connections, Metrics, and Curvature* by Clifford Taubes. There were 15 students enrolled, and the class was graded based only on problem sets. The course assistance was Robert Martinez.

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Differential geometry is mostly about taking the derivative on spaces that are not affine. When we are on a line, then we can define the derivative as

$$\nabla_e f(x) = \lim_{t\to 0} \frac{f(x+te)-f(x)}{t}.$$

But if we are on a circle, we already run into trouble because we can't add points. For instance if you are doing physics, these problems arise.

The basic objects we are going to study are smooth manifolds. The model smooth manifold of dimension n is $\mathbb{R}^n$, which comes equipped with the coordinates $(x_1,\dots,x_n)$. Any manifold is going to look locally like $\mathbb{R}^n$.

In order to tell you what a smooth manifold is, I need to tell you what a topological manifold is.

1.1 Topological spaces and manifolds

Definition 1.1. Let X be a set, and let τ be a collection of subsets of X. The pair (X,τ) is a **topological space** if

(i) $X\in\tau$ and $\emptyset\in\tau$.

(ii) If $U_\alpha\in\tau$ and $\alpha\in A$ then $\bigcup_{\alpha\in A} U_\alpha\in\tau$.

(iii) If $U_1,\dots,U_N\in\tau$ then $U_1\cap\cdots\cap U_N\in\tau$.

We say that $U\in\tau$ is an **open set**.

A topology can be bizarre, and so we give some conditions.

Definition 1.2. A topological space (X,τ) is **Hausdorff** if, for any $x,y\in X$ with $x\neq y$, there exist open sets $U,V\in\tau$ such that $x\in U$, $y\in V$, and $U\cap V=\emptyset$.

Example 1.3. A stupid example is $\tau=\{\emptyset,X\}$. If $X=\mathbb{R}^n$, then it is not Hausdorff.

Metric spaces are always Hausdorff topological spaces with τ generated by $\{B_\epsilon(p)\}_{\epsilon>0,p\in X}$.

Definition 1.4. Let $\{W_\alpha\}_{\alpha\in A}$ be an open cover of X. A **refinement** of this cover is a cover $\{V_\beta\}_{\beta\in B}$ such that for all $\beta\in B$ there exists an $\alpha\in A$ with $V_\beta\subseteq W_\alpha$.

A cover $\{W_\alpha\}_{\alpha\in A}$ is called **locally finite** if for any $x\in X$ there exists an open $U_x\ni x$ such that

$$\#\{\alpha\in A : W_\alpha\cap U_x\neq\emptyset\} < +\infty.$$

A topological space (X,τ) is called **paracompact** if every open cover has a locally finite refinement.

Definition 1.5. A **topological manifold** of dimension n is a Hausdorff, paracompact topological space such that for any $p \in X$ there exists an open set $U \subseteq X$ containing p and a homeomorphism $\varphi_U : U \to \mathbb{R}^n$.

The pair (U, φ) is called a **local coordinate chart** or a **local coordinate patch**. The set

$$\mathcal{A} = \{(U, \varphi) : U \subset X \text{ is open, } \varphi : U \to \mathbb{R}^n \text{ is a homeomorphism}\}$$

is an atlas if $X = \bigcup_{(U,\varphi)\in\mathcal{A}} U$.

Definition 1.6. M is a smooth manifold of dimension n if it has an atlas such that the transition functions

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are C^∞ homeomorphisms for all $(U, \varphi), (V, \psi) \in \mathcal{A}$.

Example 1.7. The circle S^1 cannot be covered by a single chart because it is not homeomorphic to $\mathbb{R}^1$. But if you remove one point, say the South pole, then $S^1 \setminus \{s\} \cong \mathbb{R}^1$. Doing the same thing on the North pole, you get two charts that cover S^1. You can check that the transition maps are C^∞ maps and this gives S^1 a smooth structure.

Definition 1.8. A local patch (U, φ) is compatible with the atlas $\mathcal{A}$ if for any $(V, \psi) \in \mathcal{A}$, the maps $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are C^∞.

In this case, we get a new atlas $\mathcal{A}' = \mathcal{A} \cup (U, \varphi)$ which is strictly larger than $\mathcal{A}$ unless it was already in $\mathcal{A}$. A smooth manifold is equipped with a maximal atlas. This allows us to choose our favorite cover by local coordinate charts.

1.2 Maps between manifolds

Definition 1.9. A function $f : X \to \mathbb{R}^k$ is **smooth** if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is C^∞ for every patch $(U, \varphi) \in \mathcal{A}$.

Why is this definition independent of a choice of a coordinate system? If I choose (V, ψ) then

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$$

and thus is also in C^∞. Hence it is independent of the choice of (U, φ)!

Definition 1.10. Let M and N be smooth manifolds. A map $h : M \to N$ is C^∞ if the map $\psi \circ h \circ \varphi^{-1}$ is C^∞ for any local coordinates (U, φ) of M and (V, ψ) of N. M and N are **diffeomorphic** if there exists a smooth map $h : M \to N$ with $h^{-1} : N \to M$ smooth.

It me give two key tools to construct examples of smooth manifolds.

Theorem 1.11 (1.1, Taubes, Inverse Function Theorem)**.** *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\psi : U \to \mathbb{R}^m$ be C^∞. Let $p \in U$ and suppose that the differential $\psi_*(p)$ of ψ at p is invertible. Then there exists an open $V \subseteq \mathbb{R}^m$ with $\psi(p) \in V$ and a C^∞ map $\sigma : V \to U$ such that $\sigma \circ \psi(x) = x$ on a small neighborhood of p and $\psi \circ \sigma(x) = x$.*

The next one is the Implicit Function Theorem, but I don't have enough time.

2 September 2, 2016

Last time we introduced the notion of a manifold. So a manifold M is a Hausdorff paracompact topological space such that for each $p \in M$ there is a neighborhood U and a homeomorphism $\varphi : U \to \mathbb{R}^n$. From now on, I am going to assume that all manifolds are connected.

Lemma 2.1. *If M is Hausdorff topological space which is locally Euclidean, i.e., can be covered in coordinate charts, then M is paracompact if and only if it is second countable.*

We call that (M, τ) is **second countable** if there exist open sets $\{U_i\}_{i\in\mathbb{N}}$ such that if $V \in \tau$ then there exists $G \subseteq \mathbb{N}$ such that

$$V = \bigcup_{i\in G} U_i.$$

A standard example is $M = \mathbb{R}^n$. The balls centered at rational points with rational radii form a countable base. This lemma makes it easier to prove something like that the product of two manifolds is a manifold.

2.1 Constructing new manifolds

Theorem 2.2 (Implicit function theorem)**.** *Fix $m \geq n$, and open set $U \subseteq \mathbb{R}^n$, and a C^∞ map $\psi : U \to \mathbb{R}^{m-n}$. Suppose $a \in \mathbb{R}^{m-n}$ is a regular value. Then $\psi^{-1}(a) \subseteq U$ is a smooth manifold with C^∞ structure given by "slice charts", i.e., for every $p \in \psi^{-1}(a)$ there exists a ball $B \subseteq \mathbb{R}^m$ centered at p such that the projection $\pi : B \to \ker(d\psi_p)$ restricts to $\psi^{-1}(a) \cap B$ as a coordinate chart. In addition, there exists a C^∞ map $\varphi : B \to \mathbb{R}^n$ such that $\varphi(B \cap \psi^{-1}(a))$ is a neighborhood in the n-dimensional space $(X_1, \dots, X_n, 0, \dots, 0)$.*

For example, if $\psi : \mathbb{R}^m \to \mathbb{R}$ and a is a regular value, then $\psi^{-1}(a)$ is a smooth manifold covered by charts such that $\psi^{-1}(a)$ looks locally like a hyperplane.

Definition 2.3. A value a is a **regular value** of ψ if $d\psi$ is surjective at all $p \in \psi^{-1}(a)$.

Theorem 2.4 (Sard's theorem)**.** *If $\psi : U \to \mathbb{R}^n$ is a C^∞ map, then the regular values have full measure.*

Example 2.5. Consider the map $f : \mathbb{R}^n \to \mathbb{R}$ with $x \mapsto |x|^2$. The regular values are $\mathbb{R} \setminus \{0\}$ and $f^{-1}(\{r^2\})$ is the sphere of radius r centered at the origin. In fact, for instance S^1 is indeed a smooth manifold, as I told you last time.

2.2 Submanifolds

Definition 2.6. A **submanifold** of $\mathbb{R}^m$ with dimension n is a subset Σ such that for all $p \in \Sigma$, there is an open neighborhood $U_p \subseteq \mathbb{R}^m$ and a C^∞ map $\psi_p : U_p \to \mathbb{R}^{n-m}$ with 0 as a regular value and $\Sigma \cap U_p = \psi^{-1}(0)$.

In other words, for every $p \in \Sigma$ there exists an open neighborhood $U_p \subseteq \mathbb{R}^m$ and coordinates $(x_1, \ldots, x_m)$ such that

$$\Sigma \cap U_p = \{(x_1, \ldots, x_n, 0, \ldots, 0)\}.$$

Later we will have more abstract/intrinsic defintion of what it means to be a submanifold of $\mathbb{R}^m$. But this is not useless because it tells us what a submanifold should locally look like.

Lemma 2.7. *Suppose $n \leq m$, and consider a ball $B \subseteq \mathbb{R}^n$ and an injective C^∞ map $\varphi : B \to \mathbb{R}^m$ such that $d\varphi$ is also injective everywhere. Then there exists an open $W \subseteq B$ with $\overline{W} \subseteq B$ such that $\varphi(W)$ is a smooth submanifold of $\mathbb{R}^m$ and $\varphi : W \to \varphi(W)$ is a diffeomorphism.*

Proof. Fix $p \in \varphi(W)$ and let $z = \varphi^{-1}(p)$. We need to find ψ_p as in the definition.

Since $d\varphi_z$ is injective, the linear subspace

$$K = \operatorname{im} d\varphi_z \cong \ker((d\varphi_z)^T : \mathbb{R}^m \to \mathbb{R}^n)$$

is a space of dimension $m - n$. Define the map $\lambda : W \times K \to \mathbb{R}^m$ given by

$$(x, v) \mapsto (\varphi(x) + v).$$

The map $d\lambda$ is injective and surjective at $(z, 0)$ and so by the inverse function theorem there exists a smooth map $\eta : U_p \to W \times K$ with U_p open, $p \in U_p$ such that $\eta \cdot \lambda = \mathbf{1}$ and $\lambda \cdot \eta = \mathbf{1}$.

Let $\pi : W \times K \to K$ be the natural projection map. Then $\pi \cdot \eta$ satisfies

$$(\pi \cdot \eta)^{-1}(0) = \eta^{-1}(x, 0) = \lambda(x, 0) = \varphi(x)$$

and 0 is a regular value since both $d\eta$ and $d\pi$ are surjective. Therefore $\psi_p = \pi \cdot \eta$ works. $\square$

Consider for example the map

$$(\varphi, \psi) \mapsto ((1 + \rho \cos \varphi) \cos \psi, (1 + \rho \cos \varphi) \sin \psi, \rho \sin \varphi)$$

which parametrizes the torus for $\rho < 1$. So a torus is a submanifold of $\mathbb{R}^2$.

Definition 2.8. For manifold M, a subset $Y \subseteq M$ is a submanifold if for any $p \in Y$, there exist a neighborhood $U \subseteq M$ and coordinates $\varphi : U \to \mathbb{R}^m$ such that $\varphi(U \cap Y)$ is a submanifold of $\mathbb{R}^m$.

If $f : M \to N$ is a C^∞ map and $Y \subseteq M$ is a submanifold then $f \mid_Y$ is also smooth.

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So last time we had some strategies for constructing manifolds, and the definition of a submanifold. Today I want to tell you what the tangent space to the manifold is. I am going to give three definitions and prove that they are all equivalent if there is time.

3.1 The tangent space

Suppose M is a manifold of dimension n, and let $p \in M$.

Definition 3.1. A **curve** in M through p is a C^∞ map $q : (-\epsilon, \epsilon) \to M$ such that $q(0) = p$.

Our goal is to define the tangent vector to q at p. If $M = \mathbb{R}^n$, then we can just define it as

$$\frac{dq}{dt}(0) = \lim_{t \to 0} \frac{q(t) - q(0)}{t}.$$

Here $q(t) - q(0)$ makes sense because $M = \mathbb{R}^n$ is a vector space. So this is *very special*. In general we can't do this, so we need to instead use the fact that M is locally Euclidean.

Definition T1 (Index Shuffling Definition)**.** *Choose a coordinate patch* (U, φ) *with* $p \in U$*. Suppose we have a path* $\varphi(q(t)) : (-\epsilon, \epsilon) \to \mathbb{R}^n$ *with* $\varphi(q(t)) = (x^1(t), \ldots, x^n(t))$*. Define*

$$q'(0) = \frac{d}{dt}|_{t=0}\varphi(q(t)).$$

This is not *intrinsic because it requires a choice of* φ*. If* $\psi = (y^1, \ldots, y^n)$ *is another patch near* p*, then* $\psi(q(t)) = \psi \circ \varphi^{-1} \circ \varphi(q(t))$ *and so*

$$\frac{d\psi(q(t))}{dt} = \sum_{i=1}^{n} \frac{dx^i}{dx^j}(\psi(q(t))\frac{dx^j}{dt}, \qquad \frac{dx^j}{dt} = \frac{d}{dt}[\varphi(q(t))^j].$$

A ***tangent vector*** *at* p *is an equivalence class* $[(V, (U, \varphi)]$*, where* V *is a vector in* $\mathbb{R}^n$ *and* (U, φ) *is patch, and*

$$(V, (U, \varphi)) \sim (W, (\tilde{U}, \psi)) \text{ if } W^j = \frac{\partial y^j}{\partial x^i}(p)V^i.$$

The advantage of this definition is that it is explicit and so good for computations. But it is not great conceptually.

Definition T2 (Equivalence class of curves)**.** *The idea is that if* $q(t)$ *and* $r(t)$ *are curves thorugh* p*, then either* $q'(0) = r'(0)$ *is either true in all coordinate systems or false in all coordinate systems. Define* $q(t) \sim t(t)$ *if there exists a patch* (U, φ) *such that* $\varphi'(r(0)) = \varphi'(q(0))$*. Then a* ***tangent vector*** *at* p *is an equivalence class* $[q(t)]$*.*

This definition is better conceptually because we have hidden all differentiations, but we don't know how to add tangent vectors. We would certainly want the tangent vector to be an vector space.

Before getting into the third definition, let me give some motivational speech. In $\mathbb{R}^n$, let $V \in \mathbb{R}^n$ be a vector and $p \in \mathbb{R}^n$ be a point. Moreover let $f : U \to \mathbb{R}$ with $p \in U$ be a C^1 function. The directional derivative is

$$\vec{V} f(p) = \nabla_{\vec{V}} f(p) = \nabla f(p) \cdot \vec{V}.$$

We can further recover this vector from the operation by pluggin in the coordinate functions.

We are now going to extract the properties of the derivation. $D_{\vec{V}}$ satisfies

(1) $D_{\vec{v}} : \{C^1 \text{ functions defined near } p\} \to \mathbb{R}$.

(2) $D_{\vec{v}}(\alpha f + \beta g) = \alpha D_{\vec{v}} f + \beta D_{\vec{v}} g$ for all $\alpha, \beta \in \mathbb{R}$ (Linear).

(3) $D_{\vec{v}}(fg) = g D_{\vec{v}} f + f D_{\vec{v}} g$ (Leibniz).

The first condition is not rigorous, so we make this rigorous.

Definition 3.2. Define

$$C_p^\infty = \{(f, U) : p \in U \text{ open }, f : U \to \mathbb{R} \text{ is } C^\infty\} / \sim$$

where the equivalence relation is $(f, U) \sim (g, V)$ if there exists an open $W \subseteq U \cap V$ with $p \in W$ such that $f|_W = g|_W$. The elements of C_p^∞ are called **germs**.

One can check that $[(f, U)] + [(g, V)] = [(f + g, U + V)]$ and $\alpha[(f, U)] = [(\alpha f, U)]$ for $\alpha \in \mathbb{R}$ and $[(f, U)] \cdot [(g, V)] = [(fg, U \cap V)]$. This makes C_p^∞ into an associate commutative algebra.

Definition T3 (Derivations)**.** *A* ***tangent vector*** *V at p is an operator $V : C_p^\infty \to \mathbb{R}$ such that*

(1) $V(\alpha f + \beta g) = \alpha V f + \beta V g$.

(2) $V(fg) = f V g + g V f$.

This V is called a ***derivation****.*

The third definition is the most useful. The **tangent space** at p is

$$T_p M = \text{vector space of tanget vectors at } p.$$

Example 3.3. If there is a local coordinates $(x^1, \ldots, x^n)$, then $\partial/\partial x^i|_p$ for $1 \le i \le n$ are derivations defined by

$$\left(\frac{\partial}{\partial x^i}\Big|_p\right) f = \frac{\partial f}{\partial x^i}(p).$$

For this reason, we always denote $\partial/\partial x^i$ the basis for the tangent space.

3.2 Equivalences

Let $p \in M$ and $q : (-\epsilon, \epsilon) \to M$ be a curve through p. Define a derivation

$$q_*(0)f = \frac{d}{dt}\Big|_{t=0} f(q(t)).$$

Theorem 3.4. *(a) $q_*(0)$ is a derivation.*
(b) If q and r are equivalent curves then $q_*(0) = r_*(0)$.
(c) If D is a derivation then there exists a curve q such that $q_*(0) = D$.

Proof. (a) Choose coordinates $\varphi = (x^1, \ldots, x^n)$. Then

$$q_*(0)f = \frac{d}{dt}\Big|_{t=0} f \circ \varphi^{-1}(\varphi(q(t))) \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \frac{dx^i}{dt}(\varphi(q(0))).$$

So $q_*(0)$ is a derivation by the property of the derivative.

(b) If $q \sim r$ then

$$\frac{d}{dt}\Big|_{t=0} \varphi(q(t)) = \frac{d}{dt}\Big|_{t=0} \varphi(r(t)).$$ □

Next time I will prove (c).

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There were three different ways of looking at tangent vectors. There was the index suffling, equivalence classes of curves, and derivations.

Theorem 4.1. *Let $q : (-\epsilon, \epsilon) \to M$ be a curve through $p \in M$.*

(a) $q_*(0)$ is a derivation.

(b) If r is an equivalent curve then $r_*(0) = q_*(0)$.

(c) If D is a derivation at p then there exists a curve q through p such that $q_*(0) = D$.

Last time we proved (a) and (b).

Proof of (c). Fix a coordinate patch (U, φ) near p. This is going to induce coordinates $\varphi = (x^1, \dots, x^n)$ with $\varphi(p) = 0$. Then x_i is a C^∞ function near p and then can define $a^i = D(x^i) \in \mathbb{R}^n$. We are then going to define

$$\tilde{D} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}$$

and guess that $\tilde{D} = D$.

We need to show that $\tilde{D}f = Df$ for all C^∞ map f defined near p. First of all $D(\alpha) = \tilde{D}(\alpha) = 0$ for all $\alpha \in \mathbb{R}$. By linearity, $D(\sum \beta_j x^j) = \tilde{D}(\sum \beta_j x^j)$ for every $\beta_j \in \mathbb{R}$. Note that

$$D(x^k x^l) = x^k(p) + D(x^l) + x^l(p) D(x^k) = 0 = \tilde{D}(x^k x^l).$$

Given a $f \in C^\infty$ defined near p, we can write

$$f = f(0) + \sum_{j=1}^{n} \beta_j x^j + \sum_{k,l=1}^{n} x^k x^l H_{k,l}(x)$$

where $H_{k,l}$ are smooth functions. This is possible by Hadarard's lemma. Since we can write

$$f = f(0) \sum_{j=1}^{n} \beta_j x^j + \sum_{l=1}^{n} x^l \tilde{H}_l(x)$$

where $\tilde{H}_l(0) + \beta_l = (\partial f / \partial x^l)(0)$. If $\beta_l = (\partial f / \partial x^l)(0)$ then $\tilde{H}_l(0) = 0$. Apply Hadamard's lemma again we get the equation.

So then because the quadratic terms go to zero,

$$D(f) = D(\sum \beta_j x^j) = \tilde{D}(\sum \beta_j x^j) = \tilde{D}(f)$$

for every $f \in C^\infty$ defined near p. Now $\tilde{D} = \sum a_j \partial / \partial x^j$ so $\tilde{D} = q_*(0)$ where $q(t) = \varphi^{-1}(a_1 t, \dots, a_n t)$. $\square$

4.1 The differential

Suppose we have a map $h : M \to N$. Then we can **push-forward** tangent vectors from M to N.

One way is to do it using local coordinates. Suppose we have coordinates $(x^1, \dots, x^m)$ near p and $(y^1, \dots, y^n)$ near $h(p)$. Then $\psi \circ h \circ \varphi^{-1}$ is a map that looks like

$$\psi \circ h \circ \varphi^{-1}(x^1, \dots, x^m) = (h^1(x_1, \dots, x_m), \dots, h^n(x^1, \dots, x^m)).$$

Then we can use are usual definition of the derivative as

$$dh(V) = \begin{bmatrix} \frac{\partial h^1}{\partial x^1}(p) & \cdots & \frac{\partial h^1}{\partial x^m}(p) \\ \vdots & \cdots & \vdots \\ \frac{\partial h^n}{\partial x^1}(p) & \cdots & \frac{\partial h^n}{\partial x^m}(p) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

where $V = \sum_{i=1}^m V^i \partial/\partial x^i$. We can check that $dh_p V \in T_{h(p)}N$, i.e., that this is independent of the choice of coordinates. This is a consequence of the chain rule.

We can use the equivalence of curves definition. If $q(t)$ is a curve through p, then $h(q(t))$ is a curve through $h(p)$. Then we can define

$$dh_p : [q(t)] \mapsto [h(q(t))].$$

To use the derivations definition of the tangent vector, we can pull back functions. If f is a C^∞ is a function defined near $h(p)$, then $f \circ h$ is a C^∞ function defined near p. So we set

$$dh_p D(f) = D(f \circ h).$$

4.2 Immersions and submersions

Definition 4.2. A C^∞ map $h : M \to N$ is

(1) an **immersion** if $dh_p : T_pM \to T_{h(p)}N$ is injective at every $p \in M$. If h is also a homeomorphism onto $h(M)$ with the subspace topology, then h is an **embedding**.

(2) a **submersion** if $dh_p : T_pM \to T_{h(p)}N$ is surjective for every $p \in M$.

Both of these definitions make sense locally, e.g., we can have local embeddings etc.

$N \subseteq M$ is a submanifold if and only if the inclusion map $i : N \hookrightarrow M$ is an embedding. This gives you a useful way to check whether something is a submanifold.

4.3 Vector fields

Definition 4.3. A C^∞ **vector field** on an open set $U \subseteq M$ is a map $U \ni p \mapsto V(p) \in T_pM$ such that for all $p \in U$, there exists local coordinates $(\tilde{U}, (x^1, \ldots, x^n))$ near p such that

$$V = \sum_{i=1}^{n} a^i(x) \frac{\partial}{\partial x^i}$$

with a^i in C^∞.

We note that this makes $TM = \bigcup_{p \in M} T_pM$ into a smooth manifold, because we have described what the smooth functions are.

Suppose we are given two C^∞ vector fields X and Y over U, and a C^∞ function $f : U \to \mathbb{R}$. Then $Yf : U \to \mathbb{R}$ is another smooth function, and then we can define $(XY)(f) = X(Y(f))$. Is XY a vector field? The problem is that the Leibniz rule fails. But the observation is that the error term is symmetric with respect to X and Y. So $XY - YX$ is a vector field.

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A C^∞ vector field on $U \subseteq M$ is a map $U \mapsto TM$ with $p \mapsto V(p) \in T_pM$ such that for every $p \in U$ there exist local coordinates $(V, \psi = (x_1, \dots, x_n))$ with $p \in V$ such that

$$V(p) = a_1(x)\frac{\partial}{\partial x_1} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$$

with all a_i smooth.

Given X, Y smooth fector fields on U with $p \in U$ and $f \in C_p^\infty(M)$, we could try to define $(XY)f = X(Y(f))$ as a new vector field. But this doesn't work because it doesn't satisfy the Leibniz rule. We have

$$(XY)(fg) = X(fY(g) + gY(f)) = fX(Y(g)) + X(f)Y(g) + X(g)Y(f) + gX(Y(f)).$$

There are these junk $X(f)Y(g)$ that is not what we want. So we modify the definition.

Definition 5.1. The **Lie bracket** of C^∞ vector fields X and Y is

$$[X, Y] = XY - YX.$$

It is easy to see that this satisfies the Leibniz rule.

5.1 Flow of a vecor field

Let V be a smooth vector field on M. For each point $p \in M$, we can consider the **flow** of p (or trajectory) under V. Intuitively it is the trajectory when you drop a particle at p. So its velocity at a point is the vector field at that point. This is a map $\delta(t, p) : (-\delta, \delta) \to M$ such that $\gamma(0, p) = p$ and

$$\frac{\partial \varphi}{\partial t}(s, p) = V(\varphi(s, p)).$$

Theorem 5.2. *Let V be a C^∞ vector field on M. For every $p \in M$ there exists an open set $U \subseteq M$ with $p \in U$ and $\delta > 0$ along with a C^∞ map $\varphi(t, x) : (-\delta) \times U \to M$ such that for every $x \in U$ the curve $\varphi(t, x) : (-\delta, \delta) \to M$ is the flow from p along V.*

Note that φ_t is one-to-one by local existence and uniqueness of ODEs. That is, for a fixed t, $\varphi_t(p) = \varphi(t, p)$ is a diffeomorphism onto its image. This is because you can flow by $-t$ to get to where you started. This is called the local flow.

