Math 230br - Advanced Differential Geometry

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This course was taught by Karsten Gimre, on Tuesdays and Thursdays from 1 to 2:30pm. The class was based on a series of papers, with Brendle's *Ricci Flow and the Sphere Theorem* as a main reference.

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1 January 23, 2018

I'm going to be talking about geometric flows, in particular Ricci flows. Let M be a compact smooth manifold, let g_t be a one-parameter family of Riemannian metrics on M, where the parameter $t \in I$ is in some open connected interval.

Definition 1.1. The family g_t is called a **Ricci flow** if

$$\frac{\partial g_t}{\partial t} = -2\operatorname{Ric}_{g_t}.$$

More pedantically, for every $p \in M$ and $v, w \in T_pM$,

$$\frac{d}{dt}(g_t(v,w)) = -2\operatorname{Ric}_{g_t}(v,w).$$

Theorem 1.2 (classical analysis)**.** *(1) If g_t and $\tilde{g}_t$ are Ricci flows on I and $g_\tau = \tilde{g}_\tau$ for some $\tau \in I$, then $g_t = \tilde{g}_t$ for all $t \in I$.*

(2) If g is a Riemannian metric, then there exists an $\epsilon > 0$ and a Ricci flow g_t for $t \in (0,\epsilon)$ with $\lim_{t\to 0} g_t = g$.

(3) If $\tilde{g}_t$ is another such Ricci flow in (2), then $g_t = \tilde{g}_t$ for all $t \in (0,\epsilon)$.

So the space of Ricci flows in the space of Riemannian metrics is a foliation by parametrized (directed) 1-dimensional curves. The vague idea is that the topology of the foliation reflects the topology of M.

Here are some successes of this idea:

- uniformization theorem for surfaces
- "sphere theorems" for higher-dimensional manifolds (e.g., the space of positive Ricci curvature metrics on S^3 has a natural fiber bundle structure over the space of constant curvature metrics)
- Poincaré conjecture: if $\dim M = 3$ and $\pi_1(M) = 0$ then $M \cong S^3$
- Geometrization conjecture: uniformization theorem for 3-manifolds
- Generalized Smale conjecture (e.g., if (M^3, g) is has curvature -1, then $\operatorname{Isom}(M^3, g) \hookrightarrow \operatorname{Diff}(M^3)$ is a homotopy equivalence)

There are some adjacent open problems as well. Let M and N be manifolds with M compact, $F_t : M \to N$ be a one parameter family of immersions. Call F_t a **mean curvature flow** if

$$\frac{dF_t}{dt}(p) = \vec{H}(p).$$

Then we have a similar "classical analysis" theorem, and here are the open problems:

- "sphere theorems" in this setting: the hope is that we should be able to construct "special Lagrangians" in "Calabi–Yau manifolds"
- structure of diffeomorphism and symplectomorphism groups. Ex) if $n, m \geq 2$ and $f : S^n \to S^m$ is area-decreasing on every tangent 2-plan,e then f is homotopy to a constant. Take the graph in $S^n \times S^m$ and deform it.
- Bridgeland stability of the derived Fukaya category

1.1 Riemannian geometry

Fix a Riemannian manifold (M, g).

Theorem 1.3. *There exists a unique connection ∇ on TM that is torsion-free ($\nabla_X Y - \nabla_Y X = [X, Y]$) and $\nabla g = 0$ ($X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$).*

Concretely, take local coordinates $x^1, \dots, x^n$. Define

$$g_{ij} = g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big), \quad g^{ij} = (g_{ij})^{-1},$$

and then

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\Big(\frac{\partial g_{jl}}{\partial x^j} + \frac{\partial g_{il}}{\partial g_j} - \frac{\partial g_{ij}}{\partial x^l}\Big).$$

Then the connection is given by

$$\nabla_X Y = X^i \frac{\partial Y^j}{\partial X^i} \frac{\partial}{\partial X^j} + X^i Y^j \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Note that $\nabla_X Y$ depends algebraically on X, and on 0th and 1st derivatives of Y.