Theorem 5.3. *If X and Y are local C^∞ vector fields on M with $p \in M$. Let φ_t be the local flow of X near p. Then*

$$[X, Y](p) = -\lim_{t \to 0} \frac{\varphi_{t*}Y - Y}{t}(\varphi_t(p)).$$

Proof. Given f defined near p that is C^∞, consider $f(\varphi_t(q)) - f(q) = th(t,q)$ with $h(0,q) = Xf(q)$ by Hadamard's lemma.

Then

$$\varphi_{t*}Y(f(\varphi_t(p))) = Y(f(p)) = tY(h(t,p)) + Yf(p),$$

and

$$\begin{aligned}\lim_{t\to 0}\Big[\frac{\varphi_{t*}Y - Y}{t}\Big]f(\varphi_t(p)) &= \frac{tY(h(t,p)) + Yf(p) - Yf(\varphi_t(p))}{t} \\ &= -\Big[\frac{Y(f(\varphi_t(p))) - Y(f(p))}{t}\Big]Yh(t,p) \\ &= -XY(f)(p) + YXf(p) = -[X,Y]f.\end{aligned}$$

$\square$

5.2 Partitions of unity

Let M be a manifold. A **partition of unity** is a collection of C^∞ functions $f_\alpha : M \to [0,1]$ such that

(1) $\{\operatorname{supp} f_\alpha\}$ is locally finite,

(2) $\sum_{\alpha\in A} f_\alpha \equiv 1$ on M.

We call that a partition of unity $\{f_\alpha\}_{\alpha\in A}$ is **subordinate** to a cover $\{U_j\}_{j\in J}$ if for each $\alpha \in A$ there is some $j \in J$ such that $\operatorname{supp} f_\alpha \subseteq U_j$.

Theorem 5.4 (Existence)**.** *For any open cover $\{U_j\}_{j\in J}$,*

(a) there is a countable partition of unity $\{f_i\}_{i\in\mathbb{N}}$ subordinate to $\{U_j\}$ such that each $\operatorname{supp} f_i$ *is compact.*

(b) there is a partition of unity $\{\tilde f_j\}_{j\in J}$ such that $\operatorname{supp}\tilde f_j \subseteq U_j$ *and at most countably many $\tilde f_j$ are not identically* 0*.*

This is going to be a homework.

5.3 Lie groups

Definition 5.5. A **Lie group** is a manifold G which is a group such that the multiplication and inverse are both C^∞ maps.

Definition 5.6. Let $\mathcal{M}_n(\mathbb{R})$ be the set of $n \times n$ matrices with entries in $\mathbb{R}$. This is just $\mathbb{R}^{n^2}$. The **general linear group** is defined as

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) : A \text{ is invertible with } \det A \neq 0\}.$$

First of all $\mathrm{GL}_n(\mathbb{R})$ is a group with the usual multiplication $A \cdot B = AB$. These are smooth, because everything is dividing a polynomial by a polynomial.

We also define

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A = 1\}.$$

To show that this is a manifold, it suffices to shosw that 1 is a regular value of the det function. We claim that

$$d \det|_M = \det(M)\,\mathrm{Tr}(M^{-1} dM).$$

Lastly, we define

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I\}.$$

6 September 14, 2016

Definition 6.1. A Lie group G is C^∞ manifold equipped with C^∞ maps $m : G \times G \to G$ and $\cdot^{-1} : G \to G$ satisfying the group axioms.

Why would you want to look at Lie groups? The symmetry of objects like the sphere has a structure of a group, and also has a smooth structure. So these objects occur in nature, especially if you are a physicist. Also, if a group has a smooth structure, you can differentiate a map $\phi : G \to H$ to get a map $D\phi : T_{\mathrm{id}}G \to T_{\mathrm{id}}H$. If there is a group action $G \times M \to M$ that is smooth, then any tangent vector of G naturally gives a vector field on M.

Consider

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = 1\}.$$

Is this a manifold? We would need the submersion theorem, which is a version of the implicit function theorem.

Theorem 6.2. *If $f : M \to N$ is a C^∞ submersion, then for every $n \in \mathbb{N}$, $f^{-1}(n) \subseteq M$ is a C^∞ manifold.*

Let's show $\mathrm{O}(n) = \psi^{-1}(a)$ for some a and ψ. We set

$$\psi : \mathrm{GL}(n) \to \mathrm{Sym}_n(\mathbb{R}), \qquad A \mapsto A^T A.$$

This is a submersion, because

$$d\psi_m = (dm)^T m + m^T dm$$

and if you plug in $a = mh/2$ then $d\psi_m(a) = h$.

Definition 6.3. Define $\mathrm{SO}(n) \subseteq \mathrm{O}(n)$ as the set of A such that $\det A = 1$.

You can show that $\mathrm{SO}(n)$ is a connected component of $\mathrm{O}(n)$. So $\mathrm{SO}(n)$ is a Lie group.

6.1 Complex Lie groups

We define

$$M(n; \mathbb{C}) = \{n \times n \text{ matrices over } \mathbb{C}\} \cong \mathbb{R}^{2n^2}.$$

Note that multiplication $M(n;\mathbb{C}) \times M(n;\mathbb{C}) \to M(n;\mathbb{C})$ is C^∞.

Like in the real case, we define

$$\mathrm{GL}(n, \mathbb{C}) = \{M \in M(n;\mathbb{C}) : \det M \neq 0\}.$$

Then $\mathrm{GL}(n;\mathbb{C})$ is a open subset and thus a Lie group.

There is an equivalent definition for the general linear group.

Definition 6.4. An **almost complex structure** is an element $J \in M(2n;\mathbb{R})$ such that $J^2 = -1$.

For example,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

are almost complex structures. More generally, given any basis $e_1, \dots, e_{2n}$ of $\mathbb{R}^{2n}$, the map $J : e_{2k-1} \mapsto e_{2k}, e_{2k} \mapsto -e_{2k-1}$ is an almost complex structure.

Definition 6.5. We define

$$M_J = \{m \in M(2n;\mathbb{R}) : mJ - Jm = 0\}$$

and call it matrices that **intertwine** J. Then we define

$$G_J = \{m \in M_J : \det_{\mathbb{R}} m \neq 0\}.$$

This G_J is Lie group, and in fact, $G_J \cong \mathrm{GL}(n,\mathbb{C})$ as Lie groups.

Theorem 6.6. *There is an isomorphism $G_J \cong \mathrm{GL}(n;\mathbb{C})$.*

Proof. Fix a $\mathbb{R}$-linear isomorphism $f : \mathbb{R}^{2n} \to \mathbb{C}$ such that $if = fJ$. This induces a map

$$M(n;\mathbb{C}) \to M_J, \qquad A \mapsto f^{-1}Af.$$

which is a bijection.

Now to prove the theorem, it suffices to show $\det_{\mathbb{C}} A \neq 0$ if and only if $\det_{\mathbb{R}} fAf^{-1}$. Inside $\mathrm{GL}(n;\mathbb{C})$ there exists an open and dense set of diagonalizable matrices. Such a matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$and then $\det_{\mathbb{C}} A = \lambda_1 \cdots \lambda_n$. Moreover, $f^{-1}Af$ is now diaganolizable over $\mathbb{R}$ and the eigenvalues are $\lambda_1, \dots, \lambda_n, \bar\lambda_1, \dots, \bar\lambda_n$. So $\det \mathbb{R} f^{-1}Af = |\det_{\mathbb{C}} A|^2$ on a open dense set, so the identity holds on all of $M(n;\mathbb{C})$. □

7 September 16, 2016

We are trying to learn about Lie groups, basically by realizing as the inverse image of a regular value.

7.1 More examples of Lie groups

The **special linear group** over $\mathbb{C}$ is defined as

$$\mathrm{SL}(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) : \det_{\mathbb{C}} A = 1\}.$$

We claim that this is a manifold. Consider the map $\det_{\mathbb{C}} : M(n,\mathbb{C}) \to \mathbb{C}$. Clearly $\mathrm{SL}(n,\mathbb{C})$ is the preimage of 1. The derivative is det is given by

$$d\det_M : A \mapsto \det(M)\operatorname{Tr}(M^{-1}A).$$

To show surjectivity, for any c take $A = c/(n\det M)M^{-1}$.

Let's do another example. The **unitary group** is defined as

$$U(n) = \{A \in M(n,\mathbb{C}) : AA^\dagger = I\}$$

where $A^\dagger = \overline{A}^T$. Like in the case of $O(n)$, we use the space of Hermitian matrices as the image of $\psi : M(n,\mathbb{C}) \to \mathrm{Herm}(n)$ given by $A \mapsto AA^\dagger$. The derivative is given by

$$d\psi = (dA)A^\dagger + A(dA)^\dagger.$$

We need to show that this is surjective. Given any Hermitian matrix M, we take $G = M(A^\dagger)^{-1}/2$. Then

$$d\psi(G) = \frac{1}{2}(M(A^\dagger)^{-1}A^\dagger + A(A^{-1}M^\dagger)) = M.$$

We define **special unitary group** as

$$SU(n) = \{A \in U(n) : \det_{\mathbb{C}} A = 1\}.$$

How would we prove this? We could try to show that 1 is the regural value of $\det_{\mathbb{C}} : U(n) \to \mathbb{S}^1$, but this is hard because we don't know the tangent space of $U(n)$. Instead we look at the map

$$\psi : M(n,\mathbb{C}) \to \mathrm{Herm}(n) \times \mathbb{R}, \qquad A \mapsto (AA^\dagger, (i/2)(\det A - \overline{\det A})).$$

Then $SU(n)$ is some union of connected components, because the preimage is the matrices that have determinant ± 1. The derivative is given as

$$d\psi_A = ((dA)A^\dagger + A(dA)^\dagger, \Im(\operatorname{Tr}(A^{-1}dA)\det A)).$$

If we try $\tilde{b} = M(A^\dagger)^{-1}/2$, we get

$$d\psi_A(\tilde{b}) = (M, \Im(\mathrm{Tr}(M) \det A/2)) = (M, 0).$$

This is not quite good, so we add something in the kernel of $\tilde{M} \mapsto \tilde{M}A^T + A\tilde{M}$. Particularly, we will take $b = \tilde{b} + i\tilde{c}A/n$. Then

$$d\psi_A(b) = (M, \Im(\tilde{c} \det A))$$

and so we can choose and appropriate $\tilde{c}$ to finish.

7.2 Vector bundles

Let M be a C^∞ manifold with real dimension n. Then a **vector bundle** of rank m over M is another manifold E with dimension $n + m$ together with

(1) a C^∞ map $\pi : E \to M$, called the projection map,

(2) a C^∞ map $\hat{0} : M \to E$, called the zero section,

(3) a multiplication map $\mu : \mathbb{R} \times E \to E$ satisfying $\pi(\mu(r, v)) = \pi(v)$, $\mu(r, \mu(r', v)) = \mu(rr', v)$, $\mu(1, v) = v$, and $\mu(r, v) = v$ implies $r \neq 1$ or $v \in \mathrm{im}(\hat{0})$, and

(4) for any point $p \in M$, an open set $U \subseteq M$ with $p \in U$ and a map $\lambda_U : \pi^{-1}(U) \to \mathbb{R}^m \times U$ such that $\lambda_U : \pi^{-1}(x) \to \mathbb{R}^n$ is a diffeomorphism for every $x \in U$ and $\lambda_U(\mu(r, v)) = r\lambda_U(v)$ for every $x \in U$.

8 September 19, 2016

Last time we started learning about vector bundles $\pi : E \to M$. There are maps $\lambda_n : \pi^{-1}(U) \to \mathbb{R}^n$ where $U \subseteq M$ are open sets. For a subset $W \subseteq M$, we write $E|_W = \pi^{-1}(W)$. In the case where W is a point x, the set $E|_x = \pi^{-1}x$ is called the **fiber** of E over x. If $U \subseteq M$ is an open subset such that a diffeomorphism $E|_U \cong U \times \mathbb{R}^n$ exists, this $\rho_U = (\pi, \lambda_U)$ is called a **local trivialization**.

8.1 Local charts on a vector bundle

Let $\phi : U \to \mathbb{R}^m$ be local coordinates on $U \subseteq M$ and admits a local trivialization λ_U. Then I get a diffeomorphism $E|_U \to \mathbb{R}^m \times \mathbb{R}^n$ given by $v \mapsto (\psi(\pi(v)), \lambda_U(v))$. This gives a coordinate chart on E.

Proposition 8.1. *The fiber of E over p has a canonical vector spaces structure.*

Proof. Let $p \in M$ be an arbitrary point and let $\varphi_U : E|_U \times \mathbb{R}^n$ be a local trivialization around p. Now define for $v, v' \in \pi^{-1}(p)$,

$$v + v' = \varphi_n^{-1}(p, \lambda_U(v) + \lambda_U(v')).$$

We need to verify that this is independent of the choice of λ_U, i.e., λ and λ' are two such maps then

$$\lambda'(\lambda^{-1}(e + e')) = \lambda'(\lambda^{-1}(e)) + \lambda(\lambda^{-1}(e')).$$

This is saying that the map $\varphi'_U \circ \varphi_U^{-1}$ over p is in a linear map $\mathbb{R}^n \to \mathbb{R}^n$.

Note that for any $r \in \mathbb{R}$ and $e \in \mathbb{R}^n$,

$$\lambda'\lambda^{-1}(r, e) = \lambda'(\mu(r, \lambda^{-1}(e))) = r\lambda'\lambda^{-1}(e).$$

Then by the following lemma, $\lambda'\lambda^{-1}$ is a linear map. $\square$

Lemma 8.2. *If a smooth map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\psi(rv) = r\psi(v)$ for all $r \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then ψ is linear.*

Proof. The derivative of ψ is

$$\psi_*|_{tv}(v) = \frac{d}{dr}\Big|_{r=t} \psi(rv) = \frac{d}{dr}\Big|_{r=t} r\psi(v) = \psi(v).$$

So ψ is equal to its derivative, and it implies that ψ is linear. $\square$

8.2 Cocycle definition

A **vector bundle** of rank n is also given by the following data:

(1) a locally finite open cover $\{U_\alpha\}$ of M,

(2) for any α, β, a C^∞ map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$ called "transition functions" that satisfy

(i) $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$,

(ii) $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. (The **cocyle condition**.)

Given this data, we can define

$$E = \bigcup_{\alpha \in A} U_\alpha \times \mathbb{R}^n / \sim,$$

where $(p, v_\alpha) \sim (p', v_\beta)$ if and only if $p = p'$ and $v_\alpha = g_{\alpha\beta}(p) v_\beta$.

Example 8.3. The **trivial bundle** of rank n is simply $M \times \mathbb{R}^n$.

Example 8.4. Let $^1 = \{(\cos\theta, \sin\theta)\}$, and consider $E \subseteq S^1 \times \mathbb{R}^2$ given by

$$E = \left\{ (\rho, (v_1, v_2)^T) : \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} \right\}.$$

This vector bundle is called the **Möbius band**. This vector bundle is not trivial.

9 September 21, 2016

Let M be a manifold of dimension m and let $\Lambda = \{p_1, \dots, p_k\}$ be points in M. For each p_l, let $\hat{U}_l$ be a coordinate patch with $\varphi_l : \hat{U}_l \to \mathbb{R}^m$. We are going to assume that $\hat{U}_l \cap \hat{U}_j = \emptyset$ for $l \neq j$ and $\varphi_l(p_l) = 0$. Let $U_l = \varphi_l^{-1}(B_1(0))$. So these are just small open balls around $p_1, \dots, p_k$.

For each l choose a C^∞ map $g_l : S^{m-1} \to \mathrm{GL}(n, \mathbb{R})$, and let $U_0 = M - \Lambda$. Then $U_0, U_1, \dots, U_k$ cover M and the only overlaps are $U_0 \cap U_l$ with $1 \leq l \leq k$. Also no three overlap, so the cocycle condition is trivially met. Specify

$$g_{0,l} : U_0 \cap U_l \to \mathrm{GL}(n, \mathbb{R}); \qquad x \mapsto g_l\Big(\frac{\varphi_p(x)}{|\varphi_p(x)|}\Big).$$

This specifies a vector bundle of rank n.

9.1 The tangent bundle

Let M be a manifold of dimension m, and consider a local coordinate patch $(U, \varphi = (x^1, \dots, x^m))$. Then

$$\Big\{ \frac{\partial}{\partial x^i}\Big|_p : 1 \leq i \leq m \Big\}$$

span T_pM for all $p \in U$. So over U, we have a trivial bundle of rank m spanned by $\partial/\partial x^i$.

If $(V, \psi = (y^1, \dots, y^m))$ is another coordinate patch, then $(\partial/\partial y^j)|_p = (\partial x^i/\partial y^j)(\partial/\partial x^i)|_p$ by the chain rule. This gives us a map

$$\Big\{ \frac{\partial x^i}{\partial y^j} \Big\}_{i,j} : U \cap V \to \mathrm{GL}(m\mathbb{R}).$$

So the **tangent bundle** $TM = \bigcup_{p \in M} T_pM$ is a vector bundle, with the obvious projection map $\pi : (v, p) \mapsto p$ for $v \in T_pM$.

Example 9.1. The tangent bundle of $\mathbb{R}^n$ is $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Likewise, for an open set $U \subseteq \mathbb{R}^n$, its tangent bundle is $TU = U \times \mathbb{R}^n$.

Example 9.2. Consider a function $f : \mathbb{R}^2 \supseteq U \to \mathbb{R}$ and let M be given by $\Phi : U \to M$ with $(x, y) \mapsto (x, y, f(x, y))$. The tangent vectors are

$$\frac{\partial \Phi}{\partial x} = \Big(1, 0, \frac{\partial f}{\partial x}\Big), \qquad \frac{\partial \Phi}{\partial y} = \Big(0, 1, \frac{\partial f}{\partial y}\Big).$$

So the tangent bundle is $TM \cong M \times \mathbb{R}^2$.

Example 9.3. What is the tangent bundle of $\mathrm{SL}(n, \mathbb{R})$? The special linear group is defined as

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det A = 1\},$$

and so the tangent bundle can be described as

$$T\,\mathrm{SL}(n,\mathbb{R}) = \{(A,B) : A \in \mathrm{SL}(n,\mathbb{R}), B \in \ker d\psi_A\}.$$

The derivative $d\psi$ is given by $d\psi_A(B) = \mathrm{Tr}(A^{-1}B)$, and thus we can write the whole thing as

$$T\,\mathrm{SL}(n,\mathbb{R}) = \{(A, AC) : A \in \mathrm{SL}(n,\mathbb{R}), \mathrm{Tr}(C) = 0\} \subset M(n,\mathbb{R}) \times M(n,\mathbb{R}).$$

If $V/\mathbb{R}$ is a finite dimensional vector space, we can define the dual $V^* = \mathrm{Hom}(V,\mathbb{R})$. Let $e_1, \dots, e_n$ be a basis of V, where $n = \dim V$. Then we can define $e_1^*, \dots, e_n^*$ as $e_i^*(e_j) = \delta_{ij}$. These $e_1^*, \dots, e_n^*$ is a basis for V^*.

Let $p \in M$ and with local coordinates $(x^1, \dots, x^m)$ near p. We have a basis $\{\partial/\partial x^1, \dots, \partial/\partial x^m\}$ for T_pM.

Given a smooth function f defined near p, we define a linear functional

$$df : T_pM \to \mathbb{R}; \qquad v \mapsto v(f).$$

Then $df_p \in T_p^*M$. If $f = x^j$, then $dx^j(\partial/\partial x^i) = \delta_{ij}$. So $\{dx^1, \dots, dx^m\}$ is a basis T_r^*M for all r near p.

If $\{y^1, \dots, y^m\}$ is another set of coordinates, then by the chain rule,

$$dy^i = \frac{\partial y^i}{\partial x^k} dx^k.$$

Then we see that

$$T^*M = \{(\alpha, p) : \alpha \in T_p^*M\}$$

is a vector bundle because $\partial y^i/\partial x^k : U \cap V \to \mathrm{GL}(n,\mathbb{R})$.

That equation about dy^i is true because

$$dy^i\left(\sum_l \frac{\partial x^l}{\partial y^k}\frac{\partial}{\partial x^l}\right) = dy^i\left(\frac{\partial}{\partial y^k}\right) = \delta_{ik}.$$

Then you can multiply the inverse matrix to get the equation.

10 September 23, 2016

Let M be a manifold and $E \to M$ and $F \to M$ be vector bundle. A **homomorphism** $E \to F$ is a C^∞ map $h : E \to F$ such that for every $p \in M$, $h : E_p \to F_p$ is linear. We will denote

$$\operatorname{Hom}(E, F) = \{\text{homomorphisms } E \to F\}.$$

Proposition 10.1. $\operatorname{Hom}(E, F)$ *is a* C^∞ *vector bundle over* M.

Proof. I will leave this is as an exercise. □

10.1 Sections of a vector bundle

Definition 10.2. A **section** of E is a C^∞ map $s : M \to E$ such that such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[dr, "\mathrm{id}"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Given an open set $U \subseteq M$, we denote

$$\Gamma(U, E) = \{\text{sections of } E|_U\}.$$

$\Gamma(U, E)$ is linear, by the natural pointwise addition $(s_1 + s_2)(p) = s_1(p) + s_2(p)$. In fact, if $f : U \to \mathbb{R}$ is a smooth function, then $(fx)(p) = f(p)s(p)$ defines $fs \in \Gamma(U, E)$. This gives $\Gamma(U, E)$ a $C^\infty(U)$-module structure.

There is a local description of the sections. Let $U_\alpha \subset M$ be an open set such that there is a local trivialization $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}$. For any $s \in \Gamma(U_\alpha, E)$, we can write

$$\varphi_\alpha \circ s : U_\alpha \to (U_\alpha \times \mathbb{R}^n); \quad x \mapsto (x, s_\alpha(x))$$

for a C^∞ map $s_\alpha : U_\alpha \to \mathbb{R}^n$. If we have another local trivialization $U_\beta \subseteq M$ with φ_β, then on $U_\alpha \cap U_\beta$ we have $s_\beta = \varphi_\beta \circ s$ and $s_\alpha = \varphi_\alpha \circ s$. Then

$$s_\alpha = \varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ s = g_{\alpha\beta} s_\beta,$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$ is the transition map.

Lemma 10.3. *For every open set* $U \subseteq M$, $\dim_{\mathbb{R}} \Gamma(U, E) = \infty$.

Proof. Take $V \subset U$ an open set such that $\varphi_V : E|_V \to V \times \mathbb{R}^n$ is a local trivialization. Choose an open $\tilde{V}$ with $\overline{\tilde{V}}$ compact and $\overline{\tilde{V}} \subseteq V$. Let $S : V \to \mathbb{R}^n$ be any smooth map. There is a C^∞ function $\rho : X \to \mathbb{R}$ such that $\rho \equiv 1$ on $\tilde{V}$ and $\rho \equiv 0$ on $X \setminus V$. Then $\rho s \in \Gamma(M, E)$. Because s can be any smooth map, it is infinite-dimensional. □

Definition 10.4. A set of sections $\{s_\alpha\}_{1\leq\alpha\leq n}$ define a **basis** (or a **frame**) for $E|_U$ if for all $p \in U$, $\{s_\alpha(p)\}$ is a basis for E_p.

Notice that if there exists a frame for $E|_U$ then $E|_U \cong U \times \mathbb{R}^n$, because you can map $(p, V) \mapsto (p, V^\alpha)$ where $V = \sum V^\alpha s_\alpha(p)$. There is something going on with the existence of non-vanishing global sections.

Sections of $TM|)U$ are just smooth vector fields.

Definition 10.5. A section of T^*M is called a 1-**form**.

Recall that if $f : M \to \mathbb{R}$ is C^∞, then we get $df \in \Gamma(M, T^*M)$ defined by

$$df(p)(V) = Vf(p).$$

In local coordinates, $\{x^1, \ldots, x^m\}$ on $p \in U$, then $V = a^i \frac{\partial}{\partial x^i}$[1] is sent to

$$df(V) = a^i \frac{\partial f}{\partial x^i}.$$

Then $T^*M|_U$ has frame $dx^1, \ldots, dx^m$. In general, a section $\Gamma(U, T^*M)$ is a linear combination of $\alpha_i dx^i$ where $\alpha_i : U \to \mathbb{R}$ are smooth functions.

10.2 The algebra of vector bundles

The motto is "any operation which produces new vector spaces out of all vector spaces can be applied to vector bundles".

Definition 10.6. A vector bundle $S \to M$ is a subbundle of E if there is an injective bundle map $S \hookrightarrow E$.

For example, if $M \subseteq \mathbb{R}^n$, then $TM \subseteq T\mathbb{R}^n|_M = M \times \mathbb{R}^n$. So $E \hookrightarrow M$ is always a subbundle of $M \times \mathbb{R}^N$ for N sufficiently large, if M is compact.

Similarly we can take the quotient bundle. If V is a vector bundle and $M \subseteq V$ is a subspace, then

$$V/M = \{[v] : v \in V, v_1 \sim v_2 \text{ if } v_1 - v_2 \in W\}.$$

Then V/W is a vector space is $\dim(V/W) = \dim V - \dim W$. If $E \to M$ is a vector bundle, we can define $Q = M/E$ locally as $Q_p = M_p/E_p$.

[1] Einstein Summation: Sum over repeated upper and lower indices.

11 September 26, 2016

If we have a vector bundle $E \to M$, then a subbundle is a vector bundle S with an injective bundle map $S \hookrightarrow E$. We want to define the quotient bundle $Q = E/M$. Fiberwise this has to be $Q_p = E_p/S_p$ for each $p \in M$. To show that it is a bundle we need to give a local description.

Let $U \subseteq M$ be an open set with a local trivialization $\varphi_U : E|_U \to U \times \mathbb{R}^n$ and also a local trivialization $\varphi_U|_S : S|_U \to U \times \mathbb{R}^k$ for $k < n$. Let $\langle\ ,\ \rangle$ denote the usual inner product on $\mathbb{R}^n$ and let $\pi_S : \mathbb{R}^n \times \mathbb{R}^k$ be the orthogonal projection onto S. Over U let $\{s_1, \dots, s_k\}$ be a local frame for $S|_U$. Choose $\{\tau_1, \dots, \tau_{n-k}\}$ such that $\{s_1, \dots, s_k, \tau_1, \dots, \tau_{n-k}\}$ is a frame for E. These $\tau_1, \dots, \tau_k$ can be computed using $\langle\ ,\ \rangle$ by Gram-Schmidt. Then a section of Q over U is $\sum_{i=1}^{n-k} a_i(x)\tau_i$ for $a_i \in C^\infty(U, \mathbb{R})$.

11.1 Duals and Homs

Let $E \to M$ be a vector bundle. Its **dual** $E^* \to M$ should have fibers $(E^*)_p = (E_p)^*$. Suppose $\varphi_U : E|_U \to U \times \mathbb{R}^n$ is a trivialization. Define

$$\varphi_U^* : E^*|_U \to U \times (\mathbb{R}^n)^*; \quad \varphi_U^*(\ell \in E_p^*) = (p, \hat{\ell})$$

where $\hat{\ell}$ is defined by

$$\langle \hat{\ell}, \varphi_U(e) \rangle = \ell(e)$$

for every $e \in E_p$.

This can be described in another way. If $e_1, \dots, e_n$ is a local frame for $E|_U$ we can define maps

$$e_i^* : U \to \bigcup_{p \in U} E_p^*; \quad p \mapsto e_i(p)^*.$$

We then declare e_1^* to be a smooth local frame for E^*. If $\sigma_1 \in \Gamma(U_1, E)$ then we can write $\sigma_1 = \sigma_1^\alpha e_\alpha$ for $\sigma_1^\alpha : U_1 \to \mathbb{R}$. This gives a map

$$\varphi_{U_1} : E|_{U_1} \to U \times \mathbb{R}^n; \quad \sigma(p) \mapsto (p, \sigma_1^\alpha(p)).$$

If $\psi = \psi_{1,\beta} e_\beta^*$ is a smooth section of $E^*|_{U_1}$, (i.e. $\psi_{1,\beta} : U_1 \to \mathbb{R}$ are C^∞) then

$$\psi(\sigma) = \psi_{1,\alpha}\sigma_1^\alpha = \langle \psi_1, \sigma_1 \rangle.$$

So the induced map on E^* is

$$\phi_{U_1}^* : E^*|_{U_1} \to U_1 \times \mathbb{R}^n; \quad (p, \psi(p)) \mapsto (p, \psi_{1,\beta}).$$

This is exactly what we said before.

What happens when we change frame? If $\{f_1, \dots, f_m\}$ is a frame for $E|_{U_2}$ then on $U_1 \cap U_2$ we can write $f_i = g_i^k e_k$. Then $g_{12} = (g_i^k)$ is the transition matrix. If $\vec{\sigma}_2$ are the coordinates of σ in the $\{f\}$ frame so that $\sigma = \sigma_2^\beta f_\beta$. Then the vectors are related by

$$\vec{\sigma}_1 = g_{12}\vec{\sigma}_2 \text{ where } g_{12} : U_1 \cap U_2 \to \mathrm{GL}(n, \mathbb{R}).$$

Byy definition, $\psi(\sigma) = \langle \vec{\psi}_2, \vec{\sigma}_2 \rangle = \langle \vec{\psi}_1, \vec{\sigma}_1 \rangle$, so

$$\langle \vec{\psi}_1, \vec{\sigma}_1 \rangle = \langle \vec{\psi}_1, g_{12}\vec{\sigma}_2 \rangle = \langle \vec{\psi}_2, \vec{\sigma}_2 \rangle$$

for all $\vec{\sigma}_2$. In matrix multiplication, this is $\psi_1^T g_{12}\sigma_2 = \psi_2^T \sigma_2$. So $\psi_1 = (g_{12}^{-1})^T \psi_2$, where $(g_{12}^{-1})^T : U_1 \cap U_2 \to \mathrm{GL}(n, \mathbb{R})$. So if g_{12} are the transition functions for E then $(g_{12}^{-1})^T$ are the transitions for E^*.

Corollary 11.1. $E^{**} = E$.

Let us now look at bundles of Homs. Let $E \to M$ and $F \to M$ be vector bundles. Fiberwise, we must have $\mathrm{Hom}(E, F) = \bigcup_p \mathrm{Hom}(E_p, F_p)$. Let $U \subseteq M$ such that we have frames $\{e_1, \ldots, e_r\}$ for $E|_U$ and $\{f_1, \ldots, f_k\}$ for $F|_U$. Let

$$t_{ij} : E|_U \to F|_U; \quad t_{ij}(e_j) = f_i.$$

For every $p \in U$, $t_{ij}(p)$ forms a basis for $\mathrm{Hom}(E_p, F_p)$. Declare that t_{ij} is a C^∞ local frame. This gives an isomorphism $\mathrm{Hom}(E, F)|_U \cong U \times \mathbb{R}^{rk}$. Then a section $\sigma \in \Gamma(U, \mathrm{Hom}(E, F))$ corresponds to a matrix m in $\{e\}$ and $\{f\}$.

Suppose $\{\tilde{e}\} = \{\tilde{e}_1, \ldots, \tilde{e}_r\}$ and $\{\tilde{f}\} = \{\tilde{f}_1, \ldots, \tilde{f}_k\}$ are two other local frames with σ corresponding to $\tilde{m}$. There exist matrices $g^F : \{\tilde{f}\} \to \{f\}$ and $g^E : \{\tilde{e}\} \to \{e\}$. Then the matrices m and $\tilde{m}$ are related by

$$m = g^E \tilde{m} (g^F)^{-1}.$$

The transition functions are C^∞ linear maps.

Let $\mathbb{R} : M \times \mathbb{R} \to M$ be the trivial bundle. Then $E^* = \mathrm{Hom}(E, \mathbb{R})$.

12 September 28, 2016

Last time we had a rather detailed discussion of the construction of dual bundles and Hom bundles.

12.1 Direct sums, tensor products and powers

If V and W are vector spaces, then the **tensor product** is $V \otimes W = \mathrm{Hom}(W^*, V)$. Explicitly, if $v_1, \dots, v_k$ is a basis for V and $w_1, \dots, w_r$ is a basis for W, then I can consider the symbols

$$v_i \otimes w_j : W^* \to V; \quad \ell \mapsto \ell(w_j)v_i.$$

Then $V \otimes W = \mathrm{span}\{v_i \otimes w_j\}$. Note that

$$v_i \otimes w_j + v_k \otimes w_j = (v_i + v_k) \otimes w_j, \quad v_i \otimes w_j + v_i \otimes w_k = v_i \otimes (w_j + w_k).$$

If E and F are vector bundles over M, then we can define $E \otimes F \to M$, with $\mathrm{rank}(E \otimes F) = \mathrm{rank}(E) + \mathrm{rank}(F)$. If you want to define the tensor product in the more abstract way, you have to check that this is a vector bundle.

If V and W are vector spaces we define its **direct sum** as $V \oplus W \approx (v, w) \in V \times W$ with

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad r(v, w) = (rv, rw).$$

In this case we can construct $E \oplus F$ in the obvious way. Change of frame matrices will look like

$$\begin{bmatrix} g_E & 0 \\ 0 & g_F \end{bmatrix}.$$

If V is a vector spaces,

$$V^* \otimes V^* \cong \{\text{bilinear maps } V \times V \to \mathbb{R}\},$$

with the identification being $(\ell_1 \otimes \ell_2)(v_1, v_2) = \ell_1(v_1)\ell_2(v_2)$. Likewise,

$$\otimes_k V^* = V^* \otimes \cdots \otimes V^* \cong \{k\text{-linear maps } V \times \cdots \times V \to \mathbb{R}\}.$$

If $E \to M$ is a vector bundle, then $E^{\otimes k} \to M$ is a vector bundle.

A k-linear map $f : \times_k V \to \mathbb{R}$ is **symmetric** if

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = f(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

This is a subspace $\mathrm{Sym}^k(V^*) \subseteq \otimes_k V^*$. Here is an exercise:

$$\mathrm{Sym}^k(V^*) \cong \{\text{Homogeneous polynomials of degree } k \text{ on } V\}.$$

You can check that $\mathrm{Sym}^k(E^*) \to M$ is a vector bundle.

12.2 Antisymmetric powers and forms

A k-linear map $f : \times_k V \to \mathbb{R}$ is **anti-symmetric** if

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Denote by $\bigwedge^k V^* \subseteq \otimes_k V^*$ the anti-symmetric k-linear maps on $\times_k V$. If $\dim V = n$ and $k > n$, then $\bigwedge^k V^* = \{0\}$. If $k \leq n$ then $\dim(\bigwedge^k V^*) = \binom{n}{k} = n!/(n-k)!k!$.

We can describe $\bigwedge^k V^*$ in terms of a basis. Let $v_1, \dots, v_n$ be a basis for V, and let $v_1^*, \dots, v_n^*$ be the dual basis. Denote $v_{\alpha_1}^* \wedge \cdots \wedge v_{\alpha_k}^*$ to be the antisymmetric k-linear map such that

$$\begin{cases} (v_{\alpha_1}^* \wedge \cdots \wedge v_{\alpha_k}^*)(v_{\alpha_1}, \dots, v_{\alpha_k}) = 1, \\ (v_{\alpha_1}^* \wedge \cdots \wedge v_{\alpha_k}^*)(v_{\beta_1}, \dots, v_{\beta_k}) = 0 \quad \text{if } \{\beta_1, \dots, \beta_k\} \neq \{\alpha_1, \dots, \alpha_k\}. \end{cases}$$

These maps form a basis for $\bigwedge^k V^*$ provided we take $\alpha_1 < \cdots < \alpha_k$. For example, $\bigwedge^2 V^*$ is spanned by $v_i^* \wedge v_j^* = v_i^* \otimes v_j^* - v_j^* \otimes v_i^*$.

There is a canonical homomorphism

$$\bigwedge^k V^* \otimes \bigwedge^r V^* \to \bigwedge^{k+r} V^*; \quad f_1 \otimes f_2 \mapsto f_1 \wedge f_2.$$

We call this map the **wedge product**.

Definition 12.1. $\bigwedge$ is called the **wedge product**. If $E \to M$ is a vector bundle of rank n then $\bigwedge^k E \to M$ is a vector bundle for $1 \leq k \leq n$.

The transition functions will be quite complicated in general. If the special case $n = \operatorname{rank}(E)$, the exterior power $\det(E) = \bigwedge^n E$ is a line bundle. If $\{e_1, \dots, e_n\}$ is a local frame for E and $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ is another frame with $g\{\tilde{e}\} = \{e\}$, then

$$e_1 \wedge \cdots \wedge d_n = \det g \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n.$$

Definition 12.2. A section of $\bigwedge^k T^*M$ is called a k**-form**. Locally $\bigwedge^k T^*M$ is generated by $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ for $i_1 < \cdots < i_k$.

13 September 30, 2016

13.1 Push-forwards and pull-backs

If we have a C^∞ map $h : M \to N$ and $E \to N$ is a vector bundle, then we get a bundle $h^*E \to M$. The idea is to construct a bundle whose local sections are pulled back from N. A special case of this is, if $i : M \hookrightarrow N$ for some submanifold M of N, then i^*E is $E|_M$.

Explicitly, if $U \subseteq N$ is open with a local trivalization $\varphi_U : E|_U \to U \times \mathbb{R}^n$, then $h^*E|_{h^{-1}(U)} = h^{-1}U \times \mathbb{R}^n$. If U_1 and U_2 are open sets with $g_{21} = \varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \to \mathrm{GL}(n, \mathbb{R})$, then $g_{21} \circ h = h^{-1}(U_1) \cap h^{-1}(U_2) \to \mathrm{GL}(n, \mathbb{R})$ is the transition function for h^*E. In this case, $h^*E|_P = E|_{h(p)}$.

In terms of local sections, if $\{e_1, \ldots, e_n\}$ is a local frame for $E|_U$, then $\{e_1 \circ h, \ldots, e_n \circ h\}$ is a local frame for $h^*E|_{h^{-1}(U)}$.

If we have a map $h : M \to N$, with $v \in T_pM$, then $v \mapsto h_*v \in T_{h(p)}N$ is the push-forward, defined as $h_*v(f) = v(f \circ h)$ as a derivation. If $\alpha \in T^*_{h(p)}N$ then we can define the **pull-back** $h^*\alpha \in T^*_pM$ defined by $h^*\alpha(v) = \alpha(h_*v)$. The pull-back extends to $(T^*N)^{\oplus k}$ and $\bigwedge^k T^*N$. This is defined as

$$h^*\alpha(v_1, \ldots, v_k) = \alpha(h_*v_1, \ldots, h_*v_n).$$

There are some properties:

(1) For any $f \in \Gamma(U, \mathbb{R}) = \Gamma(U, \bigwedge^0 T^*N)$, $h^*f = f \circ h$ by definition.

(2) $h^*df = d(h^*f)$ because

$$(h_*V)(f) = V(f \circ h) = V(h^*f) = d(h^*f)(V).$$

(3) For a k-form $\omega \in \Gamma(U, \bigwedge^k T^*N)$ and a function $f \in \Gamma(U, \mathbb{R})$,

$$h^*(f\omega) = (h^*f)(h^*\omega).$$

Here is a local description. Suppose $\omega \in \Gamma(U, \bigwedge^k T^*N)$ and $(x^1, \ldots, x^m)$ are local coordinates on N. Then the elements $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ with $i_1 < i_2 < \cdots < i_k$ form a local frame for $\bigwedge^k T^*N|_U$. Then

$$h^*(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = (h^*dx^{i_1}) \wedge \cdots \wedge (h^*dx^{i_k}) = d(x^{i_1} \circ h) \wedge \cdots \wedge d(x^{i_k} \circ h).$$

Lemma 13.1. *If $h : M \to N$ and $g : N \to Z$ are smooth, then $(g \circ h)^* = h^* \circ g^*$.*

Proof. This is because we have defined everything in terms of smooth functions, and it is true for smooth functions. $\square$

What we're really saying is that there's a canonical map $h^*(\bigwedge^k T^*N) \hookrightarrow \bigwedge^k T^*M$.

13.2 Forms and vector fields on a Lie group

Recall that if G is a Lie group, then we have diffeomorphisms $L_g : G \to G$ defined by $a \mapsto ga$ and $R_g : G \to G$ defined by $a \mapsto ag$. For each $g \in G$, there is also the $\mathrm{conj}_g : G \to G$ given by $a \mapsto gag^{-1}$.

Definition 13.2. A 1-form ω on G is **left-invariant** if for all $g \in G$, $L_g^*\omega = \omega$. (Note that if $\omega \in T_{ga}^*G$ then $L_g^*\omega \in T_a^*G$.) Similarly ω is **right-invariant** if $R_g^*\omega = \omega$ for all $g \in G$.

A vector field V is **left-invariant** if $(L_g)_*V = V$, and right invariant if $(R_g)_*V = V$.

The space of left-invariant 1-forms is isomorphic to T_e^*G.

Lemma 13.3. *There exists a global frame of left/right invariant 1-forms.*

Proof. Let $\tilde\omega_1, \dots, \tilde\omega_n$ be a basis of T_e^*G. Define $\omega_i(g) = L_{g^{-1}}^*\tilde\omega_1$. Note that $\omega_1(g)$ are C^∞ since the multiplication map $G \times G \to G$ is smooth. □

This shows that $T^*G \cong T_e^*G \times G$ is trivial.

Example 13.4. Take $M(n,\mathbb{R})$ and fix $q \in M(n,\mathbb{R})$ for $q \neq 0$. Define for $m \in \mathrm{GL}(n,\mathbb{R})$,

$$\omega_q|_m = \mathrm{Tr}(qm^{-1}dm),$$

i.e., if $A \in T_m \mathrm{GL}(n,\mathbb{R})$ then $\omega_q(A) = Tr(qm^{-1}A)$. Then I claim that ω_q is left-invariant. To show this we have to compute

$$(L_g^*\omega_q)|_m(A) = \omega_q|_{gm}((L_g)_*A).$$

Let $\gamma(t) = m + tA$ so that $\gamma(0) = m$ and $\gamma'(0) = A$. Then

$$(L_g)_*A = \frac{d}{dt}\bigg|_{t=0} g(m+tA) = gA,$$

and so

$$(L_g^*\omega_q)|_m A) = \omega_q|_{gm}((L_g)_*A) = \mathrm{Tr}(q(gm)^{-1}gA) = \mathrm{Tr}(qm^{-1}A) = \omega_q|_m(A).$$

14 October 3, 2016

We were talking about pull-backs of k-forms and push-forwards of vector fields.

For the Lie group $G = \mathrm{GL}(n,\mathbb{R})$ and $m \in G$, the tangent space is $T_mG = M(n,\mathbb{R})$. For $q \in M(n,\mathbb{R})$, we defined $\omega_q|_m(A) = \mathrm{Tr}(qm^{-1}A)$, which we denote as $\omega_q|_m = \mathrm{Tr}(qm^{-1}dm)$.

Lemma 14.1. $L_g^*\omega_q = \omega_q$.

On the other hand, it is not right invariant, because $(R_g^*\omega_q)|_m(A) = \mathrm{Tr}(qm^{-1}g^{-1}Ag)$.

14.1 The exponential map

For $a \in M(n,\mathbb{R})$, define

$$\exp(a) = \mathbf{1} + a + \frac{a^2}{2} + \cdots + \frac{a^n}{n!} + \cdots .$$

Lemma 14.2. *(i)* $\exp(a)$ *converges if* $a \in B_\epsilon(0)$ *for* $0 < \epsilon \ll 1$.

(ii) $\exp : B_\epsilon(0) \to U \subseteq \mathrm{GL}(n,\mathbb{R})$ *is a diffeomorphism.*

(iii) $\exp(-a) = \exp(a)^{-1}$.

Proof. (i) Let us first assume $a = P^{-1}DP$ for some diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Then $\exp(a) = P^{-1}(\exp(D))P$ and

$$\exp(D) = \begin{pmatrix} \exp(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \exp(\lambda_n) \end{pmatrix}.$$

In general, put $a = P^{-1}JP$ for a Jordan normal form J. Then $\exp(a) = P^{-1}\exp(J)P$ and by direct computation $\exp(J)$ converges if $\lambda_1, \ldots, \lambda_n$ are sufficiently small.

(ii) Since $(\exp)_*|_0 a = a$, the map exp is locally a diffeomorphism by the inverse function theorem.

(iii) You can formally check this as

$$\exp(a)\exp(-a) = \mathbf{1} + a - a + a^2 - a^2 + \cdots = \mathbf{1}.$$

□

So there is a map exp that goes from a neighborhood of $0 \in T_eG = M(n,\mathbb{R})$ to a neighborhood of $\mathbf{1} \in G$. It makes sense to pull-back to get $(\exp_0)^*\omega_q$ to get a 1-form on $T_{\mathbf{1}}\,\mathrm{GL}(n,\mathbb{R})$ at least on $B_\epsilon(0)$.

Lemma 14.3. *For any $a \in B_\epsilon(0)$ and $V \in T_a(T_{\mathbf{1}} \mathrm{GL}(n,\mathbb{R})) = T_a M(n,\mathbb{R})$, we have*

$$(\exp_0)^* \omega_q|_a(v) = \int_0^1 \mathrm{Tr}(qe^{-sa} v e^{sa}) ds.$$

Proof. Consider $a \in B_\epsilon(0) \subseteq M(n,\mathbb{R})$. Let $\gamma(t) = a + t\vec{v}$ so that $\gamma(0) = a$ and $\gamma'(0) = \vec{v}$. We need to compute

$$\omega_q\Big|_{e^a}\Big(\frac{d}{dt}\Big| \exp(a + t\vec{v})\Big).$$

This is hard to compute because a and v don't commute. So we are going to consider a family of curves and see how the one form varies along the way.

Consdier $\gamma_s(t) = \exp_0(s(a+tv))$ with $\gamma_0(t) = \mathbf{1}$ and $\gamma_1(t) = \exp(a+tv)$. We want to compute $(\exp_0)^*\omega_q|_{e^{sa}}(sv)$ for $s = 1$. This is

$$(\exp_0)^*\omega_q|_{e^{sa}}(sv) = \omega_q\Big|_{e^{sa}}\Big(\frac{d}{dt}\Big|_{t=0} \exp(s(a+tv))\Big) = \mathrm{Tr}\Big(qe^{-sa}\Big(\frac{d}{dt}\Big|_{t=0} e^{s(a+tv)}\Big)\Big).$$

So

$$\begin{aligned}
&\frac{d}{ds}(\exp_0)^*\omega_q|_{e^{sa}}(sv)\\
&\quad = -\mathrm{Tr}\Big(qae^{-sa}\frac{d}{dt}\Big|_{t=0} e^{s(a+tv)}\Big) + \mathrm{Tr}\Big(qe^{-sa}\frac{d}{dt}\Big|_{t=0}(a+tv)e^{s(a+tv)}\Big)\\
&\quad = -\mathrm{Tr}\Big(qae^{-sa}\frac{d}{dt}\Big|_{t=0} e^{s(a+tv)}\Big) + \mathrm{Tr}(qe^{-sa}Ve^{sa}) + \mathrm{Tr}\Big(qe{-}saa\frac{d}{dt}\Big|_{t=0} e^{s(a+tv)}\Big)\\
&\quad = \mathrm{Tr}(qe^{-sa}Ve^{sa}).
\end{aligned}$$

Then we get the lemma by integration. □

Lemma 14.4. *The maps $\exp_0 : T_{\mathbf{1}}\mathrm{SO}(n) \to \mathrm{SO}(n)$ and $\exp_0 : T_{\mathbf{1}}\mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ are defined and has the same formulas.*

Proof. Exercise. □

14.2 Complex vector bundles

Definition 14.5. A **complex vector bundle** E with rank $\mathrm{rk}_{\mathbb{C}} E = n$ is a real vector bundle of rank $2n$ together with endomorphism $J : E \to E$ such that $J^2 = -\mathbf{1}$.

If V is a vector space over $\mathbb{R}$ with an endomorphism $J : V \to V$ such htat $J^2 = -\mathbf{1}$, then V defines a vector space over $\mathbb{C}$.

Definition 14.6 (Alternative definition)**.** A **complex vector bundle** over M is a manifold E with a C^∞ map $\pi : E \to M$ such that for any $p \in M$ there exists an open neighborhood $U \subseteq M$ and a map $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^n$ such that $g_{uv} = \varphi_U \circ \varphi_V^{-1} : U \cap V \to \mathrm{GL}(n,\mathbb{C})$.

To show that the definitions are equivalent, consider an open set $U \subseteq M$ such that there exists a frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ such that $Je_i = e^{n+i}$ and $Je_{n+i} = -e_i$. Then we can identify $E|_U \cong \mathbb{C}^n \times U$ by this frame, in the same way we identified $\mathbb{R}^{2n} \cong \mathbb{C}^n$. If we have another frame $\{\tilde{e}_1, \ldots, \tilde{e}_{2n}\}$, and $g\{e\} = \{\tilde{e}\}$ then $gJ\{e\} = \tilde{J}\{\tilde{e}\} = \tilde{J}g\{e\}$. This implies that $g : U \to \mathrm{GL}(n, \mathbb{C})$.

That the alternative definition implies the first definition is just the restriction of scalars.

15 October 5, 2016

Let us look at examples of complex vector bundles. Consider a vector bundle $E \to M$, and look at the trivial $\mathbb{C}$ bundle $\mathbb{C} \to M$. Then

$$E \otimes_{\mathbb{R}} \mathbb{C} \to M$$

is a complex vector bundle with $\mathrm{rk}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{rk}_{\mathbb{R}}(E)$. This is not really an interesting example.