Definition 1.4. The **Riemann curvature tensor** is defined as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

From the formula it looks as if $R(X, Y)Z$ depend on 1st derivatives of X and 2nd derivatives of Z, but actually all derivatives cancel and depends algebraically on all three. In particular, you can calculate

$$R(X, Y)Z = R_{ijk}{}^l X^i Y^j Z^k \frac{\partial}{\partial X^l}$$

as

$$R_{ijk}^l \frac{\partial \Gamma_{jk}^l}{\partial x^i} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \Gamma_{ip}^l \Gamma_{jk}^p - \Gamma_{jp}^l \Gamma_{ik}^p.$$

Here, R_{ijk}^l is a dependent on $g, \partial g, \partial^2 g$.

Definition 1.5. The **Ricci tensor** is defined as

$$\mathrm{Ric}(Y, Z) = \mathrm{tr}(X \mapsto R(X, Y)Z).$$

You can also define

$$\mathrm{Ric}(Y, Z) = \sum_{i=1}^n g(R(e_i, Y)Z, e_i)$$

for $e_1, \dots, e_n$ a local orthonormal basis of vector fields. Or you can write $\mathrm{Ric}(Y, Z) = R_{jk} Y^j Z^k$ where $R_{jk} = R_{ijk}{}^i$.

Definition 1.6. We define the **scalar curvature** as

$$R = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)$$

or $R = g^{jk} R_{jk}$.

Let me introduce one convenient notation. We are going to denote lowing and raising indices with the metric implicitly. For instance, $R_i{}^j$ denotes

$$R_i{}^j = g^{jk} R_{ik}.$$

Likewise, $R_i{}^{jk}{}_l$ denotes $g^{jp} g_{kq} g_{lr} R_{ipq}{}^r$.

Here are the basic properties of the Riemann curvature tensor:

- $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$
- $R_{ijkl} + R_{jkil} + R_{kijl} = 0$
- $R_{ij} = R_{ji}$
- $\nabla_i R_{jklp} + \nabla_j R_{kilp} + \nabla_k R_{ijlp} = 0$
- $2\nabla_i R_j{}^i = \nabla_j R$

Definition 1.7. For $P \subseteq T_pM$ a 2-dimensional plane, the **sectional curvature** is

$$K(p) = g(R(e_1, e_2)e_2, e_1)$$

for e_1, e_2 an orthonormal basis of P.

Definition 1.8. g has **positive Ricci curvature** if $\mathrm{Ric}(X, X) > 0$ for $X \neq 0$.

Theorem 1.9 (Hamilton 1982)**.** *Let M be a compact 3-manifold, and g_0 a metric with positive Ricci curvature. Let g_t be the Ricci flow with $\lim_{t\to 0} = g_0$, and maximally extend it so that ϵ is as large as possible (denoted by T). Then necessarily $T < \infty$ and*

$$\lim_{t\to T} \frac{1}{4(T-t)} g_t$$

is a metric of constant sectional curvature 1, where the convergence is in C^∞ as locally defined matrix-valued functions.

2 January 25, 2018

Today we're talking about the maximum principle. Let (M, g) be a compact smooth manifold, with a Riemannian metric. Let $u : M \times [0, T) \to \mathbb{R}$ be a 1-parameter family satisfying

$$\frac{\partial u}{\partial t} = \Delta u.$$

Here, Δu is defined in the following way. ∇ is the connection on TM, and ∇u is the vector field $g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}$. We also have du a 1-form $\frac{\partial u}{\partial x^i} dx^i$, and then

$$|\nabla u|^2 = du(\nabla u) = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}.$$

The **Hessian** $\operatorname{Hess} u = \nabla \nabla u = \nabla(du)$ is a 2-tensor. If α is a k-tensor, we can define $\nabla \alpha$ as a $k+1$-tensor in general. So

$$\nabla(du)(X, Y) = X(du(Y)) - du(\nabla_X Y) = X(Y(u)) - (\nabla_X Y)u.$$

IN locally coordinates, we will have

$$\operatorname{Hess} u\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}.$$

This is the simplest modification of $\partial^2 u$ to get a tensor.