Let $\Sigma \subseteq \mathbb{R}^3$ be a surface. Let $\vec{n}$ be the unit normal vector field on σ. Then $T_p\Sigma = \{v \in \mathbb{R}^3 : v \cdot \vec{n}(p) = 0\}$. Define a complex structure

$$J \in \mathrm{End}(T\Sigma); \quad J(p)v = n(p) \times v.$$

Then $J^2 = n \times (n \times v) = -v$ and so $T\Sigma$ has the structure of a complex vector bundle. Thus the tangent bundle $T\Sigma$ has the structure of a complex vector bundle with $\mathrm{rk}_{\mathbb{C}}(T\Sigma) = 1$.

We can think of algebraic operations, and the moral is the everything that works for $\mathbb{C}$-vector spaces also works for complex vector bundles. But we need to be careful when comparing $\mathbb{R}$ and $\mathbb{C}$ structures. For instance, if E and F are $\mathbb{C}$-vector bundles, and $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$ are the underlying $\mathbb{R}$-bundles, then $\mathrm{Hom}(E, F) \neq \mathrm{Hom}(E_{\mathbb{R}}, F_{\mathbb{R}})$. Think about when a linear map $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ descend to a map $\mathbb{C}^n \to \mathbb{C}^n$. Likewise, $\mathrm{rk}_{\mathbb{C}}(E \otimes_{\mathbb{C}} F) = \mathrm{rk}_{\mathbb{C}}(E)\,\mathrm{rk}_{\mathbb{C}}(F)$. On the other hand, $\mathrm{rk}_{\mathbb{R}}(E_{\mathbb{R}} \otimes_{\mathbb{R}} F_{\mathbb{R}}) = 4\,\mathrm{rk}_{\mathbb{C}}(E)\,\mathrm{rk}_{\mathbb{C}}(F)$.

If $E_{\mathbb{R}}$ has the complex structure J, then $(-J)$ is also a complex structure. So we can define $\bar{E}$ to be the $\mathbb{C}$ vector bundle defined by $(-J)$. As an exercise, prove that if $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$ are transition functions for E, then $\bar{g}_{\alpha\beta}$ are transition functions for $\bar{E}$.

15.1 Metrics on vector bundles

Definition 15.1. A **metric** on $\mathbb{R}^n$ is a bilinear map $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $g(u, u) > 0$ for $u \neq 0$ and $g(u, w) = g(w, u)$.

Definition 15.2. A **metric** on E is a section $g \in \Gamma(M, \mathrm{Sym}^2(E^*))$ such that for all $p \in M$, g_p is a metric on $E_p \cong \mathbb{R}^n$.

Said another way, g is an assignment to each p of a metric on E_p such that if $e_p, f_p \in \Gamma(M, E)$ then the map $p \mapsto g_p(e_p, f_p)$ is C^∞.

Lemma 15.3. *Metrics always exist.*

Proof 1. If M is compact, there is a map $E \hookrightarrow \mathbb{R}^N$ where $\mathbb{R}^N \to M$ is the trivial bundle. We can put a metric on M via restriction by choosing a metric on $\mathbb{R}^N$. □

Proof 2. Take an locally finite open cover $\{U_\alpha\}$ such that $E|_{U_\alpha}$. Let g_α be a metric on $\mathbb{R}^n$, viewed as a metric on $E|_{U_\alpha}$. Take a partition of unity χ_α

subordinate to the U_α, then take

$$g = \sum_\alpha \chi_\alpha g_\alpha. \qquad \square$$

Definition 15.4. A **hermitian metric** on $\mathbb{C}^n$ is a bilinear form $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that

- $g(u,v) = \overline{g(v,u)}$,
- $g(u,u) > 0$ for $u \neq 0$,
- $g(u,cv) = cg(u,v)$ and $g(cu,v) = \bar{c}g(u,v)$.

Definition 15.5. A **hermitian metric** on $E \to M$ is a section of $\bar{E}^* \otimes E^*$ that restricts to each fiber as a hermitian metric.

Check how it *must* transform undr change of frame. For $\langle v,w\rangle = v^\dagger Aw$ and a transformation $T : E_p \to E_p$, we have $\tilde{v} = Tv$ and $\tilde{w} = Tw$. So

$$v^\dagger Aw = \langle v,w\rangle = \langle \tilde{v},\tilde{w}\rangle = \tilde{v}^\dagger \tilde{A}\tilde{w}$$

so $\tilde{A} = (T^\dagger)^{-1}AT^{-1}$.

Lemma 15.6. *Hermitian metrics always exist.*

Proof. It is the same. $\square$

Let us look at the relation between g on $E_\mathbb{R}$ and on $E_\mathbb{C}$. Let $E_\mathbb{R}$ be a vector bundle with J a complex structure, and $g_\mathbb{R}$ a metric.

Lemma 15.7. *A hermitian metric on E is defined by $g_\mathbb{R}$ on $E_\mathbb{R}$ provided that $g_\mathbb{R}(u,Jv) = -g_\mathbb{R}(Ju,v)$ for all u,v.*

Proof. Linear algebra. $\square$

Conversely, if g is hermitian, how do we find $g_\mathbb{R}$?

Lemma 15.8. *$E \subset E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ in the following way: let e be a local section of $E_\mathbb{R}$, then E is generated in $E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ as $e - \sqrt{-1}Je$.*

Note that $J(e - \sqrt{-1}Je) = \sqrt{-1}(e - \sqrt{-1}Je)$. So E_p is naturally identified with the $+\sqrt{-1}$ eigenspace of

$$J : (E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C})_p \to (E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C})_p.$$

Likewise $\overline{E}$ is the $-\sqrt{-1}$ eigenspace of J. Given $g_\mathbb{R}$ we define

$$u = e - \sqrt{-1}je, \quad v = \tilde{e} - \sqrt{-1}J\tilde{e}.$$

16 October 7, 2016

16.1 Structure group and orientable bundles

A vector bundle $E \to M$ has **structure group** (or **gauge group**) $G \subseteq \mathrm{GL}(n,\mathbb{R})$ if there exists a covering of M by open sets U_α such that $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$ and $\varphi_\alpha^{-1} \circ \varphi_\beta : U_\alpha \cap U_\beta \to G \subseteq \mathrm{GL}(n,\mathbb{R})$. In other words, there exist local frames $\{e_\alpha\}$ on U_α such that if $g_{\alpha\beta} : \{e_\beta\} \to \{e_\alpha\}$ then $g_{\alpha\beta} \in G$.

For example, if $G = \{1\}$ then E is trivial. If E has a metric, then we can reduce to structure group to $O(n)$ (or $U(n)$ in the $\mathbb{C}$-case). This because we can take orthonormal frames. A more subtle question is, when can we reduce to $\mathrm{SO}(n)$ (or $\mathrm{SU}(n)$)?

Definition 16.1. Say that a bundle $E \to M$ is **orientable** if there exist trivializations $\{e_k\}$ on U_α with $\bigcup_\alpha U_\alpha = M$ such that $g_{\alpha\beta} : \{e_\beta\} \to \{e_\alpha\}$ have $\det g_{\alpha\beta} > 0$.

Note that E is orientable if and only if the structure group reduces to $\mathrm{SO}(n)$ (or $\mathrm{SU}(n)$).

Lemma 16.2. *E is orientable if and only if $\bigwedge^n E = \det(E) \cong M \times \mathbb{R}$.*

Proof. Let g be a metric on E. Cover M by open sets U_α with frames $\{e_\alpha\}$ such that $\deg g_{\alpha\beta} > 0$ and $\{e_\alpha\}$ are orthonormal. Then $g_{\alpha\beta}\ in\, \mathrm{O}(n)$, and so $g_{\alpha\beta} = 1$. If $e_1 \wedge \cdots \wedge e_n$ are the induced local trivializations of $\bigwedge^n E$, then since

$$e_1^\alpha \wedge \cdots \wedge e_n^\alpha = \det g_{\alpha\beta} e_1^\beta \wedge \cdots \wedge e_n^\beta = e_1^\beta \wedge \cdots \wedge e_n^\beta,$$

$\bigwedge^n E$ has a global non-vanishing section.

If $\bigwedge^n E \cong M \times \mathbb{R}$, then there exists a $\eta \in \Gamma(M, \bigwedge^n E)$ such that $\eta(p) \neq 0$ for any $p \in M$. If $\{e_1^\alpha, \dots, e_n^\alpha\}$ is an orthonormal frame for $E|_{U_\alpha}$, $\eta|_{U_\alpha} = f_\alpha e_1^\alpha \wedge \cdots \wedge e_n^\alpha$. Then $f_\alpha : U_\alpha \to \mathbb{R} \setminus \{0\}$. Then define

$$\tilde{e}_1^\alpha = \frac{f_\alpha}{|f_\alpha|} e_1^\alpha, \quad \tilde{e}_j^\alpha = e_j \text{ for } 2 \leq j \leq n.$$

This gives me a bunch of new frames. Let $\tilde{g}_{\alpha\beta} : \{\tilde{e}_\beta\} \to \{\tilde{e}_\alpha\}$. Then

$$\det \tilde{g}_{\alpha\beta} = \det(T_\alpha g_{\alpha\beta} T_\beta^{-1}) = \frac{|f_\beta|}{f_\beta} \frac{f_\alpha}{|f_\alpha|} \det g_{\alpha\beta}.$$

On the other hand,

$$\eta|_{U_\alpha \cap U_\beta} = f_\alpha e_1^\alpha \wedge \cdots \wedge e_n^\alpha = f_\beta e_1^\beta \wedge \cdots \wedge e_n^\beta$$

and so $f_\alpha \det g_{\alpha\beta} = f_\beta$. This implies $\det \tilde{g}_{\alpha\beta} = |f_\beta|/|f_\alpha|$. Since $g_{\alpha\beta} \in O(n)$, we further have $\det \tilde{g}_{\alpha\beta} = 1$. ☐

Similarly, a complex vector bundle $E \to M$ is orientable if $\bigwedge^n E \cong M \times \mathbb{C}$.

Definition 16.3. A manifold M is **orientable** if and only if TM is orientable.

16.2 Induced metrics on bundles

Suppose E is a $C^\infty(\mathbb{R})$ vector bundle with a metric h. If $\{e_1, \ldots, e_n\}$ is a local frame, then we define $h_{ij} = \langle e_i, e_j \rangle$. In the $\mathbb{C}$-case, we write $h_{\bar{\alpha}\beta} = \langle e_\alpha, e_\beta \rangle$. If $\sigma_1(p), \sigma_2(p) \in E_p$ then we can write

$$\sigma_1(p) = \sigma_1^i(p) e_i(p), \quad \sigma_2(p) = \sigma_2^j(p) e_j(p).$$

Then we can simply write

$$\langle \sigma_1, \sigma_2 \rangle(p) = \sigma_1^i h_{ij} \sigma_2^j.$$

Likewise in the $\mathbb{C}$-case we have

$$\langle \sigma_1, \sigma_2 \rangle = \overline{\sigma_1^\alpha} h_{\bar{\alpha}\beta} \sigma_2^\beta.$$

If e_i is a local section of E, then $\sigma^i = \langle e_i, \bullet \rangle$ is a local section of E^*. Then

$$\sigma^i = \sum_j h_{ij} e_j^*.$$

Since h is positive definite, this gives an isomorphism $E_p \cong E_p^*$. We define the metric on E^* such that this map is an isometry. In other words, if h^* is the metric on E^*, then

$$\langle \sigma^i, \sigma^j \rangle_{h^*} = \langle e_i, e_j \rangle.$$

Proposition 16.4. *In the frame e_i^*, $(h^*)_{il} = (h^{-1})_{il} = h^{il}$.*

Proof. We have

$$h_{il} = \langle \sigma^i, \sigma^l \rangle_{h^*} = \sum_{j,p} h_{ij} h_{lp} \langle e_j^*, e_p^* \rangle = \sum_{j,p} h_{ij} h_{lp} (h^*)_{jp} = (hh^*h^T)_{il}.$$

So $h^* = h^{-1}$ because $h^T = h$. $\square$

17 October 12, 2016

Last time we were discussing metrics on vector bundles. If h is a metric on E, then h^{-1} is a metric on E^*. If h is a metric on E and g is a metric on F, then we get a metric on $E \otimes F$ as follows: if $\{e_1, \dots, e_n\}$ is a local frame for E and $\{f_1, \dots, f_r\}$ is a local frame for F, then define

$$\langle e_i \otimes f_\alpha, e_j \otimes f_\beta \rangle = \langle e_i, e_j \rangle_h \langle f_\alpha f_\beta \rangle_g,$$

i.e., if $\{e_i\}$ is orthonormal and $\{f_\alpha\}$ is also orthonormal then we declare $\{e_i \otimes f_\alpha\}$ is orthonormal. In terms of local fames,

$$(h \otimes g) = h_{ij} g_{\alpha\beta}.$$

Since other bundles like $\bigwedge^r E$ or $\mathrm{Sym}^r E$ sits inside $E^{\otimes r}$, this induces metric on $\bigwedge^r E$ or $\mathrm{Sym}^r E$.

17.1 Metrics on the tangent bundle

Let's assume we have (M, g) a connected **Riemannian manifold**, i.e., a manifold M with a choice of g.

Definition 17.1. A curve $\gamma : [a, b] \to M$ is **piecewise** C^∞ if there exist $a = t_0 < t_1 < \cdots < t_k = b$ such that γ is continuous and $\gamma|_{(t_i, t_{i+1})}$ is C^∞.

Definition 17.2. Define

$$d(p, q) = \inf_\gamma \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g(\gamma'(t), \gamma'(t))} dt$$

where inf is over all piecewise C^∞ curves $\gamma : [0, 1] \hookrightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Proposition 17.3. *d is a metric on M.*

Proof. Clearly it is symmetric. The triangle inequality is also clear since if γ_1 is a curve from p to q and γ_2 is a curve from q to r then

$$(\gamma_2 \circ \gamma_1)(t) = \begin{cases} \gamma_1(2t) & 0 \le t < 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

has length the length of γ_1 plus the length of γ_2. Finally we need to show that $p \neq q$ implies $d(p, q) > 0$. Choose a coordinate patch (U, φ) such that $q \in U$, $p \notin U$ and $\varphi : U \to B_1(0)$ with $q \mapsto 0$. If $\gamma(t)$ connects p to q there exists a last time T such that $\gamma(T) \in \partial U$. In local coordinates $(x^1, \dots, x^n)$, there exist a $c > 0$ such that $g_{ij} > c\delta_{ij}$. Then the length of $\gamma(t)|_{[T,1]}$ is at least c. $\square$

Lemma 17.4. *The metric topology on (M, d) is identical to the topology of M.*

Proof. Locally $\delta_{ij} C^{-1} < g_{ij} < C\delta_{ij}$ for some $C < +\infty$. $\square$

17.2 Geodesics

A natural question is, is there a C^∞ curve γ such that the length of γ is $d(p,q)$? If such a γ exists, then any variation will increase the length of the curve.

Suppose we have a curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Consider a variation $\gamma_\epsilon(t) = \gamma(t) + \epsilon c(t)$ for some $c : [0,1] \to \mathbb{R}^n$ with $c(0) = c(1) = 0$. We can assume that $c(t)$ is orthogonal to $\gamma'(t)$, i.e., $\langle c(t), \gamma'(t)\rangle_g = 0$ with $|\gamma'| = 0$.

Let $g = g_{ij}$. We need to compute

$$g_{\gamma_\epsilon(t)}(\dot\gamma_\epsilon(t), \dot\gamma_\epsilon(t)) = g_{\gamma_\epsilon(t)}(\dot\gamma, \dot\gamma) + 2\epsilon g_{\gamma_\epsilon(t)}(\dot\gamma, \dot c) + O(\epsilon^2).$$

The derivative with respect to ϵ at $\epsilon = 0$ is

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} g_{\gamma_\epsilon(t)}(\dot\gamma_\epsilon, \dot\gamma_\epsilon) = \frac{\partial}{\partial x^k} g_{ij}\dot\gamma^i\dot\gamma^j c^k + 2g_{ij}\dot\gamma^i\dot c^j.$$

So by the chain rule,

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_0^1 \sqrt{g_{\gamma_\epsilon(t)}(\dot\gamma_\epsilon, \dot\gamma_\epsilon)}dt = \int_0^1 \frac{g_{ij}\dot\gamma^i\dot c^j + \frac{1}{2}\partial_k g_{ij}\dot\gamma^i\dot\gamma^j c^k}{\sqrt{g(\dot\gamma,\dot\gamma)}}dt = 0.$$

Since $g_{ij}\dot\gamma^i c^j = 0$ because $\dot\gamma$ and c are orthogonal, we can take d/dt of both sides and get

$$\partial_k g_{ij}\dot\gamma^i\dot\gamma^k c^j + g_{ij}\ddot\gamma^i c^j + g_{ij}\dot\gamma^i\dot c^j = 0.$$

So we have

$$0 = \int_0^1 \left[-(g_{ij}\ddot\gamma^j + \partial_k g_{ij}\dot\gamma^j\dot\gamma^k) + \frac{1}{2}\partial_i g_{jk}\dot\gamma^j\dot\gamma^k\right] c^i dt.$$

This implies that for all i,

$$g_{ij}\ddot\gamma^j + \partial g_{ij}\dot\gamma^j\dot\gamma^k - \frac{1}{2}\partial_i g_{kj}\dot\gamma^j\dot\gamma^k = 0.$$

Multiplying by g^{il}, we get

$$\ddot\gamma^l + \frac{1}{2}g^{il}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk})\dot\gamma^j\dot\gamma^k = 0.$$

We now define the **Christoffel symbols**

$$\Gamma^l_{kj} = \frac{1}{2}g^{il}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}).$$

A curve $\gamma(t)$ solving $\ddot\gamma^l + \Gamma^l_{kj}\dot\gamma^k\dot\gamma^j = 0$ is called a **geodesic**.

18 October 14, 2016

For a Riemannian manifold (M, g), a curve $\gamma : [a, b] \to M$ is a geodesic if

$$\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0, \quad \text{where } \Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}).$$

Geodesics are critical points for the length function $\gamma \mapsto \int_a^b \sqrt{g(\gamma', \gamma')}dt$.

Theorem 18.1. *For every $p \in M$, there exists and open neighborhood $U \subseteq T_pM$, $0 \in U$, and a number $\epsilon > 0$ with a map $\gamma : (-\epsilon, \epsilon) \times U \to M$ such that for all $v \in U$, $\gamma(t, v)$ is the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$.*

A geodesic need not be defined on $\mathbb{R}$. For example, take an open subset of $\mathbb{R}^2$ with g_{Euc}. This is a Riemannian manifold but it does not always have geodesics defined on $\mathbb{R}$.

Example 18.2. Take $(\mathbb{R}^n, g_{\text{Euc}})$. In standard coordinates $(x^1, \dots, x^n)$, we have $g_{ij} = \delta_{ij}$. So $\Gamma^k_{ij} = 0$ and so the geodesic equation is $\ddot{\gamma}^k = 0$. That is, the geodesics are $\gamma(t) = p + t\vec{v}$.

As an homework, you will need to compute the geodesics of (S^n, g_{round}). You can use the symmetries of S^n instead of computing all the Christoffel symbols and then solving the differential equaitons.

18.1 Geodesics on $\mathrm{SO}(n)$

Let us compute the geodesics of $\mathrm{SO}(n)$. We have $\mathrm{SO}(n) \subseteq \mathrm{GL}(n) \subseteq M(n)$, and then

$$T_{\mathbf{1}} \mathrm{SO}(n) = \{a \in M(n) : a^T = -a\}.$$

Now fix a basis $a_j \in M(n, \mathbb{R})$ for $1 \le j \le n^2$ such that

$$a_j = -a_j^T \text{ for } 1 \le j \le \frac{n(n-1)}{2}, \quad a_j = a_j^T \text{ for } \frac{n(n-1)}{2} < j \le n^2.$$

Define an inner product on $M(n)$ by $\langle a, b\rangle = \mathrm{Tr}(a^T b)$, which is the Euclidean inner product. Then $\langle a_j, a_k\rangle = 0$ if a_j is antisymmetric and a_k is symmetric. By Gram-Schmidt, we can assume that a_j are orthonormal.

Lemma 18.3. *$g = \langle\ ,\ \rangle|_{T_m \mathrm{SO}(n)}$ is both left and right invariant.*

Proof. Given $m \in \mathrm{SO}(n)$, it suffices to show

$$(L_m^*)g|_{\mathbf{1}}(A, B) = \mathrm{Tr}(A^T B), \quad (R_m^*)g|_{\mathbf{1}}(A, B) = \mathrm{Tr}(A^T B).$$

We have

$$(L_m^* g)(A, B) = g|_m(mA, mB) = \mathrm{Tr}((mA)^T mB) = \mathrm{Tr}(A^T m^T mB) = \mathrm{Tr}(A^T B).$$

Likewise,

$$(T_m^* g)(A, B) = g_m^{-1}(Am^{-1}, Bm^{-1}) = \mathrm{Tr}((m^{-1})^T A^T B m^{-1}) = \mathrm{Tr}(A^T B). \quad \square$$

Definition 18.4. Such metrics are called **bi-invariant**.

Recall we have defined 1-forms $\omega^i|_m = \mathrm{Tr}(a^i m^{-1} dm)$. Define $\tilde{g} = \sum \omega^i \otimes \omega^i$.

Proposition 18.5. $\tilde{g}|_{\mathrm{SO}(n)} = g$.

Proof. Check at $\mathbf{1}$ and then use left-invariance. □

Theorem 18.6. *For $a \in T_{\mathbf{1}} \mathrm{SO}(n)$, the curve $t \mapsto me^{at} = \gamma(t)$ is a geodesic in $\mathrm{SO}(n)$ for the bi-invariant metric with $\gamma(0) = m \in \mathrm{SO}(n)$ and $\gamma'(0) = ma \in T_m \mathrm{SO}(n)$.*

Proof. It suffices to prove for $m = \mathbf{1}$ because g is left-invariant. Also, it suffices to show that $at \in T_{\not{1}} \mathrm{SO}(n)$ is a geodesic for $\exp^* g = \hat{g}$.

First of all $\ddot{\gamma} = 0$. So we need to show $\Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = \Gamma^k_{ij} v^i v^j = 0$, where $a = \sum v^i a_i$. Recall that

$$\Gamma^k_{ij} = \frac{1}{2}\hat{g}^{lk}(\partial_i \hat{g}_{lj} + \partial_j \hat{g}_{li} - \partial_l \hat{g}_{ij}).$$

We have

$$\partial_i \hat{g}_{lj}\dot{\gamma}^j\dot{\gamma}^i = \frac{d}{dt}(\hat{g}_{lj}\dot{\gamma}^j)$$

and also

$$\hat{g}(v,w)|_a = \int_0^1\int_0^1 \mathrm{Tr}(e^{-sa^T} v^T e^{sa^T} e^{ra} w e^{-ra}) ds dr$$

by repeating the calculation from before. So

$$\begin{aligned}
\hat{g}(v,\dot{\gamma})_{\gamma(t)} = \hat{g}(v,a)_{at} &= \int_0^1\int_0^1 \mathrm{Tr}(e^{-sta^T} v^T e^{sta^T} e^{rat} a e^{-rat}) drds \\
&= \int_0^1\int_0^1 \mathrm{Tr}(e^{-sta^T} v^T e^{sta^T} a) drds \\
&= \int_0^1\int_0^1 \mathrm{Tr}(v^T e^{sta^T} e^{-sta^T} a) drds = \mathrm{Tr}(v^T a).
\end{aligned}$$

So

$$\frac{d}{dt} g(v,\dot{\gamma})_{\gamma(t)} = 0$$

for all v, and this implies that $\frac{d}{dt}(\hat{g}_{lj}\dot{\gamma}^j) = 0$.

Now we would like to prove $\partial_l \hat{g}_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$. This is equivalent to $\nabla_v \hat{g}(a,a)_{at} = 0$ for any $v \in T_{\mathbf{1}} \mathrm{SO}(n)$. We have

$$\begin{aligned}
\hat{g}(a,a)_v &= \int_0^1\int_0^1 \mathrm{Tr}(e^{sv} a^T e^{-sv} e^{rv} a e^{-rv}) drds \\
&= \int_0^1\int_0^1 \mathrm{Tr}(e^{(s-r)v} a^T e^{(r-s)v} a) drds.
\end{aligned}$$

We will continue next time. □

19 October 17, 2016

Let us finish the computation of the geodesics of $\mathrm{SO}(n)$.

Theorem 19.1. *The geodesics (with respect to the bi-invariant metric) through $\mathbf{1}$ with tangent vector a is e^{at}.*

Last time we reduced the theorem to proving $\nabla_v \tilde{g}(a,a)|_{at} = 0$ for all v, where $\tilde{g} = \exp_{\mathbf{1}}^* g$. For any $v \in T_{\mathbf{1}} \mathrm{SO}(n)$,

$$\begin{aligned} \tilde{g}(a,a)|_v &= \int_0^1 \int_0^1 \mathrm{Tr}(e^{sv} a^T e^{-sv} e^{rv} a e^{-rv}) ds dr \\ &= \int_0^1 \int_0^1 \mathrm{Tr}(e^{(s-r)v} a^T e^{(r-s)v} a) dr ds. \end{aligned}$$

Take $v_\epsilon = at + \epsilon w$ and let us evaluate $(d/d\epsilon)|_{\epsilon=0} \tilde{g}(a,a)_{v_\epsilon}$. Then

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} e^{(s-r)(at+\epsilon w)} = (s-r) d(\exp_{\mathbf{1}})_{(s-r)at} w,$$

where $d(\exp_{\mathbf{1}})_{(s-r)at} : T_{(s-r)at} T_{\mathbf{1}} \mathrm{SO}(n) \to T_{\mathbf{1}} \mathrm{SO}(n)$. Let

$$M^+(r,s) = d(\exp)_{(s-r)at} w, \quad M^-(r,s) = d(\exp)_{(r-s)at} w = M^+(s,r).$$

Then

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \tilde{g}(a,a)_{v_\epsilon} &= \int_0^1 \int_0^1 \mathrm{Tr}((s-r) M^+ a^T e^{(r-s)at} a) dr ds \\ &\quad + \int_0^1 \int_0^1 \mathrm{Tr}(e^{(s-r)at} a^T (r-s) M^- a) dr ds. \end{aligned}$$

Here

$$\mathrm{Tr}(M^+ a^T e^{(r-s)ta} a) = \mathrm{Tr}(e^{(r-s)ta} M^T a^T a).$$

Because $M^+ = d\exp_{(r-s)ta} w \in T_{(s-r)ta} \mathrm{SO}(n)$, we have $e^{(r-s)ta} M^+ \in T_{\mathbf{1}} \mathrm{SO}(n)$. This shows that $e^{(r-s)ta} M^+$ is antisymmetric, but $a^T a$ is symmetric. So the trace is just zero. Likewise, the second terms is zero.

19.1 Gaussian coordinates

Theorem 19.2. *For every $p \in M$ there exist a $\epsilon, \delta > 0$ such that, for every $v \in B_\epsilon(0) \subseteq T_p M$, the unique geodesic $\gamma(t)$ with $\gamma(0) = p$ and $\gamma'(0) = v$ exists for all $t = (-\delta, \delta)$.*

Lemma 19.3. *If $\gamma(t)$ is a geodiesic with $\gamma'(t) = v$, then for $c \in \mathbb{R}$ the curve $\tilde{\gamma}(t) = \gamma(ct)$ is a geodesic with $\tilde{\gamma}'(0) = cv$.*

Corollary 19.4. *For every $p \in M$ there exists an $\epsilon > 0$ such that $\gamma(t,v)$ for any $v \in B_0(\epsilon) \subseteq T_pM$ the geodesic with $\gamma(0) = p$ and $\gamma'(0) \in v$ exists with $t \in (-1,1)$.*

Definition 19.5. The **exponential map** $\exp_p : B_\epsilon(0) \to M$ is the map $v \to \gamma(1,v) = \exp_p(v)$ where $\exp_p(tv)$ is the geodesic $\exp_p(0) = p$ and $(d/dt)\exp_p(vt)|_{t=0} = v$.

For $\mathrm{SO}(n)$, the map $\exp_p : B_\epsilon(0) \to M$ is C^∞ (by homework). Because $d(\exp_p)_0 : T_0T_pM \to T_pM$ given by $v \mapsto v$ is invertible, by the inverse function theroem, there is a neighborhood $B_\delta(0) \subseteq T_pM$ such that $\exp_p : B_\delta(0) \to U \subseteq M$ is a diffeomorphism.

Fix an orthonormal basis $\{v^1, \dots, v^n\}$ on T_pM. Then we have $\mathrm{id} : (T_pM, g_p) \cong (\mathbb{R}^n, \langle \bullet, \bullet \rangle)$. Since $\exp_p : B_0(0) \to U$ is a diffeomorphism, we get coordinates on U by

$$q \longrightarrow \exp_p^{-1}(q) \xrightarrow{\mathrm{id}} (a^1, \dots, a^n).$$

These coordinates are called **Gaussian coordinates** or **normal coordinates**.

Theorem 19.6. *Let $(x^1, \dots, x^n)$ be Gaussian coordinates at $p = (0, \dots, 0)$. Then*

$$g_{ij}(p) = \delta_{ij}, \quad \partial_k g_{ij}(p) = 0.$$

Proof. First $\partial/\partial x^i|_p = v^i$ so $g_{ij}(p) = \delta_{ij}$. Now $g(x^1, \dots, x^n) = \exp_p^* g$.

Let $\tilde{\Gamma}^i_{jk}$ be the Christoffel symbols of $\exp^* g$. We claim that $\tilde{\Gamma}^i_{jk}(p) = 0$ if and only if $\partial g = 0$. By definition of the exp map, the curve $t \mapsto vt$ is a geodesic in T_pM with respect to the metric $\exp^* g$. Then

$$\frac{d^2}{dt^2}(vt) = 0, \quad \frac{d}{dt}(vt) = v.$$

Then by the geodesic equation, $\tilde{\Gamma}^k_{ij} v^i v^j = 0$ for all k. Since this holds for all $v \in T_0T_pM$, we conclude $\tilde{\Gamma}^k_{ij}(0) = 0$ for all i, j, k. Then

$$\tilde{\Gamma}^k_{ij}(0) = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = 0.$$

Then $\partial_i g_{jk} = 0$. □

Corollary 19.7. *There is no geometric invariants of g involving $\partial_i g_{jk}$.*

20 October 19, 2016

20.1 Gauss's lemma

Lemma 20.1 (Gauss)**.** *For a point $p \in M$ and a vector $v \in T_pM$ such that $\exp_p(v)$ is defined. Let $w \in T_v(T_pM) \cong T_pM$. Then*

$$\langle d(\exp_p)_v v, d(\exp_p)_v w\rangle_{g(\exp_p(v))} = \langle v, w\rangle_p.$$

Proof. First assume $w = \lambda v$. It suffices to prove that

$$\langle d(\exp_p)_v v, d(\exp_p)_v v\rangle = \langle v, v\rangle.$$

If we let $\gamma(t) = \exp_p(vt)$ then $d(\exp_p)_v v = \gamma'(1)$. Because γ has constant speed,

$$\langle \gamma'(1), \gamma'(1)\rangle = \langle \gamma'(0), \gamma'(0)\rangle = \langle v, v\rangle.$$

Now assume that $\langle v, w\rangle = 0$. Take a curve $v(s)$ in T_pM with $v(0) = v$ and $v'(0) = w$ with $|v(s)| = \text{const}$. Consider $f(t, s) = \exp_p(tv(s))$ defined for $(t, s) = A = \{0 \le t \le 1, -\epsilon \le s \le \epsilon\}$. Note that $f(t, s_0)$ is a geodesic for a fixed s_0. Also

$$\langle d(\exp_p)_v v, d(\exp_p)_v w\rangle = \Big\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\Big\rangle\Big|_{\substack{t=1\\s=0}}.$$

We claim that this quantity is independent of t. Fix (t_0, s_0) and choose normal coordinates near $p_0 = f(t_0, s_0)$. Then

$$\Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle = g_{ij}(f(t, s))\frac{\partial f^i}{\partial t}\frac{\partial f^j}{\partial s}.$$

Take d/dt and evaluate at (t_0, s_0). Then because $\partial_l g_{ij} = 0$ since we are working in normal coordinates,

$$\frac{d}{dt}\Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle(t_0, s_0) = g_{ij}(p_0)\frac{\partial^2 f^i}{\partial t^2}\frac{\partial f^j}{\partial s} + g_{ij}(p_0)\frac{\partial f^i}{\partial t}\frac{\partial^2 f^j}{\partial t \partial s}.$$

Because $f(t, s_0)$ is a geodesic, f^i is linear in t. Thus $(\partial^2 f/\partial t^2)(t_0, s_0) = 0$.

Now we have one term left, and by the same argument, we further have

$$\frac{d}{dt}\Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle(t_0, s_0) = g_{ij}(p_0)\frac{\partial f^i}{\partial t}\frac{\partial^2 f^j}{\partial s \partial t} = \frac{1}{2}\frac{d}{ds}\Big\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\Big\rangle.$$

Now since $f(t, s_0)$ is a geodesic,

$$\Big\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\Big\rangle(t_0, s_0) = \Big\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\Big\rangle(0, s_0) = |v(s_0)| = \text{const}.$$

This shows that

$$\Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle(t, s) = \Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle(0, s).$$

So

$$\Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle(1, 0) = \lim_{t\to 0}\Big\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\Big\rangle(t, 0) = \Big\langle v, \lim_{t \to 0} d(\exp_p)_{tv} tw\Big\rangle = 0. \qquad \square$$

Corollary 20.2. *Let* $(x^1, \ldots, x^n)$ *be normal coordinates. Let* $r = \sqrt{\sum |x^i|^2}$ *and let* $\theta^1, \ldots, \theta^{n-1}$ *be coordinates on* S^{n-1}*. Then the metric* $(\exp_p)^* g$ *can be written as*

$$(\exp_p)^* g = dr^2 + r^2 K_{ab} d\theta^a d\theta^b.$$

Proof. The only nontrivial thing is that the radical vector $\partial/\partial r$ is orthogonal to $\{r = \text{const}\}$ with respect to $(\exp_p)^* g$, but that is what Gauss proves. □

21 October 21, 2016

If $0 \in V \subset T_pM$ is open such that $\exp_p : V \to \exp_p(V)$ is a diffeomorphism, then we say that $U = \exp_p V$ is a **normal neighborhood**. (This just saying that U has normal coordinates.) If $\overline{B_\epsilon(0)} \subseteq V$, then $B_\epsilon(p) = \exp_p(B_\epsilon(0))$ is called the **normal ball** and $S_\epsilon(p) = \partial B_\epsilon(p)$ is called the **normal sphere**. By Gauss's lemma, $S_\epsilon(p)$ is orthogonal to the radial geodesics.

21.1 Geodesics are locally length minimizing

Theorem 21.1. *If $B_\epsilon(p)$ is a normal ball, and $\gamma : [0,1] \to B_\epsilon(p)$ is any geodesic, $\gamma(0) = p$, and if $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = \gamma(1)$, then* $\mathrm{length}(c) \geq \mathrm{length}(\gamma)$ *with equality if and only if $c([0,1]) = \gamma([0,1])$.*

Proof. Assume $c([0,1]) \subseteq B = B_\epsilon(p)$. Then since $\exp_p$ is a diffeomorphism, we can write

$$c(t) = \exp_p(r(t)v(t)),$$

where $r(t) > 0$ and $|v(t)| = 1$. It might be that $c(t_1) = p$ for $t_1 > 0$ so $v(t)$ is not well-defined, but then we can consider $c|_{[t_1,1]}$ instead. So we can assume $c(t) \neq p$ for $t > 0$. Write $f(s,t) = \exp_p(sv(t))$ so that $c(t) = f(r(t),t)$. Then

$$\frac{dc}{dt} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial f}{\partial t}$$

and so by Gauss's lemma, $\langle \partial f/\partial r, \partial f/\partial t \rangle = 0$. So

$$\Big|\frac{dc}{dt}\Big|^2 = |r'|^2 \Big|\frac{\partial f}{\partial r}\Big|^2 + \Big|\frac{\partial f}{\partial t}\Big|^2 = |r'|^2 \Big\langle \frac{d}{dr}\exp_p(rv), \frac{d}{dt}\exp_p(rv)\Big\rangle + \Big|\frac{\partial f}{\partial t}\Big|^2 \geq |r'|^2.$$

So for any $\delta > 0$,

$$\int_\delta^1 |c'| dt \geq \int_\delta^1 |r'| dt \geq \Big|\int_\delta^1 r' dt\Big| = r(1) - r(\delta).$$

Taking the limit as $\delta \to 0$, we get that $\mathrm{length}(c) \geq r(1) = \mathrm{length}(\gamma)$. If equality fholds, then $\partial f/\partial t = 0$ so $v(t) = v(0)$. That is, c is a reparametrization of γ.

If $c([0,1]) \subsetneq B$, then the distance from p to the first point getting outside B is at least $\epsilon > \mathrm{length}(\gamma)$. $\square$

21.2 Globally length minimizing curves are geodesics

The question I want to ask is:

> When can we find *long* legnth minimizing geodesics?

Theorem 21.2. *Fore very point $p \in M$, there exists a neighborhood $p \in W \subseteq M$ and $\delta > 0$ such that, for each $q \in M$ there exists a diffeomorphism $\exp_q : B_\delta(0) \to M$ onto its image, and $\exp_q(B_\delta(0)) \supseteq W$, i.e., W is a normal neighborhood of every $q \in W$.*

If this condition is satisfied, we call W a **totally normal neighborhood**.

Proof. By the existence theorem for geodesics, there exists an $V \subseteq M$, $p \in V$ and $\epsilon > 0$ such that for every $q \in V$ the map $\exp_q$ is defined on $B_\epsilon(0) \subseteq T_qM$. Consider

$$U = \{(q, w) \in TM : q \in V, |w| < \epsilon\}, \quad F : (q, w) \mapsto (q, \exp_q w) \in M \times M.$$

Then

$$dF_{(p,0)} = \begin{pmatrix} \mathbf{1} & * \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

and so F is a local diffeomorphism near $(p, 0)$. Then there exists a $\tilde{U} \subseteq U$, in particular,

$$\tilde{U} = \{q \in \tilde{V}, w \in T_qM, |w| < \delta\}$$

for an open neighborhood $\tilde{V} \subset M$ of p, such that $F : \tilde{U} \hookrightarrow M \times M$ is a diffeomorphism. Take $W \subseteq M$ such that $W \times W \subseteq F(\tilde{U})$. Then $F(\{q\} \times B_\delta(0)) \supseteq q \times W$ and so $\exp_q B_\delta(0) \supseteq W$. $\square$

Corollary 21.3. *If $\gamma : [a, b] \to M$ is piecewise differetiable and $|\gamma'| = 1$ (where this makes sense) and for any other curve c connecting $\gamma(a)$ to $\gamma(b)$ we have* $\text{length}(c) \geq \text{length}(\gamma)$ *the γ is a geodesic and so C^∞.*

Proof. Take $t \in [a, b]$. Let W be a totally normal neighborhood of $\gamma(t)$. There exists an open interval $t \in I \subseteq [a, b]$ such that $\gamma|_I : I \to W$. If γ_I connects points $p, q \in W$, then $B_\delta(p) \ni q$ so there is a radial geodesic $\tilde{\gamma}$ joining p to q. Now $\text{length}(\tilde{\gamma}) = \text{length}(\gamma)$, because γ is length minimizing. Then $\gamma(I) = \tilde{\gamma}(I)$ but $|\gamma'| = 1$ so γ is a geodesic locally. Thus it is a geodesic and so γ is C^∞. $\square$

22 October 22, 2016

22.1 Completeness

Definition 22.1. A Riemannian manifold (M, g) is **geodesically complete** if for every $p \in M$ and $v \in T_pM$, $\exp_p(v)$ is defined, i.e., geodesics exist for all time.

Theorem 22.2 (Hopf-Rinnow)**.** *Let (M, g) be a Riemannian manifold and $p \in M$. Then the following are equivalent:*

(a) $\exp_p$ is defined on all of T_pM.

(b) The closed and bounded sets in (M, g) are compact.

(c) M is complete as a metric space.

(d) M is geodesically complete.

(e) There exist a sequence of compact sets $K_n \subseteq M$ with $K_n \subseteq K_{n+1}$ such that if $q_n \in M \setminus K_n$ then $d(p, q_n) \to \infty$ as $n \to \infty$.

Moreover, any of (a)-(e) imply:

(f) For any $q \in M$ there exist a geodesic γ joining p to q with length $\gamma = d(p, q)$.

Proof. The hardest thing is (a) $\Rightarrow$ (f). Let $B_\delta(p)$ be a normal ball, and let $x_0 \in \partial B_\delta(p)$ be the point where $d(q, \partial_\delta(p)) = d(q, x_0)$ is achieved. We can write $x_0 = \exp_p(\delta v)$ where $v \in T_pM$ with $|v| = 1$. This gives our candidate geodesic $\gamma(t) = \exp_p(vt)$ which exists for all time.

Let $r = d(p, q)$ and let

$$A = \{t \in [0, r] : d(\gamma(t), q) = r - t\}.$$

Clearly $0 \in A$ and A is closed by continuity. Now we show that A is open. Then it follows that $A = [0, r]$. Let $t_1 \in A$ and $x_1 = \exp_p(t_1 v)$ so that $d(x_1, q) = r - t_1$. Then $d(p, x_1) \le t_1$ but $r = d(p, q) \le d(p, x_1) + d(x_1, q)$. So $d(p, x_1) = t_1$.

Let $B_{\delta_1}(x_1)$ be a normal ball around x_1, and let $y_1 \in \partial B_{\delta_1}(x_1)$ be such that $d(q, y_1) = d(q, \partial B_{\delta_1}(x_1))$. We claim that $d(q, y_1) = r - t_1 - \delta_1$. Fix an $\epsilon > 0$ and let C be a curve from q to x_1 with $r - t_1 \le \text{length}(c) \le r - t_1 + \epsilon$. Let $\hat{c}$ be the portion of c occurring after the first time that c intersect $B_{\delta_1}(x_1)$, and let $\hat{\gamma}$ be the geodesic connecting $c(T) \in B_{\delta_1}(x_1)$ and x_1, where T is the first time of intersection. Let $\tilde{c} = \hat{c} \cup \hat{\gamma}$ that is piecewise C^∞. Then by the triangle inequality,

$$\begin{aligned} r - t_1 &\le \text{length}(\tilde{c}) = \text{length}(\hat{\gamma}) + \text{length}(\hat{c}) = \delta_1 + \text{length}(\hat{c}) \\ &\le \text{length}(c) = r - t_1 + \epsilon. \end{aligned}$$

This shows that $\text{length}(\hat{c}) \le r - t_1 - \delta_1 + \epsilon$, and so $d(q, y_1) \le r - t_1 - \delta_1$. Since $d(q, y_1) \ge r - t_1 - \delta_1$, we get

$$d(q, y_1) = r - t_1 - \delta_1.$$

We finally have to show that y_1 is actually on the geodesic $\exp_p(tv)$. By the triangle inequality, $d(p, y_1) \le t_1 + \delta_1$ and so from $d(p, q) = r$ it follows that $d(p, y_1) = t_1 + \delta_1$. Since $d(p, y_1)$ is achieved by the curve "$\exp_p(tv)$ for $0 \le t \le t_1$ and the radial geodesic from x_1 to y_1", this curve is a smooth geodesic. So $y_1 = \exp_p((t_1 + \delta_1)v)$ and therefore $t_1 + \delta_1 \in A$.

The rest is easy. We prove (a) $\Rightarrow$ (b). Let A be closed and bounded. Then

$$A \subseteq \{\exp_p(tv) : v \in T_pM, |v| = 1, 0 \le t \le R\}$$

for some large R by (f). Then A is the image of a compact set under a continuous map $\exp_p$. This implies that A is compact.

Proving (b) $\Rightarrow$ (c) is just point-set topology.

For (c) $\Rightarrow$ (d), let $q \in M$, $v \in T_qM$, $|v| = 1$. Suppose the geodesic $\gamma(t) = \exp_q(vt)$ is defined only for $t \in [0, T)$. Since $d(\gamma(t_1), \gamma(t_2)) \le |t_1 - t_2|$, completeness implies that $\gamma(t) \to q_t$ as $t \to T$. We will finish this next time. $\square$

23 October 24, 2016

Last time we were proving (c) $\Rightarrow$ (d). Let M be a complete manifold and $q \in M$, $v \in T_qM$, $|v| = 1$. Suppose that $\exp_q(vt)$ exists on $[0, T)$. Then by completeness, $\gamma(t) \to q_T$ as $t \to T$. Take a totally normal neighborhood of q_T. Then for every $p \in W$, $\exp_p(b)$ is defined. Choose $t_* \in [0, T)$ so that $|t_* - T| < \delta/2$. Then consider $C(s) = \exp_{\gamma(t_*)}(s\gamma'(t_*))$. This $C(s)$ exists for $|s| < \delta$ and this extends the geodesic $\gamma(t)$ past T.

Finally, (d) $\Rightarrow$ (a) is obvious and (b) is equivalent to (c).

Corollary 23.1. *If (M, g) is compact, then it is complete. More interestingly, if $N \subseteq (M, g)$ with the induced metric, N is closed, and (M, g) is complete, then $(N, g|_N)$ is complete. In particular, closed submanifolds of $(\mathbb{R}^n, g_{\mathrm{Euc}})$ are complete.*

23.1 Connections

How do we do calculus on sections of vector bundles? Suppose you have an electric field on earth or something and you want to differentiate. It is not obvious how to differentiate. Let us first make a naïve attempt. Let $\pi : E \to M$ be a C^∞ vector bundle and let $\sigma \in \Gamma(U, E)$ be a section on an open set $U \subseteq M$. U has coordinates $(x^1, \dots, x^n)$, and let $\{e_1, \dots, e_r\}$ be a local frame for E. Then we can write

$$\sigma = \sigma^1 e_1 + \cdots + \sigma^r e_r$$

for C^∞ functions $\sigma^1, \dots, \sigma^r : U \to \mathbb{R}$. Can we now define

$$\frac{\partial \sigma}{\partial x^i} = \Big(\frac{\partial \sigma^1}{\partial x^i}, \dots, \frac{\partial \sigma^r}{\partial x^i}\Big)?$$

Suppose we choose instead $\{\tilde{e}_1, \dots, \tilde{e}_r\}$. Then we have a map $g : U \to \mathrm{GL}(r)$ such that $\{e\} = g\{\tilde{e}\}$, i.e., $e_i = g_i^l \tilde{e}_l$. Then $\tilde{\sigma} = g\sigma$ and so

$$\frac{\partial}{\partial x^i}\begin{pmatrix}\tilde{\sigma}^1 \\ \vdots \\ \tilde{\sigma}^r\end{pmatrix} = \frac{\partial g}{\partial x^i}\begin{pmatrix}\sigma^1 \\ \vdots \\ \sigma^r\end{pmatrix} + g\frac{\partial}{\partial x^i}\begin{pmatrix}\sigma^1 \\ \vdots \\ \sigma^r\end{pmatrix}.$$

To fix this defect, consider

$$\nabla_i = \frac{\partial}{\partial x^i} + A_i$$

where A_i is a linear operator in the $\{e\}$ frame. For the differentiation to be well-defined, we need $g\nabla_i\sigma = \tilde{\nabla}_i\tilde{\sigma}$. After some computation, this reduces to

$$A_i = g^{-1}\frac{\partial g}{\partial x^i} + g^{-1}\tilde{A}_i g.$$

Definition 23.2. A **covariant derivative** or a **connection** on E is a map

$$\nabla : C^\infty(M, E) \to C^\infty(E \otimes T^*M)$$

such that

(1) $\nabla(s + s') = \nabla s + \nabla s'$,
(2) $\nabla(fs) = f\nabla s + s \otimes df$ for any $f \in C^\infty(M, \mathbb{R})$.

Lemma 23.3. *If ∇ and ∇' are two connections, then $\nabla - \nabla' \in \mathrm{End}(E) \otimes T^*M$.*

Proof. Note that $(\nabla - \nabla')(s_1 + s_2) = (\nabla - \nabla')s_1 + (\nabla - \nabla')s_2$ by linearity. Also if $f \in C^\infty(M, \mathbb{R})$ then

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s + df \otimes s - df \otimes s = f(\nabla - \nabla')s$$

which is what we need. Then we conclude by the next lemma. □

Lemma 23.4. *Let E and E' be vector bundles and $L : E \to E'$ be such that $L(s_1 + s_2) = L(s_1) + L(s_2)$ and $L(fs_1) = fL(s_1)$ for each $f \in C^\infty(M)$. Then $L \in \mathrm{Hom}(E, E')$.*

Proof. Let $U \subseteq M$ be an open set and $\{e_1, \ldots, e_r\}$ be a local frame for E, $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ be a local frame for E'. Then we can write $L(e_i) = a_i^k \tilde{e}_k$. Then for any $\sigma \in \Gamma(U, E)$, write $\sigma = \sigma^i e_i$ with $\sigma^i : U \to \mathbb{R}$. By assumption, $L(\sigma) = \sigma^i a_i^k \tilde{e}_k$. So L is determined by (a_i^k). So L transforms as a section of $\mathrm{Hom}(E, E')$. □

This shows that the space of connections is affine, and after choice of a base point, is isomorphic to $\mathrm{Hom}(E, E \otimes T^*M)$. Next time, we will prove that connections exist and $\nabla = d + A$ for some A.

24 October 28, 2016

We were discussing connections on vector bundles. A connection is a map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that $\nabla s(v) = \nabla_v s$ is like the directional derivative for $v \in TM$. We saw that connections "should" be of the form $\nabla = d + A$ where, given a frame, A is matrix-valued 1-form.

24.1 Construction of connections

Proposition 24.1. *Let $E \to M$ be a vector bundle and let M be compact. Then there exists a connection.*

We remark that compactness is not necessary–it can be dropped.

Proof. We first assume that $E = M \times \mathbb{R}^r$ is the trivial bundle. A section of E is an r-tuple of C^∞ functions $\sigma = (f_1, \dots, f_r) : M \to \mathbb{R}^r$. Define

$$\nabla\sigma = (x, df_1(x), \dots, df_r(x)) \in \mathbb{R}^r \otimes T^*M,$$

where $x \in M$. This defines a connection.