Definition 2.1. If α is a k-tensor, we define $\Delta \alpha$ as a k-tensor as

$$\Delta \alpha = \operatorname{tr}_{12}(\nabla \nabla \alpha) = g^{ij} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \alpha.$$

So

$$\Delta u = g^{ij} \Big(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}\Big).$$

For every p, there exist normal coordinates $x^1, \dots, x^n$ such that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$. Then $\Gamma_{ij}^k(p) = 0$ and so

$$\Delta u(p) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(p).$$

2.1 Maximum principle

Theorem 2.2 (Maximum principle)**.** *Assume that* $\frac{\partial u}{\partial t} = \Delta u$. *If* $\sup_M u(-, 0) < C$, *then* $\sup_M u(-, t) < C$ *for all* $t \in (0, T)$.

Proof. Suppose that there exist x, t such that $u(x, t) \geq C$. Let $t_{\min}$ be the minimal such t. Let $x_{\min}$ be the point such that $u(x_{\min}, t_{\min}) = C$. Then $u(x, t) < C$ for all $x \in M$ and $t < t_{\min}$. Because $u(x, t_{\min}) \leq C$, we have $\nabla u(x_{\min}, t_{\min}) \leq 0$. So by the PDE, we get $\frac{\partial u}{\partial t}(x_{\min}, t_{\min}) \leq 0$.

If we had < 0, then we would have gotten a contradiction. To get around this problem, we replace u by $u_\epsilon(x,t) = u(x,t) - \epsilon t$. Then we have a different PDE

$$\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon - \epsilon.$$

Then we have a strict inequality, so we have $\sup_M u_\epsilon(-,t) < C$ and let $\epsilon \to 0$. $\square$

There are many generalizations of this. We can have more complicated equations like

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 - u^2,$$

but it would not work for equations like

$$\frac{\partial u}{\partial t} = \nabla u + |\nabla u|^2 + u^2.$$

Theorem 2.3 (Hamilton, JDG 1986)**.** *Let $\Omega \subseteq \mathbb{R}^k$ be an open subset, and let $K \subseteq \Omega$ a closed convex subset. Take a smooth function $F : \Omega \to \mathbb{R}^k$ (which can be thought of as a vector field). If for every $k \in K$, the solution of*

$$\frac{dz}{dt} = F(z),\ z(0) = k$$

has $z(t) \in K$ for all $t > 0$, then any $u : M \times [0,T) \to \mathbb{R}^k$ with

$$\frac{\partial u}{\partial t} = \Delta u + F(u)$$

with $u(p,0) \in K$ for all $p \in M$ will satisfy $u(p,t) \in K$ for all $p \in M$.

The intuition is that, at the first point moving outside of K, both ∇u and $F(u)$ point back into K.

Definition 2.4. A function ℓ is called a **support function** for K at $k \in K$ if

(i) $\ell : \mathbb{R}^k \to \mathbb{R}$ is linear with $|d\ell| = 1$,

(ii) $\ell(k) \geq \ell(x)$ for all $x \in K$.

In this case, we say $\ell \in S_k K$.

Proof. Suppose that K is compact. Denote $d_K(x) = \operatorname{dist}(x, K)$, and also define $d(t) = \sup_M d_K(u(-,t))$. We are given that $d(0) = 0$, and we want to show that $d(t) = 0$. The key claim is that

$$d_k(x) = \sup_{k \in \partial K} \sup_{\ell S_k K} \max\{\ell(x-k), 0\}.$$

Then it is easy to feed a linear function into a PDE. It follows that

$$d'(t) \leq \sup_{p,k,\ell} \frac{\partial}{\partial t} \ell(u(p,t) - k).$$

The PDE tells us that

$$\frac{d}{dt}\ell(u(p,t)-k) = \ell(\Delta u(p,t)) + \ell(F(u(p,t))) = \Delta(\ell(u(p,t)) + \ell(F(u(p,t))).$$

At supremum points, p maximizes $d(u(p,t),K)$, k is the closest point in K to p, and $\nabla\ell$ is in the direction $u(p,t)-k$.