Now consider the general case $E \to M$. Then there is map $i : E \hookrightarrow M \times \mathbb{R}^r$. Fix a (hermitian) metric H on $\mathbb{R}^r$, and let $\pi_{i(E)} = \pi$ be the orthogonal projection onto $E \subseteq M \times \mathbb{R}^r$. Given $\sigma \in \Gamma(M, E)$, define

$$\nabla^E \sigma = i^{-1}(\pi \nabla^{\mathbb{R}^r} i(\sigma)).$$

We claim that ∇^E is a connection. To show this we need to check the axioms. Clearly $\nabla^E(\sigma_1 + \sigma_2) = \nabla^E\sigma_1 + \nabla^E\sigma_2$. Also

$$\nabla^E(f\sigma) = i^{-1}\pi(i(\sigma) \otimes df + f\nabla^{\mathbb{R}^r} i(\sigma)) = \sigma \otimes df + f\nabla^E\sigma.$$

This shows that ∇^E is a connection. □

Let's see what this looks like locally. Let $\{e_1, \dots, e_k\}$ be a frame for E such htat $i(e_1), \dots, i(e_k)$ is orthonormal in $M \times \mathbb{R}^r$. Choose $s_{k+1}, \dots, s_r$ be such that $\{i(e_1), \dots, i(e_k), s_{k+1}, \dots, s_r\}$ is an orthonormal frame for $M \times \mathbb{R}^r$. Next, we have another frame $\{f_1, \dots, f_r\}$ adapted to $\nabla^{\mathbb{R}^r}$ so that f_i is the section corresponding to the $(0, \dots, 0, 1, 0, \dots, 0)$.

Let g be the map

$$g : \{f_i\} \to \{i(e_1), \dots, i(e_k), s_{k+1}, \dots, s_r\},$$

and let σ be a section of E. We can write $\sigma = \sigma^1 e_1 + \cdots + \sigma^k e_k$ and then $i(\sigma) = \sigma^1 i(e_1) + \cdots + \sigma^k i(e_k)$. Then

$$g \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_r \end{pmatrix},$$

whre v^i are the coordinates of $i(\sigma)$ in the frame $\{f_i\}$. Then

$$\nabla^{\mathbb{R}^r} i(\sigma) = d\Big(g \begin{pmatrix} (\sigma^j) \\ (0) \end{pmatrix}\Big) = dg \begin{pmatrix} (\sigma^j) \\ (0) \end{pmatrix} + gd \begin{pmatrix} (\sigma^j) \\ (0) \end{pmatrix}.$$

In order to calculuate $i^{-1}\pi$, we need to write it back in $\{i(e_1), \ldots, s_r\}$. In this frame,

$$\pi\nabla^{\mathbb{R}^r} i(\sigma) = \pi g^{-1} dg \begin{pmatrix} (\sigma^j) \\ (0) \end{pmatrix} + d\left((\sigma^j)\right).$$

Note that this has this form of $\nabla^E \sigma = d\sigma + A\sigma$ where A is a matrix-valued 1-form $\pi g^{-1} dg = A$.

Corollary 24.2. *Any connection is written in a frame a $\nabla = d + A$, where A is a matrix valued 1-form.*

Proof. If $\tilde{\nabla}$ is any connection of E, then $\tilde{\nabla} = \nabla^E + T$ where $T \in \mathrm{End}(E) \otimes T^*M$. Clearly T is given as a matrix. $\square$

24.2 Parallel transport

One reason ∇ is called a connection is that it allows us to "connect" different fibers of $E \to M$. Given $p, q \in M$, let $\gamma(t)$ be a curve $\gamma(0) = p$ and $\gamma(1) = q$. Let $\sigma_p \in E_p$. A map $\sigma(t) : [0,1] \to E$ with $\sigma(t) \in E_\gamma(t)$ is the **parallel transport** of σ_p if

$$\sigma(0) = \sigma_p \quad \text{and} \quad \nabla_{\dot{\gamma}(t)} \sigma = 0.$$

In a frame, the equation can be written as

$$\frac{d\sigma^\alpha}{dt} + A^\alpha_{i\beta} \dot{\gamma}^i \sigma^\beta(t) = 0 \qquad \text{and} \qquad \sigma^\alpha(0) = \sigma^\alpha(p).$$

By existence and uniqueness of ordinary differential equations, the parallel transport exists and is unique.

25 October 31, 2016

25.1 The Levi-Civita connection

Theorem 25.1 (Levi-Civita). *Let (M, g) be a Riemannian manifold. There is a unique connection on TM such that*

(1) $\nabla_X Y - \nabla_Y X = [X, Y]$ (torsion-free),

(2) $X\langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$ (metric compatible).

In local coordinates $(x^1, \dots, x^n)$, the connections coefficients are Γ^k_{ij} the Christoffel symbols.

Proof. It suffices to determine $\langle \nabla_X, Z\rangle_g$ for all X, Y, Z. We have

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = \langle \nabla_X Z, Y\rangle + \langle [Z, X], Y\rangle + \langle X, \nabla_Z Y\rangle, \\ Y\langle X, Z\rangle &= \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle \\ &= \langle \nabla_X Y, Z\rangle + \langle [Y, X], Z\rangle + \langle X, \nabla_X Y\rangle + \langle X, [Y, Z]\rangle. \end{aligned}$$

So we have

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\{ & X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle + \langle [X, Y], Z\rangle\}. \end{aligned}$$

This shows uniqueness.

In local coordinates,

$$\langle \nabla_{\partial/\partial x^i} \partial/\partial x^j, \partial/\partial x^k\rangle = \frac{1}{2}\{\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\} = \Gamma^k_{ij}.$$

In other words, $\nabla_{\partial/\partial x^i} \partial/\partial x^j = \Gamma^l_{ij} \partial/\partial x^l$. □

25.2 De Rham differential

Recall that for $f \in C^\infty(M, \mathbb{R})$, then $df \in T^*M$. Then we can define the de Rham differential d as

$$d : \Gamma(M, \textstyle\bigwedge^p T^*M) \to \Gamma(M, \textstyle\bigwedge^{p+1} T^*M),$$

as in the homework. Recall that $d^2 = 0$. So we have the **cohomology groups** defined as

$$H^p_{\mathrm{dR}}(M) = \frac{\ker d : \bigwedge^p T^*M \to \bigwedge^{p+1} T^*M}{\operatorname{im} d : \bigwedge^{p-1} T^*M \to \bigwedge^p T^*M}.$$

Given a connection ∇ on $E \to M$, we can define

$$d_\nabla : \textstyle\bigwedge^p T^*M \otimes E \to \bigwedge^{p+1} T^*M \otimes E$$

in the following way.

Definition 25.2. The **exterior covariant derivative** d_∇ (or d_A for $\nabla = d + A$) is defined as

(1) if $\sigma \in \Gamma(M, E)$ and $\omega \in \bigwedge^p T^*M$ then

$$d_\nabla(\sigma \otimes \omega) = \nabla\sigma \wedge \omega + \sigma \otimes d\omega,$$

(2) $d_\nabla(s_1 + s_2) = d_\nabla(s_1) + d_\nabla(s_2)$ for $s_1, s_2 \in \Gamma(M, \bigwedge^p T^*M \otimes E)$.

So if E is the trivial bundle, then this is just the de Rham derivative. In a trivialization,

$$\sigma \otimes \omega = \sigma^\alpha dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad \text{maps to}$$
$$d_\nabla(\sigma \otimes \omega) = \left(\frac{\partial}{\partial x^l}\sigma^\alpha + A^\alpha_{l\beta}\sigma^\beta\right) dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Also

$$\begin{aligned} d^2_\nabla(\sigma dx^I) &= d_\nabla(\partial_j\sigma + A_j\sigma)dx^j \wedge dx^I \\ &= \sum_{k,j}(\partial_k\partial_j\sigma + (\partial_k A_j)\sigma + A_j\partial_k\sigma + A_k\partial_j\sigma + A_kA_j\sigma)dx^k \wedge dx^j \wedge dx^I \\ &= \sum_{k<j}(\partial_k A_j - \partial_j A_k + A_kA_j - A_jA_k)\sigma dx^k \wedge dx^j \wedge dx^I. \end{aligned}$$

This is not necessarily zero! Viewing A as a matrix valued 1-form, $d_\nabla s = ds + A \wedge s$ and so $d^2_\nabla = (dA + A \wedge A) \wedge s$.

Definition 25.3. The **curvature 2-form** is defined as

$$F_\nabla = dA + A \wedge A \in \bigwedge^2 T^*M \otimes \operatorname{End}(E).$$

26 November 2, 2016

Last time, for a vector bundle $E \to M$ and a connection $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$, we defined $d_\nabla : \Gamma(E \otimes \bigwedge^p T^*M) \to \Gamma(E \otimes \bigwedge^{p+1} T^*M)$. In general, $d_\nabla^2 \neq 0$ if $\nabla = d + A$. We defined

$$F_\nabla = dA + A \wedge A = d_\nabla^2 \in \operatorname{End}(E) \otimes \textstyle\bigwedge^2 T^*M.$$

26.1 Induced connections

Given a connection $(E, \nabla) \to M$, how do we induce a connection on E^*? The idea is that we require ∇^{E^*} to be compatible with ∇^E and the map $E \otimes E^* \to C^\infty(M)$. We can demand

$$d(\tau(\sigma)) = (\nabla^{E^*}\tau)(\sigma) + \tau(\nabla^E \sigma).$$

Let's see how this looks. Fix a frame $\{e_1, \dots, e_r\}$ for E and let $\{e_1^*, \dots, e_r^*\}$ be the dual frame for E^*. We can write $\nabla^E = d + A$ in this frame. Write $\tau = \sum_\alpha \tau_\alpha e_\alpha^*$ and $\sigma = \sum_\alpha \sigma^\alpha e_\alpha$. Then

$$\begin{aligned} \frac{\partial}{\partial x^i}(\tau(\sigma)) &= \frac{\partial}{\partial x^i}\Big(\sum_\alpha \tau_\alpha \sigma^\alpha\Big) = \sum_\alpha \Big(\frac{\partial \tau_\alpha}{\partial x^i}\Big)\sigma^\alpha + \sum_\alpha \tau_\alpha \Big(\frac{\partial \sigma^\alpha}{\partial x^i}\Big), \\ \tau\big(\nabla^E_{\partial/\partial x^i}\sigma\big) &= \tau_\alpha \big(\nabla^E_{\partial/\partial x^i}\sigma\big)^\alpha = \tau_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} + \tau_\alpha A^\alpha_{i\beta}\sigma^\beta. \end{aligned}$$

So

$$\big(\nabla^{E^*}_{\partial/\partial x^i}\tau\big)_\alpha \sigma^\alpha = \frac{\partial \tau_\alpha}{\partial x^i}\sigma^\alpha - \tau_\alpha A^\alpha_{i\beta}\sigma^\beta,$$

and after switching α and β on the last component, we deduce

$$\big(\nabla^{E^*}_{\partial/\partial x^i}\tau\big)_\alpha = \frac{\partial \tau_\alpha}{\partial x^i} - \tau_\beta A^\beta_{i\alpha}.$$

In matrices, if $\tau = (\tau_1, \dots, \tau_r)$ then

$$\nabla^{E^*}_{\partial/\partial x^i}\tau = \frac{\partial}{\partial x^i}\tau - \tau A_i.$$

Now let us look at connections on tensor products. If (E, ∇^E) and (F, ∇^F) are connections, then define $\nabla^{E\otimes F}$ on $E \otimes F$ by

$$\nabla(\sigma \otimes \tau) = (\nabla^E \sigma) \otimes \tau + \sigma \otimes (\nabla^F \tau).$$

As an example, suppose we have (E, ∇^E) and $T \in \operatorname{End}(E) = E \otimes E^*$. If $\nabla^E = d + A$ then we have

$$\nabla T = dT + AT - TA.$$

Corollary 26.1 (Second Bianchi identity). *If ∇ is the connection on E and F_∇ is the curvature, then $d_\nabla F_\nabla = 0$.*

Proof. Because $F_\nabla = dA + A \wedge A$, we have

$$\begin{aligned} d_\nabla F_\nabla &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A \\ &= 0 + dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A = 0. \end{aligned}$$ □

26.2 Characteristic class

Recall that we have the trace map $\mathrm{Tr} : \mathrm{End}(E) \otimes \bigwedge^p T^*M \to \bigwedge^p T^*M$, defined by extending the map $\mathrm{Tr} : \mathrm{End}(E) \to C^\infty(M)$ linearly.

Proposition 26.2. $d\,\mathrm{Tr}(B) = \mathrm{Tr}(d_\nabla B)$.

Proof. If $\nabla = d + A$ on E then $d_\nabla B = dB + AB - BA$. Then

$$\mathrm{Tr}(d_\nabla B) = \mathrm{Tr}(dB) + \mathrm{Tr}([A, B]) = \mathrm{Tr}(dB) = d\,\mathrm{Tr}(B).$$ □

Corollary 26.3. *For any connection ∇ on E, we have $\mathrm{Tr}(F_\nabla)$ defines an element of $H^2_{\mathrm{dR}}(M)$.*

This is an example of a **characteristic class**.

Proposition 26.4. *The cohomology $[\mathrm{Tr}(F_\nabla)]_{\mathrm{dR}}$ depends only on E and not on ∇.*

Proof. Let $\nabla, \tilde{\nabla}$ be connections on E, and write $\nabla = d + A$. Write $\tilde{\nabla} = \nabla + B$ where $B \in \mathrm{End}(E) \otimes T^*M$. Then

$$\begin{aligned} F_{\tilde{\nabla}} &= dA + dB + A \wedge A + A \wedge B + B \wedge A + B \wedge B \\ &= F_\nabla + dB + A \wedge B + B \wedge A + B \wedge B. \end{aligned}$$

Write $A_i dx^i = A$ and $B_j dx^j = B$. Then the $dx^i \wedge dx_j$ component of $A \wedge B + B \wedge A + B \wedge B$ component is

$$(A_iB_j - A_jB_i) + (B_iA_j - B_jA_i) + (B_iB_j - B_jB_i) = [A_i, B_j] - [A_j, B_i] + [B_i, B_j]$$

that has trace 0. Also $\mathrm{Tr}(dB) = d\,\mathrm{Tr}(B)$. This shows that $\mathrm{Tr}(F_\nabla)$ and $\mathrm{Tr}(F_{\tilde{\nabla}})$ lies in the same cohomology class. □

27 November 7, 2016

If you have a vector bundle $E \to M$ with a connection ∇, then $\mathrm{Tr}(F_\nabla)$ defines a class inside $H^2_{\mathrm{dR}}(M)$ that is independent of ∇.

27.1 Ad-invariant functions

More generally, let $f : M(n, \mathbb{C}) \to \mathbb{C}$ be a C^∞ function such that $f(gmg^{-1}) = f(m)$ for all $g \in \mathrm{GL}(n, \mathbb{C})$, i.e., f is **Ad-invariant**. For example, $f_k(m) = \mathrm{Tr}(m^k)$. If f is Ad invariant and f is real analytic, then we can write

$$f(tm) = f_0 + tf_1(m) + t^2 f_2(m) + \cdots .$$

Then $f_k : M(n, \mathbb{C}) \to \mathbb{C}$ are Ad-invariant and homogeneous of degree k. Important examples include

$$c(tm) = \det\Big(\mathbf{1} + \frac{i}{2\pi} tm\Big), \quad \mathrm{Ch}(tm) = \mathrm{Tr}\Big(\exp\Big(\frac{it}{2\pi} m\Big)\Big).$$

Theorem 27.1 (Ad-invariant function theorem)**.** *The vector space of Ad-invariant real analytic functions, homogeneous of degree p, is the $\mathbb{C}$-linear span of*

$$\{\mathrm{Tr}(m^{k_1}) \cdots \mathrm{Tr}(m^{k_q}) : k_1 + \cdots + k_q = p\}.$$

Proof. We first claim that any C^∞ Ad-invariant function on $M(n, \mathbb{C})$ is determined by a symmetric, C^∞ function on $\mathbb{C}^n$. Note that there is a dense open subset $U \subseteq M(n, \mathbb{C})$ of diagonalizable matrices $U \subseteq M(n, \mathbb{C})$ of diagonalizable matrices on U. Then any Ad-invariant function is induces by a C^∞ function on $\mathbb{C}^n$, and vice versa. Moreover, this function is symmetric because

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So it suffices to understand functions on $\mathbb{C}^n$. Let us write

$$\prod_{j=1}^n (1 + \lambda_j u) = 1 + \sigma_1(\lambda) u + \sigma_2(\lambda) u^2 + \cdots + \sigma_n(\lambda) u^n.$$

Theorem 27.2 (Fundamental theorem of symmetric polynomials)**.** *The ring of symmetric polynomilas is generated by $\sigma_0, \ldots, \sigma_n$ as an algebra, i.e., if $\tau(\lambda)$ is any symmetric polynomial, then $\tau(\lambda) = p(\sigma_0, \sigma_1, \ldots, \sigma_n)$ for some polynomial p with $\mathbb{C}$-coefficients.*

Observe

$$\log\big(\textstyle\prod_j (1 + \lambda_j u)\big) = \displaystyle\sum_j \Big(\lambda_j u - \frac{\lambda_j^2}{2} u^2 + \cdots + (-1)^{k+1} \frac{\lambda_j^k}{k!} u^k + \cdots\Big).$$

Thus $\sum_{j=1}^n \lambda_j^p$ for $1 \le p \le n$ generate $\sigma_1, \ldots, \sigma_n$ as an algebra. So we can change bases. $\square$

27.2 More characteristic classes

Going back to our two examples of Ad-invariant functions, we have

$$c(tm) = 1 + \frac{i}{2\pi}\operatorname{Tr}(m)t - \frac{1}{(8\pi)^2}((\operatorname{Tr} m)^2 - (\operatorname{Tr} m^2))t^2 + \cdots$$
$$\operatorname{Ch}(tm) = 1 + \sum_{k\geq 1} \frac{1}{k!}\Big(\frac{i}{2\pi}\Big)^k \operatorname{Tr} m^k.$$

Theorem 27.3. *Every Ad-invariant real analytic function f defines a cohomology class $C_f(E) \in H^*_{\mathrm{dR}}(M)$, by setting*

$$[C_f(E)] = [f(F_\nabla)].$$

In particular, this is independent of the choice of ∇.

Let's say exactly what we mean. If $f_k = \operatorname{Tr}(m^k)$, then

$$f_k(F_\nabla) = \operatorname{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla) \in \textstyle\bigwedge^{2k} T^*M,$$

and

$$f_l(F_\nabla) f_k(F_\nabla) = \operatorname{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla) \wedge \operatorname{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla) \in \textstyle\bigwedge^{2(k+l)} T^*M.$$

Note that there is no ambiguity in multiplication because $\bigoplus_k \bigwedge^{2k} T^*M$ is a commutative algebra.

28 November 9, 2016

Last time, for a bundle $E \to M$ and a connection ∇ on E, for every Ad-invariant polynomial, we got a characteristic class $C_f(E) = f(F_\nabla) \in H^*_{\mathrm{dR}}(M)$. Also we showed that Ad-invariant polynomials are generated as an algebra by $m \mapsto \mathrm{Tr}(m^k)$.

28.1 Ad-invariance functions give well-defined cohomology

Proposition 28.1. $\mathrm{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla)$ *is a closed* $2k$*-form and its cohomology class is independent of the choice of* ∇ *on* E*.*

Combining this statement with what we proved last time, we get the same statement for every Ad-invariant f.

Proof. Because $F_\nabla \wedge \cdots \wedge F_\nabla \in \mathrm{End}(E) \otimes \bigwedge^{2k} T^*M$,

$$d_\nabla(F_\nabla \wedge \cdots \wedge F_\nabla) = (d_\nabla F_\nabla) \wedge \cdots \wedge F_\nabla + \cdots + F_\nabla \wedge \cdots \wedge F_\nabla \wedge (d_\nabla F_\nabla) = 0$$

by Bianchi. Now for every $B \in \mathrm{End}(E) \otimes \bigwedge^p T^*M$,

$$d\,\mathrm{Tr}(B) = \mathrm{Tr}(dB) = \mathrm{Tr}(d_\nabla B)$$

because d and d_∇ differ by a commutator. This shows that $\mathrm{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla)$ is closed.

Let $\nabla = d + A$ ad $\tilde{\nabla} = \nabla + B$ with $B \in \mathrm{End}(E) \otimes T^*M$. Define $\nabla^t = \nabla + tB$ and consider $\mathrm{Tr}(F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t})$. Then

$$\frac{d}{dt}\mathrm{Tr}(F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t}) = k\,\mathrm{Tr}(\dot{F}_{\nabla^t} \wedge F_{\nabla^t} \wedge \cdots F_{\nabla^t}).$$

Because $F_{\nabla^t} = dA + tdB + (A + tB) \wedge (A + tB)$, we get

$$\dot{F}_{\nabla^t} = dB + A \wedge B + B \wedge A + 2tB \wedge B.$$

Now we claim that $dB + A \wedge B + B \wedge A = d_\nabla B$. If $B = B_j dx^j$, we have by definition

$$d_\nabla B = \nabla_{\partial/\partial x^i} B_j dx^i \wedge dx^j = (\partial_i B_j + A_i B_j - B_j A_i) dx^i \wedge dx^j.$$

On the other hand,

$$\begin{aligned} dB = d(B_j dx^j) &= \partial_i B_j dx^i \wedge dx^j, \\ A \wedge B &= A_i B_j dx^i \wedge dx^j, \\ B \wedge A &= B_j A_i dx^j \wedge dx^i. \end{aligned}$$

So $d_\nabla B = dB + A \wedge B + B \wedge A$.

Thus

$$\frac{d}{dt}\operatorname{Tr}(F_{\nabla^t}\wedge\cdots\wedge F_{\nabla^t}) = k\operatorname{Tr}(d_\nabla B\wedge F_{\nabla^t}\wedge\cdots\wedge F_{\nabla^t}) \\ + 2tk\operatorname{Tr}(B\wedge B\wedge F_{\nabla^t}\wedge\cdots\wedge F_{\nabla^t}).$$

Note also that $d_\nabla B + 2tB\wedge B = d_{\nabla^t}B$. Finally,

$$\operatorname{Tr}(d_{\nabla^t}B\wedge F_{\nabla^t}\wedge\cdots\wedge F_{\nabla^t}) = d\operatorname{Tr}(B\wedge F_{\nabla^t}\wedge\cdots\wedge F_{\nabla^t}).$$

So

$$\operatorname{Tr}(F^k_{\tilde\nabla}) - \operatorname{Tr}(F^k_{\nabla}) = d\int_0^1 \operatorname{Tr}(B\wedge F_{\nabla^t}\wedge\cdots\wedge F_{\nabla^t})dt$$

is a boundary. □

Lemma 28.2. *If E and E' are vector bundles over M and $E\cong E'$ then $C_f(E) = C_f(E')$ for any real analytic Ad-invariant $f : M(n,\mathbb{C})\to\mathbb{C}$.*

Proof. It suffices to show for $f_k = \operatorname{Tr}(m^k)$. Let us unravel the definitions. There is a $\sigma\in\operatorname{Hom}(E',E)$ with an inverse $\sigma^{-1}\in\operatorname{Hom}(E',E)$. In order to compute $C_{f_k}(E')$, fix a connection ∇' on E'. Define ∇ on E by $\nabla s = \sigma^{-1}\nabla'\sigma(s)$. If $\nabla' = d + A'$, then

$$\nabla = \sigma^{-1}d(\sigma s + A'\sigma s) = ds + (\sigma^{-1}d\sigma + \sigma^{-1}A'\sigma)s = (d+A)s.$$

Then

$$\begin{aligned} dA &= d(\sigma^{-1}d\sigma + \sigma^{-1}A'\sigma) \\ &= d(\sigma^{-1})\wedge d\sigma + (d\sigma^{-1})\wedge A'\sigma + \sigma^{-1}dA'\sigma - \sigma^{-1}A'\wedge d\sigma. \end{aligned}$$

Here, $d\sigma^{-1} = -\sigma^{-1}d\sigma\sigma^{-1}$ because $d(\sigma\sigma^{-1}) = 0$. So

$$dA = -\sigma^{-1}d\sigma\wedge\sigma^{-1}A'\sigma + \sigma^{-1}dA'\sigma - \sigma^{-1}A\wedge d\sigma.$$

Now let us compute $F_\nabla = dA + A\wedge A$. We have

$$A\wedge A = \sigma^{-1}A'\wedge A'\sigma + \sigma^{-1}d\sigma\wedge\sigma^{-1}A'\sigma + \sigma^{-1}A'\sigma\wedge\sigma^{-1}d\sigma.$$

Therefore

$$F_\nabla = dA + A\wedge A = \sigma^{-1}(dA' + A'\wedge A')\sigma = \sigma^{-1}F_{\nabla'}\sigma.$$

So we are done by Ad-invariance. □

29 November 11, 2016

Last time we showed that if E and E' are isomorphic then $C_f(E) = C_f(E')$ for every Ad-invariant f.