Now $\ell(u)$ maximized at (p,t) shows that $\Delta(\ell(u))(p,t) \le 0$. Because the ODE starting at k stays in K, we have $F(k)$ points into K, i.e., $\ell(F(k)) \le 0$. Now

$$\begin{aligned}\ell(F(u(p,t))) &\le \ell(F(u(p,t))) - \ell(F(k)) \le |F(u(p,t)) - F(k)| \\ &\le C|u(p,t)-k| = Cd(t)\end{aligned}$$

for some uniform constant C, because we're assuming that K is compact and F is smooth, so it is uniformly Lipschitz. This all shows that $d'(t) \le Cd(t)$, and $d(0) = 0$ together with this inequality implies that $d(t) = 0$.

Let us now suppose that K is noncompact. Suppose that there exists a counterexample. Because M is compact, the image of u is contained a compact region in Ω up until the first time that u leaves K. Now use a cutoff function to modify outside this compact region. Then we also get a counterexample to the compact K setting. □

2.2 Applications of the maximum principle

Let me give some context to this. If $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ then

$$\begin{aligned}
\frac{\partial g^{ij}}{\partial t} &= 2R^{ij}, \\
\frac{\partial}{\partial t}\Gamma_{ij}^k &= -\nabla_i R_j{}^k - \nabla_j R_i{}^k + \nabla^k R_{ij}, \\
\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\
&\quad - R_i{}^p R_{pjkl} - R_j{}^p R_{ipkl} - R_k{}^p R_{ijpl} - R_l{}^p R_{ijkp}, \\
\frac{\partial}{\partial t}R_{ij} &= \Delta R_{ij} + 2R_{pijq}R^{pq} - 2R_i{}^p R_{pj}, \\
\frac{\partial R}{\partial t} &= \Delta R + 2R^{ij}R_{ij},
\end{aligned}$$

where $B_{ijkl} = -R_{pijq}R^p{}_{kl}{}^q$. The last three equations give nice contexts for maximal principles. We would need to modify the tatment for vector bundles, but this should not be hard.

Theorem 2.5 (Hamilton–Ivey, 1995–1993)**.** *Let M be a compact 3-manifold and g_t be a Ricci flow (with $t \in [0,T)$). Then there exists a constant $C = C(g_0)$ such that*

$$\lambda_1 + \lambda_2 + \lambda_3 \ge -\frac{1}{2}C, \quad \lambda_1 + Cf^{-1}\Big(\frac{\lambda_1+\lambda_2+\lambda_3}{C}\Big) \ge 0$$

where $f(x) = x\log x - x$ *and* $\lambda_1 \le \lambda_2 \le \lambda_3$ *are the eigenvalues of*

$$TM \to TM; \quad v \mapsto Rv - 2R^i{}_j v^j \frac{\partial}{\partial x^i}.$$

(This can be said to be the eigenvalues of $Rg - 2\,\mathrm{Ric}$*.)*

It can be checked that $\lambda_1 \le \lambda_2 \le \lambda_3$ are eigenvalues of $v \mapsto Rv - 2R_i{}^j v^i \frac{\partial}{\partial x^j}$ if and only if $\frac{\lambda_1+\lambda_2}{2} \le \frac{\lambda_1+\lambda_3}{2} \le \frac{\lambda_2+\lambda_3}{2}$ are eigenvalues of $v \mapsto R_i{}^j v^i \frac{\partial}{\partial x^i}$. Hamilton–Ivey can then be rearranged as

$$\frac{\lambda_2+\lambda_3}{2} \ge -\frac{\lambda_1}{2}\log\frac{-\lambda_1}{C}.$$

So if $\lambda_1 \to -\infty$ as $t \to T$, then $\frac{\lambda_2+\lambda_3}{2} \to \infty$ as $t \to T$, at a faster rate. Changing the scale to normalize $\frac{\lambda_2+\lambda_3}{2}$ at a point will make the manifold locally look nonnegatively curved. If we can change the scale and pass to some kind of limit, then the limit will have nonnegative curvature.

3 January 30, 2018

Recall that if $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ then

$$\begin{aligned}\frac{\partial}{\partial t} R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad - R_i{}^p R_{pjkl} - R_j{}^p R_{ipkl} - R_k{}^p R_{ijpl} - R_l{}^p R_{ijkp}\end{aligned}$$

where $B_{ijkl} = -R_{pijq} R^p{}_{kl}{}^q$.

3.1 Shi estimates

Theorem 3.1. *Schematically,*

$$\frac{\partial Rm}{\partial t} = \Delta Rm + g^{-1} \cdot g^{-1} \cdot Rm \cdot Rm.$$

Here, $g^{-1} \cdot g^{-1} \cdot Rm \cdot Rm$ is written as $Rm * Rm$.

As an exercise, show that

$$\nabla_i \nabla_j T_{klm} - \nabla_j \nabla_i T_{klm} = -R_{ijk}{}^p T_{plm} - R_{ikl}{}^p - R_{ijm}{}^p T_{klp}.$$

Proposition 3.2. $\dfrac{\partial}{\partial t}(\nabla Rm) = \Delta(\nabla Rm) + Rm * \nabla Rm.$

Proof. We can compute

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla Rm) &= \frac{\partial}{\partial t}(\partial Rm - \Gamma \cdot Rm) \\ &= \partial\Big(\frac{\partial}{\partial t} Rm\Big) - \frac{\partial \Gamma}{\partial t} Rm - \Gamma \frac{\partial}{\partial t}(Rm) \\ &= \nabla\Big(\frac{\partial}{\partial t} Rm\Big) - \frac{\partial \Gamma}{\partial t} Rm \\ &= \nabla(\Delta Rm + Rm * Rm) - Rm * \nabla Rm = \nabla \Delta Rm + Rm * \nabla Rm.\end{aligned}$$

Then we need to check that $\nabla \Delta Rm = \Delta \nabla Rm + Rm * \nabla Rm$. This can be computed as

$$\begin{aligned}\nabla_i \Delta Rm &= \nabla_i \nabla^p \nabla_p Rm \\ &= \nabla^p \nabla_i \nabla_p Rm + Rm * \nabla Rm \\ &= \nabla^p \nabla_p \nabla_i Rm + \nabla^p(Rm * Rm) + Rm * \nabla Rm \\ &= \Delta \nabla_i Rm + Rm * \nabla Rm.\end{aligned}$$

This proves the claim. □

Theorem 3.3 (Shi)**.** *If M is compact and g_t a Ricci flow, for $t \in [0, \tau]$, if*

$$\sup_M |Rm_{g_t}|_{g_t} \leq \frac{1}{\tau}$$

for all t, then

$$\sup_M |\nabla Rm_{g_t}|_{g_t} \le \frac{c}{\tau t^{1/2}}$$

for all t. (c depends on $\dim M$*.)*

Proof. From $\frac{\partial g^{ij}}{\partial t} = 2R^{ij}$ and $\frac{\partial}{\partial t} Rm = \nabla Rm + R_m * Rm$, we get

$$\frac{\partial}{\partial t}|Rm|^2 = Rm * Rm * Rm + 2R^{ijkl}(\nabla R_{ijkl} + Rm * Rm).$$

So

$$\begin{aligned}\Delta|Rm|^2 &= 2R^{ijkl}\Delta R_{ijkl} + 2|\nabla Rm|^2\\ &= \Delta|Rm|^2 - 2|\nabla Rm|^2 + Rm * Rm * Rm\\ &\le \Delta|Rm|^2 - 2|\nabla Rm|^2 + \frac{c}{\tau^3}\end{aligned}$$

by some Cauchy–Schwartz. In the same way, we compute

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla Rm|^2 &= Rm * \nabla Rm * \nabla Rm + 2\nabla^i R^{jklm}(\Delta\nabla_i R_{jklm} + Rm * \nabla Rm)\\ &= \Delta|\nabla Rm|^2 - 2|\nabla\nabla Rm|^2 + Rm * \nabla Rm * \nabla Rm\\ &\le \Delta|\nabla Rm|^2 + \frac{\tilde{c}}{\tau}|\nabla Rm|^2.\end{aligned}$$