29.1 The Chern class and the Chern character

Let $E \to M$ be a *complex* vector bundle with a ($\mathbb{C}$-linear connection) ∇. The **k-th Chern class** is the characteristic class defined by $\sigma_k : M(n, \mathbb{C}) \to \mathbb{C}$, i.e., $c_k(E)$ is the coefficient of t^k in $\det(\mathbf{1} + (i/2\pi)F_\nabla t)$. For instance,

$$c_0(E) = 1, \quad c_1(E) = \frac{i}{2\pi} \operatorname{Tr}(F_\nabla),$$
$$c_2(E) = -\frac{1}{8\pi}(\operatorname{Tr}(F_\nabla) \wedge \operatorname{Tr}(F_\nabla) - \operatorname{Tr}(F_\nabla \wedge F_\nabla)), \quad \cdots .$$

The **k-th Chern character class** $c_k(E)$ is the expansion $\operatorname{Tr}(\exp((i/2\pi)tF_\nabla))$. So for instance,

$$\operatorname{ch}_0(E) = \operatorname{rk}(E), \quad \operatorname{ch}_1(E) = \frac{i}{2\pi} \operatorname{Tr}(F_\nabla) = c_1(E),$$
$$\operatorname{ch}_2(E) = -\frac{1}{4\pi^2} \operatorname{Tr}(F_\nabla \wedge F_\nabla), \quad \cdots, \quad \operatorname{ch}_k(E) = \Big(\frac{i}{2\pi}\Big)^k \operatorname{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla).$$

Using them, we can define the **total Chern class** and **total Chern character** as

$$c(E) = \det(\mathbf{1} + (i/2\pi)F_\nabla) = c_0(E) + c_1(E) + \cdots + c_{\lfloor n/2 \rfloor}(E),$$
$$\operatorname{ch}(E) = \operatorname{ch}_0(E) + \operatorname{ch}_1(E) + \cdots .$$

Proposition 29.1. *Let $E, \tilde{E}$ be vector bundles. Then*

(1) $\operatorname{ch}(E \otimes \tilde{E}) = \operatorname{ch}(E) \wedge \operatorname{ch}(\tilde{E})$,

(2) $\operatorname{ch}(E \oplus \tilde{E}) = \operatorname{ch}(E) + \operatorname{ch}(\tilde{E})$.

Proof. Let ∇^E and $\nabla^{\tilde{E}}$ be connections on E and $\tilde{E}$ respectively. Define $\nabla^{E\otimes\tilde{E}}$ by

$$\nabla^{E\otimes\tilde{E}}(\sigma \otimes \tau) = (\nabla^E \sigma) \otimes \tau + \sigma \otimes (\nabla^{\tilde{E}} \otimes \tau)$$

and extend by linearity.

We now claim that $F_{\nabla^{E\otimes\tilde{E}}} = F_{\nabla^E} \otimes \mathbf{1}_{\tilde{E}} + \mathbf{1}_E \otimes F_{\nabla^{\tilde{E}}}$. We can just compute

$$\begin{aligned} d_{\nabla^{E\otimes\tilde{E}}}(\sigma \otimes \tau) &= d_{\nabla^{E\otimes\tilde{E}}}(d_{\nabla^E}\sigma \otimes \tau + \sigma \otimes (d_{\nabla^{\tilde{E}}}\tau)) \\ &= (d_E^2\sigma) \otimes \tau - d_E\sigma \wedge d_{\tilde{E}}\tau + d_E\sigma \wedge d_{\tilde{E}}\tau + \sigma \otimes (d_{\tilde{E}}^2\tau) \\ &= F_E\sigma \otimes \tau + \sigma \otimes F_{\tilde{E}}\tau. \end{aligned}$$

Let $m_E = F_E \otimes \mathbf{1}_{\tilde{E}}$ and $m_{\tilde{E}} = \mathbf{1}_E \otimes F_{\tilde{E}}$. Then $m_E m_{\tilde{E}} = m_{\tilde{E}} m_E$ and so

$$\exp\Big(\frac{i}{2\pi}(m_E + m_{\tilde{E}})\Big) = \exp\Big(\frac{i}{2\pi} m_E\Big) \exp\Big(\frac{i}{2\pi} m_{\tilde{E}}\Big).$$

Now note that

$$m_E^n = F_E \wedge \cdots \wedge F_E \otimes \mathbf{1}_{\tilde{E}}.$$

So we get $\operatorname{ch}(E \otimes \tilde{E}) = \operatorname{ch}(E) \wedge \operatorname{ch}(\tilde{E})$.

The direct sum is easier. Because $(\nabla^E \oplus \nabla^{\tilde{E}})(\sigma \oplus \tau) = \nabla^E \sigma + \nabla^{\tilde{E}} \tau$, we have $F_{E \oplus \tilde{E}} = F_E \oplus F_{\tilde{E}}$. So $\operatorname{ch}(E \oplus \tilde{E}) = \operatorname{ch}(E) + \operatorname{ch}(\tilde{E})$. □

For example,

$$\begin{aligned}
\operatorname{ch}_0(E \otimes \tilde{E}) &= \operatorname{ch}_0(E) \operatorname{ch}_0(\tilde{E}) = \operatorname{rk}(E) \operatorname{rk}(\tilde{E}) = \operatorname{rk}(E \otimes \tilde{E}), \\
\operatorname{ch}_1(E \otimes \tilde{E}) &= \operatorname{ch}_0(E) \operatorname{ch}_1(\tilde{E}) + \operatorname{ch}_0(\tilde{E}) \operatorname{ch}_1(E) = \operatorname{rk}(E) c_1(\tilde{E}) + \operatorname{rk}(\tilde{E}) c_1(E), \\
\operatorname{ch}_2(E \otimes \tilde{E}) &= \operatorname{rk}(E) \operatorname{ch}_2(\tilde{E}) + \operatorname{rk}(\tilde{E}) \operatorname{ch}_2(E) + 2 \operatorname{ch}_1(E) \wedge \operatorname{ch}_1(\tilde{E}).
\end{aligned}$$

Lemma 29.2. *All characteristic classes of Ad-invariant analytic functions are generated by* $\operatorname{ch}_0, \ldots, \operatorname{ch}_{\lfloor n/2 \rfloor}$*, or equivalently by* $c_0, \ldots, c_{\lfloor n/2 \rfloor}$ *as an algebra.*

Proof. Ad-invariant function theorem. □

In the real vector bundle cases, things are not very interesting. For instance, it is a nice exercise to show that if $E \to M$ is a real vector bundle, then $[\operatorname{Tr}(F_\nabla)] = 0$. This is not the case for a complex vector bundle. So how do we get interesting invariants? Given $E \to M$, we may consider the complexification $E \otimes \mathbb{C} \to M$. A connection ∇ on E can be extended (by $\mathbb{C}$-linearity) to $\nabla^{\mathbb{C}}$ on $E \otimes \mathbb{C}$. In other words, we define $\nabla^{\mathbb{C}}$ so that if $\sigma = \sigma_1 + i\sigma_2$ with $\sigma \in \Gamma(E)$ then $\nabla^{\mathbb{C}} \sigma = \nabla^E \sigma_1 + i \nabla^E \sigma_2$. Then define characteristic classes for E, by taking Chern classes of $E \otimes \mathbb{C}$. These are called the **Pontryagin classes**.

30 November 14, 2016

Last time, for a complex vector bundle $E \to M$ we defined the Chern classes as

$$\det\Big(\mathbf{1} + \frac{i}{2\pi} F_\nabla t\Big) = c_0(E) + c_1(E)t + \cdots + c_k(E)t^k + \cdots .$$

If E is real, then consider the complexification $E \otimes \mathbb{C}$. If ∇ is a connection onf E, then it extends to $E \otimes \mathbb{C}$, which we denote by $\nabla^{\mathbb{C}}$. Then we define the k-th Pontryagin class as

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}).$$

Note that $p_k(E) \in H^{4k}_{\mathrm{dR}}(M)$. Why do we only consider $2k$ and not k?

Proposition 30.1. *If k is odd, then $c_k(E \otimes \mathbb{C}) = 0$.*

The way we are going to prove this is using metric compatible connections.

30.1 Metric compatible connections

Definition 30.2. A connection on $E \to M$ is **compatible with a metric** H if

$$X\langle \sigma, \tau\rangle_H = \langle \nabla_X \sigma, \tau\rangle + \langle \sigma, \nabla_X \tau\rangle$$

for all vector fields X and $\sigma, \tau \in \Gamma(E)$.

Theorem 30.3. *Given a vector bundle $E \to M$ with a smooth metric H, there exists a connection on E compatible with H (although they are definitely not unique).*

Proof. Let $\tilde{\nabla}$ be one connection on E. Let $\{e_1, \dots, e_r\}$ be an orthonormal frame for E over an open set U. In this frame, $\tilde{\nabla} = d + A$ and $H = \mathbf{1}$.

We claim that ∇ is compatible with H if and only if $\nabla H = 0$, whre ∇ denotes the induced connection on $E^* \otimes E^*$. In a local frame, $\langle \sigma, \tau\rangle = \sigma^T H \tau$. If I covariantly differentiate this,

$$\nabla(\sigma^T H \tau) = (\nabla\sigma)^T H\tau + \sigma^T(\nabla H)\tau + \sigma^T H(\nabla \tau).$$

So if I want this to equal to the sum of the first and third terms, the ∇H must be zero.

In teneral, the covariant derivative acts on H by

$$\begin{aligned}(\nabla_{\partial/\partial x^l} H)_{\alpha\beta} &= \partial_l H_{\alpha\beta} - A^\gamma_{l\alpha} H_{\gamma\beta} - A^\gamma_{l\beta} H_{\alpha\gamma} \\ &= \partial_l H_{\alpha\beta} - (HA_l)_{\alpha\beta} - (HA_l)^T_{\alpha\beta} = (\partial_l H - HA_l - A_l^T H)_{\alpha\beta}.\end{aligned}$$

In an orthonormal frame $H = \mathbf{1}$, we have $\tilde{\nabla} H = -(A + A^T)$. In other words, $\tilde{\nabla}$ is metric compatible if and only if $A = -A^T$ in an orthonormal frame.

So our gaol is to find $T \in \mathrm{End}(E) \otimes T^*M$ such that $-(T+T^T) = A + A^T$ in $\{e_1, \ldots, e_r\}$ since then $\nabla = d + A + T$ has $\nabla H = 0$. Note that if we are defining T locally in one frame, and so we need to check that our local definitions glue to give a section of $\mathrm{End}(E) \otimes T^*M$. For example, $T = -A$ doesn't work because A doesn't transform like a endomorphism.

Let's guess and take $T = -(A + A^T)/2$. Then $\nabla = d + (A - A^T)/2$ in the orthonormal frame $\{e_1, \ldots, e_r\}$. Does T glue to give a global section of $\mathrm{End}(E) \otimes T^*M$? Suppose we have two open sets U_1 and U_2. Let $\{e_1^1, \ldots, e_r^1\}$ and $\{e_1^2, \ldots, e_r^2\}$ be the orthonormal frames on U_1 and U_2, and write $\tilde{\nabla} = d + A_1$ and $\tilde{\nabla} = d + A_2$. Also let $T_1 = -(A_1 + A_1^T)/2$ and $T_2 = -(A_2 + A_2^T)/2$. On $U_1 \cap U_2$, we have $g : \{e^2\} \to \{e_1\}$ given by $g : U_1 \cap U_2 \to \mathrm{O}(r)$ because $\{e^1\}$ and $\{e^2\}$ are orthonormal frames. We have

$$A_2 = g^{-1} dg + g^{-1} A_1 g$$

because $\tilde{\nabla}$ shouldn't depend on the coordinates. Since $g \in \mathrm{O}(r)$, $g^T = g^{-1}$. Then

$$\begin{aligned} A_2 + A_2^T &= (g^{-1}dg)^T + g^T A_1^T (g^{-1})^T + g^{-1}dg + g^{-1}A_1 g \\ &= (g^{-1}dg)^T + g^{-1}dg + g^{-1}(A_1^T + A_1)g. \end{aligned}$$

But $dg \in T_g \mathrm{O}(r)$ and so $g^{-1}dg \in T_{\mathbf{1}} \mathrm{O}(r)$ is anti-symmetric. So $A_2 + A_2^T = g^{-1}(A_1 + A_1^T)g$. This shows that $T \in \Gamma(M, \mathrm{End}(E) \otimes T^*M)$ and so $\nabla = \tilde{\nabla} + T$ is metric compatible. □

In this case, people abusively write

$$A + T \in \Gamma(\mathrm{lie}(\mathrm{O}(r)) \otimes T^*M).$$

Now $\nabla = d + (A - A^T)/2 = d + M$ is compatible with the metric. Then

$$\begin{aligned} F_\nabla = F_{ij} dx^i \wedge dx^j &= dM + M \wedge M = \partial_i M_j - \partial_j M_i \\ &= (\partial_i M_j - \partial_j M_i + M_i M_j - M_j M_i) dx^i \wedge dx^j. \end{aligned}$$

Then F_{ij} is anti-symmetric. In particular, if k is ood, then

$$\mathrm{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla) = \mathrm{Tr}((F_\nabla \wedge \cdots \wedge F_\nabla)^T) = (-1)^k \,\mathrm{Tr}(F_\nabla^T \wedge \cdots \wedge F_\nabla^T).$$

Thus $\mathrm{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla) = 0$.

Proposition 30.4. *Up to a factor,* $\mathrm{ch}_k(E \otimes \mathbb{C}) = (\quad) \,\mathrm{Tr}(F_\nabla \wedge \cdots \wedge F_\nabla)$.

Proof. Exercise. □

We have shown that $\mathrm{ch}_k(F_\nabla) = 0$ for k odd. Since $c_m(F_\nabla)$ are made up of ch_k, we get $c_m(F_\nabla) = 0$ for m odd.

The reason this doesn't show that odd degree Chern classes vanish is because the reduction of the gauge group is from GL to U and then the same argument only shows that the Chern class is only pure imaginary.

31 November 16, 2016

We are going to do some Riemannian geometry.

31.1 Curvature of a Riemannian manifold

Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection on TM. Then for C^∞ vector fields X, Y, Z, we can define

$$R(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Proposition 31.1. *$R(\bullet, \bullet)$ is endomorphism valued, i.e., is in* $\mathrm{End}(TM) \otimes T^*M^{\otimes 2}$.

In local coordinates $(x^1, \dots, x^n)$ near p, we can use $\{\partial/\partial x^1, \dots, \partial/\partial x^n\}$ as a frame for TM. Then we can write

$$R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k} = R_{ij}{}^l{}_k \frac{\partial}{\partial x^l}.$$

In other words, if $v = v^k \partial/\partial x^k$, then

$$R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) v = R_{ij}{}^l{}_k v^k \frac{\partial}{\partial x^l}.$$

We can also define, for $X, Y, Z, W \in \Gamma(TM)$,

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

This defines a section of $(T^*M)^{\otimes 4}$. In local coordinates,

$$R_{ijpk} = g_{lp} R_{ij}{}^l{}_k$$

Proposition 31.2 (The Bianchi identities)**.**

(1) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(2) $\nabla R = 0$

Proof. We have already proved (2). Let us do (1). Since everything is tensorical, we can check (1) at a point $p \in M$, in normal coordinates $(x^1, \dots, x^n)$. We have

$$\begin{aligned} R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k} &= (\nabla_i \nabla_j - \nabla_j \nabla_i)\frac{\partial}{\partial x^k} \\ &= \nabla_i \Big(\Gamma^l_{jk} \frac{\partial}{\partial x^l}\Big) - \nabla_j \Big(\Gamma^l_{ik}\frac{\partial}{\partial x^l}\Big) = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} \end{aligned}$$

because we are working in normal coordinates and so the evaluation of Γ at p is zero. Now

$$R_{ij}{}^l{}_k + R_{jk}{}^l{}_i + R_{ki}{}^l{}_j = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \partial_k \Gamma^l_{ij} - \partial_i \Gamma^l_{kj} = 0$$

since $\Gamma^l_{jk} = \Gamma^l_{kj}$. $\square$

Lemma 31.3 (Symmetries of R)**.**

(i) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

(ii) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$.

Proof. Homework. □

The curvature tensor has *a lot* of information.

31.2 Sectional curvature

Definition 31.4. Let $p \in M$, and let $\sigma \subseteq T_pM$ a 2-dimensional subspace. Let $\sigma = \operatorname{span}\{x, y\}$ and define

$$K(\sigma) = \frac{R(X,Y,Y,X)}{\|X \wedge Y\|^2}$$

where $\|X \wedge Y\|^2 = |X|^2|Y|^2 - \langle X, Y\rangle^2$. This $K(\sigma)$ is called the **sectional curvature**.

Proposition 31.5.

(1) $K(\sigma)$ *is independent of the choice of* $\{X, Y\}$ *spanning* σ.

(2) $K(\sigma)$ *for every* $\sigma \subseteq T_pM$ *determines* $R(p)$.

Proof. (1) We can just check that $K(\sigma)$ is not changed by $(X,Y) \to (Y,X)$ and $(X,Y) \to (\lambda X, Y)$ and $(X,Y) \to (X, Y + \lambda X)$. This can be checked.

(2) It suffices to prove the following lemma. □

Lemma 31.6. *If* V *is a vector space with an inner product* $\langle \bullet, \bullet\rangle$ *and trilinear maps* $R, R' : V \times V \times V \to V$ *such that*

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle, \quad R'(X,Y,Z,W) = \langle R'(X,Y)Z, W\rangle$$

satisfy the symmetries of the Riemannian curvatures, then $R(X,Y,Y,X) = R'(X,Y,Y,X)$ *for all* X, Y *implies* $R = R'$.

Proof. By assumption, expanding $R(X+Z,Y,Y,X+Z) = R(X+Z,Y,Y,X+Z)$ implies

$$R(X,Y,Y,X) + R(Z,Y,Y,Z) + R(X,Y,Y,Z) + R(Z,Y,Y,X) = \text{same with } R'.$$

Becuase $R(X,Y,Y,Z) = R(Y,Z,X,Y) = R(Z,Y,Y,X)$, we get

$$R(Z,Y,Y,X) = R'(Z,Y,Y,X).$$

Now let us expand $R(Z, Y+W, Y+W, X)$. From this, we get

$$\begin{aligned} R(Z,W,Y,X) - R'(Z,W,Y,X) &= R'(Z,Y,W,X) - R(Z,Y,W,X) \\ &= R(Y,Z,W,X) - R'(Y,Z,W,X). \end{aligned}$$

That is, $R - R'$ is invariant under cyclic permutations of the first three entries. So summing over a cyclic permutation and using the first Bianchi identity, we get

$$3[R(Y, Z, W, X) - R'(Y, Z, W, X)] = 0.$$

Therefore $R = R'$. $\square$

32 November 18, 2016

Last time we defined the Riemann curvature tensor as

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle, \quad R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z.$$

Then we defined

$$K(\sigma) = \frac{R(X,Y,Y,X)}{|X\wedge Y|^2} \text{ for } \sigma = \operatorname{span}(X,Y) \subseteq T_pM$$

We proved that $K(\sigma)$ determines K.

Definition 32.1. Let us say that (M,g) has **constant sectional curvature** $K(\sigma) = K$ is independent of both p and σ.

Lemma 32.2. *If (M,g) has constant sectional curvature K, then*

$$R(X,Y,Z,W) = K(\langle X,W\rangle\langle Y,W\rangle - \langle X,Z\rangle\langle Y,W\rangle)$$

Proof. This is a 4-tensor that has agrees on 2-planes and has all the symmetries. □

Definition 32.3. The **Ricci curvature** is given by

$$\operatorname{Ric}(X,Y) = \operatorname{Tr}(Z \mapsto R(Z,X)Y).$$

In local coordinates, we have

$$(\operatorname{Ric})_{ij} = R_{ki}{}^{k}{}_{j} = g^{lk}R_{kilj}.$$

Note that this is not $\operatorname{Tr}(F_\nabla)$.

Definition 32.4. The **scalar curvature** is defined as

$$R = g^{ij}(\operatorname{Ric})_{ij}.$$

32.1 Effect of curvature on geodesics

Fix a point $p \in M$, a vector $v \in T_pM$ and consider $\gamma(t) = \exp_p vt$. Our goal is to study variations of $\gamma(t)$. The general family of geodesics is

$$\alpha(t,s) = \exp_p(t(v, w(s)))$$

where $w(s) : (-\epsilon,\epsilon) \to T_pM$ with $w(0) = 0$. Then for each s_0, $\alpha(t,s_0)$ is a geodesic with initial vector $v + w(s_0)$. We are going to study the infinitesimal variations of the geodesics. We have

$$\frac{d}{ds}\Big|_{s=0} \exp_p(t(v+w(s))) = d(\exp_p)_{tv}tw'(0).$$

This gives us a vector field.

Proposition 32.5. *The vector field $J(t) = d(\exp_p)_{tv}tw$ is solves the following* ***Jacobi field equation****:*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(t) = R(\dot\gamma, J)\dot\gamma.$$

Proof. By torsion-freeness of the connection,

$$\nabla_{\dot\gamma}J - \nabla_J\dot\gamma = [\dot\gamma, J] = [d(\exp_p)_{tv}tv, d(\exp_p)_{tv}tw] = d(\exp_p)_{tv}[v, w] = 0$$

because v and w are constant on T_pM. Then

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J = \nabla_{\dot g}\nabla_J\dot\gamma = R(\dot\gamma, J)\dot\gamma + \nabla_J\nabla_{\dot\gamma}\dot\gamma + \nabla_{[\dot\gamma,J]}\dot\gamma = R_(\dot\gamma, J)\dot\gamma. \qquad \square$$

Definition 32.6. For a geodesic $\gamma : [0, a] \to M$, a vector field $J(t)$ along $\gamma(t)$ is a **Jacobi field** with initial data J_0, J_0' if $J(t)$ solves the Jacobi field equations with $J(0) = J_0$ and $J'(0) = J_0'$.

Let us show that Jacobi fields exist. Let $e_1, \dots, e_n$ be an orthonormal basis for T_pM, and let $E_1(t), \dots, E_n(t)$ be the parallel transport of $e_1, \dots, e_n$ along $\gamma(t)$. Then $\{E_i(t)\}$ is orthonormal for all $t \in [0, a]$ because the Levi-Civita connection is compatible with g. More formally, we can write

$$\frac{d}{dt}\langle E_i(t), E_j(t)\rangle = \langle \nabla_{\dot\gamma}E_i(t), E_j(t)\rangle + \langle E_i(t), \nabla_{\dot\gamma}E_j(t)\rangle = 0 + 0 = 0.$$

Now write $J(t) = \sum_{i=1}^n a_i(t)E_i(t)$. Then

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J = \sum_{i=1}^n \ddot a_i(t)E_i(t) = \sum_i a_i(t)R(\dot\gamma, E_i)\dot\gamma.$$

Taking the inner product with E_j, we get

$$\ddot g_j(t) = \sum_i a_i(t)R(\dot\gamma, E_i, \dot\gamma, E_j).$$

This is an ODE, so we can always solve it.

Theorem 32.7. *Given $\gamma(t)$, $J_0, J_0' \in T_pM$, there exists a unique Jacobi field $J(t)$ along $\gamma(t)$ with initial conditions $J(0) = J_0$ and $J'(0) = J_0'$.*

So if $J_0 = 0$ and $J_0' = w$ then $J(t) = d(\exp_p)_{vt}wt$.
Recall that Gauss's lemma says

$$\langle d(\exp_p)_v w, d(\exp_p)_v v\rangle = \langle v, w\rangle.$$

Proposition 32.8. *If $J(0) = 0$ and $J'(0) \perp \gamma'(0)$ then $J(t) \perp \gamma'(t)$ for all t.*

Proof. Note that

$$\frac{d}{dt}\langle J(t), \dot\gamma(t)\rangle = \langle \nabla_{\dot\gamma(t)}J(t), \dot\gamma(t)\rangle$$

because $\nabla_{\dot\gamma}\dot\gamma = 0$ since γ is a geodesic. Then

$$\frac{d^2}{dt^2}\langle J(t), \dot\gamma(t)\rangle = \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} J(t), \dot\gamma(t)\rangle = R(\dot\gamma, J(t), \dot\gamma, \dot\gamma) = 0.$$

Thus

$$\langle J(t), \dot\gamma(t)\rangle = \langle J(0), \dot\gamma(0)\rangle + t\langle J'(0), \dot\gamma(0)\rangle.$$

If $J(0) = 0$ and $J'(0) \perp \dot\gamma(0)$, then we immediately get $J(t) \perp \dot\gamma(t)$. □

32.2 Jacobi fields on manifolds with constant sectional curvature

Suppose (M, g) has sectional curvature $K \in \mathbb{R}$. Fix $\gamma(t)$ a geodesic with $|\dot\gamma(t)| = 1$.

Proposition 32.9. *The Jacobi field equation is*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} J = -KJ$$

if $J(0) = 0$ and $J'(0) \perp \dot\gamma$.

Proof. We have $R(\dot\gamma, J)\dot\gamma = -KJ$ since we have an explicit formula. □

For example, if $w(t)$ is parallel along $\dot\gamma(t)$ with $|w(t)| = 1$ and $w(t) \perp \dot\gamma$ then

$$J(t) = \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t)$$

is a Jacobi field $K > 0$.

33 November 21, 2016

Let (M, g) be a Riemannian manifold and $\gamma : [0, 1] \to M$ be a geodesic. Then $J(t)$ is a Jacobi vector field along $\gamma(t)$ if it satisfies the equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(t) = R(\dot\gamma, J)\dot\gamma$.