We want to do something about $|\nabla Rm|$. Write

$$F = t^2|\nabla Rm|^2 + c't|Rm|^2.$$

and take the time-derivative. Then

$$\begin{aligned}\frac{\partial F}{\partial t} &= 2t|\nabla Rm|^2 + t^2\frac{\partial}{\partial t}|\nabla Rm|^2 + C'|Rm|^2 + c't\frac{\partial}{\partial t}|Rm|^2\\ &\le \Delta F + 2t|\nabla Rm|^2 + \frac{\tilde{c}t^2}{\tau}|\nabla Rm|^2 - 2c't|\nabla Rm|^2 + c'|Rm|^2 + \frac{ct^2}{\tau^3}\\ &= \nabla F + \left(2t + \frac{\tilde{c}t^2}{\tau} - 2c't\right)|\nabla Rm|^2 + c'|Rm|^2 + \frac{cc't}{\tau^3}.\end{aligned}$$

Choosing $c' = \frac{1}{2}(\tilde{c}+3)$ gives a negative constant for $|\nabla Rm|^2$.

The upshot of all this is

$$\frac{\partial}{\partial t}F \le \Delta F + \frac{c'}{\tau^2} + \frac{cc't}{\tau^3} \le \nabla F + \frac{c}{\tau^2}.$$

By the maximum principle, we have

$$\frac{d}{dt}F_{\max}(t) \le \frac{c}{\tau^2},$$

and $t^2|\nabla Rm|^2 \le F_{\max}(t)$ finishes the proof. ∎

With the same proof with induction, we get the following real Shi estimates.

Theorem 3.4 (Shi)**.** *If*

$$\sup_M |Rm_{g_t}|_{g_t} \leq \frac{1}{\tau}$$

on $t \in [0, \tau]$*, then*

$$\sup_M |\nabla^m Rm| \leq \frac{c}{\tau t^{m/2}}$$

for all $t \in (0, \tau]$*, where* $c = c(m, \dim M)$*.*

The slogan is that the control of $|Rm|$ on some closed parameter interval extends to the control of all deriatives of Rm. Also, for $t \in [\frac{\tau}{2}, \tau]$, we can say

$$\sup_M |\nabla^m Rm|_{g_t} \leq \frac{c_{n,m}}{\tau^{1+\frac{m}{2}}}.$$

3.2 When Ricci flow fails

This was obtained before Shi's estimate, but it is a nice corollary.

Corollary 3.5 (Hamilton, 1982)**.** *Let* M *be a compact manifold, and* g_t *be a Ricci flow for* $t \in [0, T)$*. If* T *cannot be raised (and* $T < \infty$*), then*

$$\limsup_{t \to T} \sup_M |Rm| = \infty.$$

Proof. Suppose not, so that $|Rm| \leq C$ for all p and t. Then by the Shi estimates, we have uniform estimates on $|\nabla^m Rm|$. Fix a tangent vector v. Then

$$\pm \frac{d}{dt} g_t(v, v) = \mp 2 \operatorname{Ric}_{g_t}(v, v) \leq 2 |\operatorname{Ric}|_{g_t} g_t(v, v).$$

So we get

$$\left| \frac{d}{dt} \log g_t(v, v) \right| \leq 2 |\operatorname{Ric}|_{g_t}.$$

Then we get

$$\left| \frac{\log g_\tau(v,v)}{\log g_\theta(v,v)} \right| = \left| \int_\theta^\tau \log g_t(v,v) dt \right| \leq \int_\theta^\tau \left| \frac{d}{dt} \log g_t(v,v) \right| \leq \int_\theta^\tau 2 |\operatorname{Ric}|_{g_t} dt \leq C$$

for $\theta < \tau < T$.

From this, we get the estimate $\frac{1}{C} g_\theta \leq g_\tau \leq C g_\theta$. Then there exists a C^0-convergent subsequence $g_{t_i} \to g_T$ for some $t_i \nearrow T$. Then by the estimate above, we get C^0-convergence $g_t \to g_T$ as $t \to T$.

Now we can go back to the beginning and do the same argument with $\nabla^{g_0} g_t$ replacing g_t. (We need a fixed connection.) Using the first Shi estimate $|\nabla Rm| \leq C$ instead of $|\nabla Rm| \leq C$. Then we get C^1-convergence $g_t \to g_T$ as $t \to T$. Repeat the argument to get C^∞-convergence $g_t \to g_T$ as $t \to T$.