Let us now look at the special example where (M, g) has constant sectional curvature K. Then $\nabla_{\dot\gamma}\nabla_{\dot\gamma}J + KJ = 0$. If $w(t)$ is parallel along $\gamma(t)$ with $w(t) \perp \dot\gamma(t)$, then $J(t)$ is a Jacobi field with $J(0) = f(0)w(0)$ and $J'(0) = f'(0)w(0) + f(0)w'(0)$ if and only if

$$\frac{d^2 f}{dt^2} + Kf(t) = 0$$

if we set $J(0) = 0$. The solution to this differential equation is

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t) & K > 0 \\ tw(t) & K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t) & K < 0. \end{cases}$$

Let us think what this is saying. If $K < 0$, then a small variation of a geodesic oscillates around the original geodesic. This already hints that the manifold M looks somewhat like the sphere. If $K = 0$, then a small variation of a geodesic linearly drifts away from the original one. The Euclidean space $(\mathbb{R}^n, g_{\mathrm{eug}})$ is an example of this. If $K < 0$, then the variational geodesic goes away from the original one exponentially. The upper-half plane $(\mathcal{H}^2, (dx^2 + dy^2)/y^2)$ is an example.

33.1 Conjugate points

Definition 33.1. q is **conjugate** to p (along $v \in T_pM$) if q is a singular value of $\exp_p : T_pM \to M$, i.e., $q = \exp_p v$ and $d(\exp_p)_v : T_pM \to T_qM$ has non-trivial kernel.

Proposition 33.2. *q is conjugate to p along v if there exists a non-zero Jacobi field $J(t)$ along $\gamma(t) = \exp_p(vt)$ such that $J(0) = 0 = J(1)$. Moreover q is conjugate to p if and only if p is conjugate to q.*

Proof. Take any $w \in T_pM$. Then $J(t) = d(\exp_p)_{vt}tw$ is a Jacobi field with $J(0)$ and $J'(0) = w$. Then $J(1) = 0$ if and only if $w \in \ker d(\exp_p)_v$. The symmetry of the statement is obvious. □

Recall that the first variation of the arc length functional gave rise to the geodesic equation. Now let us look at the second variation of the arc length. Since geodesics may not be length minimizing, the second variation may not be positive definite.

Theorem 33.3. *Let $\gamma(t) : [a,b] \to M$ be a geodesic with unit speed. Let $Q = [a,b] \times [-\epsilon,\epsilon] \times [-\delta,\delta] \to M$ be a smooth map such that $Q(t,0,0) = \gamma(t)$. Then*

$$\frac{\partial^2}{\partial v \partial w} L(v,w)\Big|_{(v,w)=(0,0)} = \int_a^b \Big(\langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle \\ - (T\langle V,T\rangle)(T\langle W,T\rangle)\Big) dt + \langle \nabla_W V, T\rangle|_a^b,$$

where

$$L(v,w) = \int_a^b \Big| \frac{\partial Q}{\partial t}(t,v,w)\Big|_g dt,$$
$$W = \frac{\partial Q}{\partial w}(t,0,0), \quad V = \frac{\partial Q}{\partial V}(t,0,0), \quad T = \frac{\partial Q}{\partial t}(t,0,0).$$

Proof. We have $L(v,w) = \int_a^b \|T\| dt$. Then

$$\frac{\partial L}{\partial v} = \int_a^b V\sqrt{\langle T,T\rangle} dt = \int_a^b \frac{\langle \nabla_V T, T\rangle}{\|T\|} dt.$$

But $V\langle T,T\rangle = 2\langle \nabla_V T, T\rangle$ and $\nabla_V T = \nabla_T V + [V,T]$ and $[V,T] = 0$. So

$$\frac{\partial L}{\partial V} = \int_a^b \frac{\langle \nabla_T V, T\rangle}{\|T\|} dt.$$

Note that this is the same thing we got when we were talking about the geodesic equation.[2]

Now the second derivative is

$$\frac{\partial^2 L}{\partial w \partial v} = \int_a^b \left(\frac{W\langle \nabla_T V, T\rangle}{\|T\|} - \frac{\langle \nabla_T V, T\rangle W\|T\|}{\|T\|^2}\right) dt \\ = \int_a^b \left(\frac{\langle \nabla_W \nabla_T V, T\rangle + \langle \nabla_T V, \nabla_T W\rangle}{\|T\|} - \frac{\langle \nabla_T V, T\rangle\langle \nabla_W T, T\rangle}{\|T\|^3}\right) dt.$$

We can do the swapping using the same arguments, and evaluate at $u = v = 0$. Then $\nabla_T T = 0$ and $\|T\| = 1$ and so $\langle \nabla_T V, T\rangle = T\langle V,T\rangle$ and $\langle \nabla_T W, T\rangle = T\langle W,T\rangle$. Thus

$$\frac{\partial^2 L}{\partial v \partial w}\Big|_{(0,0)} = \int_a^b \Big(\langle \nabla_T \nabla_W V, T\rangle + \langle R(W,T)V, T\rangle \\ + \langle \nabla_T V, \nabla_T W\rangle - (T\langle V,T\rangle)(T\langle V,W\rangle)\Big) dt.$$

Now $\langle \nabla_T \nabla_W V, T\rangle = (d/d)\langle \nabla_W V, T\rangle$ and so finally

$$\frac{\partial^2 L}{\partial V \partial W} = \langle \nabla_W V, T\rangle|_a^b + \int_a^b \Big(\langle \nabla_T V, \nabla_T W\rangle \\ - R(W,T,T,V) - (T\langle V,T\rangle)(T\langle W,T\rangle)\Big) dt. \qquad \square$$

[2] If we require $V(a) = V(b) = 0$ and $\|\dot\gamma\| = 1$, then $\langle \nabla_T V, T\rangle = (d/dt)\langle V,T\rangle - \langle V, \nabla_T T\rangle$ and so we get that $\nabla_T T = 0$. This gives another (really the same) proof of the fact that geodesics are the critical points of L.

Note that this makes sense even when V and W are piecewise C^∞. Assume that V and W are C^∞ on $[t_i, t_{i+1}]$ where $a = t_0 < \cdots < t_n = b$, and assume also that $\langle V, T\rangle = \langle W, T\rangle = 0$. Then

$$\frac{\partial^2 L}{\partial v \partial w} = \sum_{i=0}^{n-1} \int_{t_{i-1}}^{t_i} \Big(\langle \nabla_T V, \nabla_T W\rangle - R(W, T, T, V) \Big) dt + \langle \nabla_W V, T\rangle \big|_{t_i}^{t_{i+1}}.$$

The second term telescopes. For the first term, we can integrate by parts and get

$$\begin{aligned} &\int_{t_i}^{t_{i+1}} \Big(\langle \nabla_T V, \nabla_T W\rangle - R(W, T, T, V) \Big) dt \\ &\qquad = \Delta_{t_i} \langle \nabla_T V, W\rangle - \int_{t_i}^{t_{i+1}} \Big(\langle \nabla_T \nabla_T V, W\rangle - R(T, V, T, W) \Big) dt. \end{aligned}$$

I will discuss the applications next week.

34 November 28, 2016

Recall that $p, q \in M$ are conjugate (along $v \in T_pM$) if $q = \exp_p v$ and $d(\exp_p)_v$ has non-trivial kernel. We have proved that $q \in M$ is conjugate to p if and only if there exists a non-zero Jacobi field $J(t)$ along $\exp_p vt$ such that $J(0) = J(1) = 0$.

Corollary 34.1. *If q is not conjugate to p then for any $w \in T_qM$ there exists a Jacobi field along the geodesic $g(t)$ connecting p and $q = \gamma(1)$ such that $J(0) = 0$ and $J(1) = w$.*

Proof. Let $\hat{v} \in T_pM$. Let $J_{\hat{v}}(t)$ be the Jacobi field with $J_{\hat{v}}(0) = 0$ and $\dot{J}_{\hat{v}}(0) = \hat{v}$. Consider the map $T_pM \to T_qM$ given by $\hat{v} \mapsto J_{\hat{v}}(1)$. This map is linear and has no kernel. Thus it is an isomorphism. $\square$

34.1 Second variation of arc length

If $\gamma : [0, a] \to M$ is a geodesic with $\|\dot{\gamma}\| = 1$, we want to compute the second variation or arc-length. Given piecewise C^∞ vector fields V, W along $\gamma(t)$ such that $V(0) = 0, W(0) = 0$ and $V(a) = 0, W(a) = 0$ and $V \perp \dot{\gamma}, W \perp \dot{\gamma}$.

The second variation, we have shown last time, can be computed as

$$I(V, W) = \sum_{i=0}^{n} \langle \Delta_{t_i} \nabla_{\dot{\gamma}} V, W\rangle - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - R(\dot{\gamma}, V)\dot{\gamma}, W\rangle,$$

where V, W are C^∞ on $[t_i, t_{i+1}]$ and $\Delta_{t_i} \nabla_{\dot{\gamma}} V = \lim_{t \to t_i^+} \nabla_{\dot{\gamma}} V - \lim_{t \to t_i^-} \nabla_{\dot{\gamma}} V$.

We can consider $I(V, W)$ as a symmetric bilinear form on piecewise C^∞ vector fields along $\gamma(t)$. (Here, we don't necessarily have to put assumptions at end points or being perpendicular to $\dot{\gamma}$.)

Proposition 34.2. *The null space of I is exactly the set of Jacobi fields along $\gamma(t)$ which vanish at the end points.*

Proof. If $V(t)$ is a Jacobi field along $\gamma(t)$, then

$$I(V, W) = \sum_{i=0}^{n} \langle \Delta_{t_i} \nabla_{\dot{\gamma}} V, W\rangle - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \langle \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - R(\dot{\gamma}, V)\dot{\gamma}, W\rangle = 0$$

because V is C^∞ and solves the Jacobi equation.

Conversely, Take $0 = t_0 < \cdots < t_n = a$ such that $V|_{[t_i, t_{i+1}]}$ is C^∞. Let $f(t) : [0, a] \to \mathbb{R}$ be C^∞ with $f \geq 0$ and $f(t_i) = 0$. As a test vector field, take

$$W = f(t)[-\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V + R(\dot{\gamma}, V)\dot{\gamma}].$$

Then we get

$$I(V, W) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t) \|\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - R(\dot{\gamma}, V)\dot{\gamma}\|^2 dt = 0.$$

So $\nabla_{\dot\gamma}\nabla_{\dot\gamma}V - R(\dot\gamma, V)\dot\gamma = 0$ on $[t_i, t_{i+1}]$. So $I(V, W) = \sum_{i=0}^n \langle \Delta_{t_i}\nabla_{\dot\gamma}V, W\rangle$. Let W be any C^∞ vector field so that $W(t_i) = \nabla_{t_i}\nabla_{\dot\gamma}V$. Then $I(V, W) = \sum_{i=0}^n \|\Delta_{t_i}\nabla_{\dot\gamma}V\|^2$ and so $\Delta_{t_i}\nabla_{\dot\gamma}V = 0$. So $\nabla_{\dot\gamma}V$ is continuous.

Now we claim that V is C^∞ and solves the Jacobi equation. This is because the Jacobi equation is determined by $V(t), \dot V(t)$ and since $\nabla_{\dot\gamma}V$ is continuous, the two solutions around a singular point actually have to agree. This is basically appealing to the existence and uniqueness of ODEs. $\square$

Corollary 34.3. *I has non-trivial null space if and only if $\gamma(0)$ is conjugate to $\gamma(a)$. The dimension of the null space is equal to the **order of the conjugacy**.*

34.2 First index lemma

Lemma 34.4 (First index lemma)**.** *Suppose γ is a unit speed geodesic between $p = \gamma(0)$ and $q = \gamma(a)$. Suppose there are no points on γ conjugate to p. Let W be a piecewise C^∞ vector field along γ such that $W(p) = 0$. Then*

(i) there exists a unique Jacobi field V along γ such that $V(p) = 0 = W(p)$ and $V(q) = W(q)$.

(ii) $I(V, V) \le I(W, W)$ with equality if and only if $V = W$.

Theorem 34.5 (Bonnet–Myers)**.** *Let (M^n, g) be a compact Riemannian manifold. If there exists a $H \in \mathbb{R}_{>0}$ such that*

(i) $\mathrm{Ric}(x, x) \ge (n-1)H$ for all unit vectors x, or

(ii) $K \ge H$,

then every geodesic γ of length $(\gamma) > \pi/\sqrt{H}$ has conjugate pints, and $\dim(M) = \sup\{d(p, q) : p, q \in M\} \le \pi/\sqrt{H}$*. In particular, M is compact.*

I'll prove both of these next time.

35 November 30, 2016

Lemma 35.1 (First index lemma). *Suppose we have $\gamma : [0,a] \to M$ be a unit speed geodesic connecting the points $p = \gamma(0)$ and $q = \gamma(a)$. Suppose there are no points on γ conjugate to p. If W is any piecewise C^∞ vector field along γ with $W(p) = 0$, then*

(1) there is a unique Jacobi field V along γ such that $V(p) = W(p) = 0$ and $V(q) = W(q)$, and

(2) $I(V,V) \le I(W,W)$ with equality if and only if $V = W$.

Proof. (1) Since q is not conjugate to p, we can find a unique V.

(2) Let $\{V_i\}_{i=1}^N$ be a basis of T_qM. Extend V_i to Jacobi fields $J_i(t)$ along $\gamma(t)$ such that $J_i(0) = 0$. For each $t \in (0,a)$, the $J_i(t)$ are linearly independent. This is because if $\sum_{i=1}^N \alpha_i J_i(t)$ vanishes at $t = t_0$ and $t = 0$, then p is conjugate to $\gamma(t_0)$. This contradicts our assumption.

We now claim that $W = \sum f_i(t) J_i(t)$ for f_i piecewise C^∞. Because $J_i(0) = 0$, we can write $J_i(t) = tA_i(t)$ for A_i smooth vector fields linearly independent on $[0,a]$. Here, $A_i(0)$ are linearly independent because $A_i(0) = J_i'(0)$. Then $W(t) = \sum_i q_i(t) A_i(t)$ where $q_i(t)$ are piecewise C^∞. But now, $W(0) = \sum_i q_i(0)A_i(0) = 0$ and so $q_i(0) = 0$ for all i. Then we can write $q_i(t) = tf_i(t)$ with $f_i(t)$ piecewise C^∞. This implies that $W(t) = \sum f_i(t) J_i(t)$.

Because V is a Jacobi field that agrees with W at a, we get $V = \sum_i f_i(a) J_i(t)$. Then

$$I(V,V) = \sum_{i,j} f_i(a) f_j(a) \langle J_i'(a), J_j(a)\rangle.$$

Likewise we have $\nabla_{\dot\gamma} W = \sum_i \dot f_i(t) J_i(t) + \sum_i f_i(t) \dot J_i(t) = A + B$ and so

$$\begin{aligned} I(W,W) &= \int_0^a \langle \nabla_{\dot\gamma} W, \nabla_{\dot\gamma} W\rangle dt + \int_0^a \langle R(\dot\gamma, W)\dot\gamma, W\rangle dt \\ &= \int_0^a \langle A, A\rangle + \langle A, B\rangle + \langle B, A\rangle + \langle B, B\rangle + R(\dot\gamma, W, \dot\gamma, W). \end{aligned}$$

The term $\langle B, B\rangle$ looks like

$$\begin{aligned} \int \langle B, B\rangle &= \sum_{i,j} \int_0^a f_i f_j \langle \dot J_i, \dot J_j\rangle = \sum_{i,j} \int_0^a f_i f_j \Big[\frac{d}{dt}\langle J_i, \dot J_j\rangle - \langle J_i, \ddot J_j\rangle\Big] \\ &= \sum_{i,j} f_i(a) f_j(a) \langle J_i(a), \dot J_j(a)\rangle - \sum_{i,j} \int_0^a (\dot f_i f_j + \dot f_j f_i)\langle J_i, \dot J_j\rangle - \int R(\dot\gamma, W, \dot\gamma, W). \end{aligned}$$

Also

$$\int_0^a \langle A, B\rangle dt = \int_0^a \sum_{i,j} \dot f_i f_j \langle \dot J_i, J_j\rangle$$

cancels one of the terms.

We note that $\langle J_i, \dot{J}_j\rangle = \langle \dot{J}_i, J_j\rangle$ because

$$\frac{d}{dt}(\langle J_i, \dot{J}_j\rangle - \langle \dot{J}_i, J_j\rangle) = \langle J_i, \ddot{J}_j\rangle - \langle \ddot{J}_i, J_j\rangle = R(\dot\gamma, J_j, \dot\gamma, J_i) - R(\dot\gamma, J_i, \dot\gamma, J_j) = 0$$

and $\langle J_i, \dot{J}_j\rangle - \langle \dot{J}_i, J_j\rangle = 0$ at $t = 0$. So

$$\sum_{i,j}\int_0^a f_i\dot{f}_j\langle J_i, \dot{J}_j\rangle dt = \sum_{i,j}\int_0^a f_i\dot{f}_j\langle \dot{J}_i, J_j\rangle dt = \int_0^a \langle B, A\rangle dt.$$

Therefore we get

$$I(W, W) = I(V, V) + \int_0^a \langle A, A\rangle dt$$

and $A = 0$ if and only if $f_i(t) = 0$, which means $W = V$. □

35.1 Bonnet–Myers theorem

Theorem 35.2 (Bonnet–Myers)**.** *Let M^n be a complete Riemannian manifold with $H \in \mathbb{R}_{>0}$ such that*

(i) (Myers) $\mathrm{Ric}(x, x) \geq (n-1)H$ *for all unit vectors x.*

(ii) (Bonnet) $K \geq H$.

Then $\mathrm{diam}(M) \leq \pi/\sqrt{H}$ *and in particular, M is compact.*

Note that (i) is a stronger result than (ii) because if $K \geq H$ then $\mathrm{Ric}(x, x)$ is a sum of $n-1$ sectional curvatures.

Proof. Take $p, q \in M$ and let $d(p, q) = l$. Take $\gamma : [0, l] \to M$ a unit speed geodesic connecting p and q. Let $E_i(t)$ be an orthonormal frame of parallel vectors along $\gamma(t)$. Let $W_i(t) = \sin(\pi t/l)E_i(t)$. (This will serve as the model Jacobi field from space of constant curvature $K = l^2$.) We have

$$I(W_i, W_i) = \int_0^l \sin\Big(\frac{\pi}{l}t\Big)^2\Big[\frac{\pi^2}{l^2} - R(\dot\gamma, E_i, \dot\gamma, E_i)\Big]dt.$$

So if $K \geq H$ then

$$I(W_i, W_i) \leq \int_0^l \sin\Big(\frac{\pi}{l}t\Big)^2\Big[\frac{\pi^2}{l^2} - H\Big].$$

If $l > \pi/\sqrt{H}$, then $I(W_i, W_i) < 0$. This contradicts the fact that γ is length minimizing.

For Myers, we sum over i. Then

$$\sum_{i=1}^{n-1} I(W_i, W_i) = \int \sin\Big(\frac{\pi}{l}t\Big)\Big[(n-1)\frac{\pi^2}{l^2} - \mathrm{Ric}(\dot\gamma, \dot\gamma)\Big]dt < 0$$

if $l > \pi/\sqrt{H}$ where $\mathrm{Ric}(\dot\gamma, \dot\gamma) \geq (n-1)H$. Then $I(W_i, W_i) < 0$ for some i and so γ is not length minimizing. □

36 December 2, 2016

36.1 Covering spaces

Definition 36.1. Let B be a topological space. $\tilde{B}$ is a **covering space** with covering map $\pi : \tilde{B} \to B$ if it is continuous and surjective and for each $p \in B$ there exists a neighborhood $U \subseteq B$ such that $\pi^{-1}(U) = \bigcup_\alpha V_\alpha$ with $V_\alpha \subseteq B$ open and $\pi|_{V_\alpha} : V_\alpha \to U_\alpha$ a homeomorphism.

For example, the map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a covering map. The map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z} \cong S^1 \times S^1$ is also a covering map.

Proposition 36.2 (Curve lifting)**.** *If $\alpha : [0, l] \to B$ is curve and $\pi : \tilde{B} \to B$ is a covering map, then for any $\tilde{p}_0 \in \tilde{B}$ with $\pi(\tilde{p}_0) = \alpha(0)$, there is a unique lift $\tilde{\alpha} : [0, l] \to \tilde{B}$ such that $\pi(\tilde{\alpha}) = \alpha$ and $\tilde{\alpha}(0) = \tilde{p}_0$.*

Definition 36.3. $\alpha_0, \alpha_1 : [0, l] \to B$ with $\alpha_0(0) = \alpha_1(0) = p$ and $\alpha_0(l) = \alpha_1(l) = q$ are **homotopic** if there exists a continuous map $H : [0, 1] \times [0, 1] \to B$ such that $H(\bullet, 0) = \alpha_0$, $H(\bullet, 1) = \alpha_1$, and $H(0, \bullet) = p$, $H(l, \bullet) = q$.

Proposition 36.4 (Homotopy lifting)**.** *If $\pi : \tilde{B} \to B$ are local homeomorphisms with path lifting, then we can lift homotopies. So if α_0 and α_1 are homotopic, then they lift to homotopic curves provided $\tilde{\alpha}_0(0) = \tilde{\alpha}_1(0) = \tilde{p}_0$.*

Corollary 36.5. *The cardinality $\#\pi^{-1}(p)$ is independent of p if $\pi : \tilde{B} \to B$ is a covering map and B is connected.*

Definition 36.6. B if **simply connected** if all curves connecting any $p, q \in B$ are homotopic.

The sphere S^n is simply connected if $n \geq 2$ and S^1 is not.

Proposition 36.7. *If $\pi : \tilde{B} \to B$ is local homeomorphism with path lifting, then if B is simply connected and $\tilde{B}$ is path connected, then π is a homeomorphism.*

Proof. We need to show that π is one-to-one. If $\pi(p_1) = \pi(p_2)$, then consider a path connecting p_1 and p_2. The image of this curve is homotopic to the constant map. This implies that the path is homotopic to the constant map. So $p_1 = p_2$. □

Corollary 36.8. *If $\pi : \tilde{B} \to B$ is a covering map and $\tilde{B}$ is path connected, $\tilde{B}$ is simply connected, then π is a homeomorphism.*

Proposition 36.9. *Let $\pi : \tilde{B} \to B$ be a local homeomorphism with path lifting and let $\tilde{B}$ be locally path connected. If B is locally simply connected, then π is a covering map.*

Proof. For $p \in B$, let $V \ni p$ be a simply connected neighborhood. Let $\pi^{-1}(V) = \bigcup_\alpha \tilde{V}_\alpha$ where the $\tilde{V}_\alpha$s are the path components. We claim that $\pi(\tilde{V}_\alpha) = V$. If $q \in V \setminus \pi(\tilde{V}_\alpha$, then a path connecting q and a point in $\pi(\tilde{V}_\alpha)$ can be connected by a path in V. The lift has to be in $\pi^{-1}(V)$ and so we get a contradiction. This shows that $\pi|_{\tilde{V}_\alpha} : \tilde{V}_\alpha \to V$ is a covering map. Then By the proposition, it is a homeomorphism. Since $\tilde{V}_\alpha$ are disjoint, we see that π is a covering map. □

36.2 Hadamard's theorem

Theorem 36.10 (Hadamard)**.** *Let (M, g) be a complete metric with $K \leq 0$. Then for every $p \in M$, $\exp_p : T_pM \to M$ is a covering map.*

Corollary 36.11. *If (M, g) is complete, simply connected, and $K \leq 0$, then $M \cong \mathbb{R}^n$.*

For example, $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ has a flat metric, and $\mathbb{R}^n \to T^n$ is a covering map. For S^n, this cannot be true for $n \geq 2$.

Lemma 36.12. *The map $\exp_p : T_pM \to M$ is a local diffeomorphism.*

Proof. We just need to show that $\ker d(\exp_p)_v = \emptyset$ for $v \in T_pM$. In other words, we need to show that if $\gamma(t) = \exp_p vt$ and $J(t)$ is a nonzero Jacobi field along $\gamma(t)$ with $J(0) = 0$, then $J(t) \neq 0$ for $t > 0$. We have

$$\frac{d}{dt}|J(t)|^2 = 2\langle J, \dot{J}\rangle,$$

and then

$$\frac{d^2}{dt^2}|J(t)|^2 = 2\langle \dot{J}, \dot{J}\rangle - 2K(\dot{\gamma}, J)\|J, \dot{\gamma}\| > 0.$$

Because $d|J(t)|^2/dt|_{t=0} = 0$, we get $|J(t)| > 0$ for $t > 0$. This implies that there are no conjugate points. So $\exp_p$ is a local diffeomorphism. □

Thus $(\mathbb{R}^n, \exp_p^* g) \to (M, g)$ is a local isometry.

Lemma 36.13. *If $(N, h) \to (M, g)$ is a local isometry between complete manifolds, then π is a covering map.*

Proof. We need to verify the path lifting property. Given $\alpha : [0, l] \to M$, we need to lift α uniquely on $[0, t_0)$ for $t_0 > 0$. If $t_\alpha \to t_0$, then $l_{\bar{\alpha}}(\bar{\alpha}(0), \bar{\alpha}(t_k))_N = \underline{l_\alpha(\alpha(0)}, \underline{\alpha(t_k))}_M \leq C$. Because M is complete, bounded sets are compact. Sow $\overline{\alpha(t_k)} \to \overline{\alpha(t_0)} \in M$. Now use local path lifting. □

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