Local existence theorem shows that there exists a Ricci flow $\tilde{g}_t$ on $t \in [T, T + \epsilon)$ with $\tilde{g}_T = g_t$. Then putting g_t and $\tilde{g}_t$ together gives a longer Ricci flow. $\square$

It is typical that $T < \infty$. Recall that $\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2$. Then some linear algebra gives $|\mathrm{Ric}|^2 \geq \frac{1}{n} R^2$. So

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n} R^2.$$

Applying the maximal principle gives

$$\frac{dR_{\min}}{dt} \geq \frac{2}{n} R_{\min}^2$$

and solving this differential equation gives

$$\min_M R(-,t) \geq \frac{n\alpha}{n - 2\alpha t}$$

for $\alpha = \min_M R(-,0)$. The conclusion is that if $\alpha > 0$, then $T < \frac{n}{2\alpha} < \infty$.

As we'll see, in many cases ("sphere theorems") we have not just $|Rm| \to \infty$ somewhere but actually $|Rm| \to \infty$ uniformly. In other examples, e.g., "neckpinching" on S^3, the blowup of $|Rm|$ will only happen on an equatorial S^2. Next time we will apply the vector bundle version of Hamilton's maximal principle to the evolution equation.

4 February 1, 2018

Today we are going to looking at applications of Hamilton's full maximum principle.

4.1 Hamilton's maximum principle

Theorem 4.1 (Hamilton maximum principle). *Let M be a compact manifold and $V \to M$ be a vector bundle, with a metric h on V, h-compatible connections A_t on V, and g_t on M. Let $K \subseteq V$ be a closed, invariant under A_t-parallel transport, fiberwise convex set. Let F be a vector field on V, tangent to the fibers. If the solutions of $\frac{dz}{dt} = F(z)$ preserves K for $z \in K$, then if*

$$\frac{\partial u}{\partial t} = \Delta u + F(u)$$

and $u(-,0) \in K$ then $u(-,t) \in K$ for $t > 0$. (Here, $\Delta u = g^{ij}(\nabla_i^A \nabla_j^A u - \nabla^A_{\nabla_i g_j} u)$.)

We want to apply this to

$$\frac{\partial}{\partial t} Rm = \Delta Rm + \cdots .$$

The technical problem is that we need a fixed metric. So there is something called a "Uhlenbeck trick". We take a 1-parameter family $f_t : TM \to TM$ by

$$\frac{d}{dt} f_t(v) = R_i{}^j v^i \frac{\partial}{\partial x^j}$$

and $f_t(v) = v$.

Proposition 4.2. $g_t(f_t(v), f_t(w)) = g_0(v, w)$.

Proof. We check that $\frac{d}{dt}$ of the left hand side is 0. This is because $\frac{d}{dt} g = -2\,\mathrm{Ric}$. ☐

A cool observation is that if we pull back the Riemann tensor, we get

$$\frac{\partial}{\partial t} Rm_{g_t}(f_t(v), f_t(w), f_t(w), f_t(x)) = \frac{\partial Rm}{\partial t}(\cdot) + \mathrm{Ric} * Rm.$$

Then the four terms we had in this $\frac{\partial}{\partial t} Rm$ cancels out with this $\mathrm{Ric} * Rm$s we have. That is, pulling by f_t not only makes the theorem applicable, but also simplifies the equation. I don't know of a deep reason this happens.

Let us define

$$Q(Rm)_{ijkl} = 2(B_{ijkl} - (k \leftrightarrow l) + (j \leftrightarrow k) - (j \to l \to k \to j)),$$

with $B_{ijkl} = R_{pijq} R^p{}_{kl}{}^q$. Also define

$$C_B(\mathbb{R}^n) = \left\{ \begin{array}{c} \text{multilinear } R : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \text{with the algebraic symmetries of the Riemann tensor} \end{array} \right\}.$$

This is the "algebraic space of curvature tensors".

Definition 4.3. For $Rm \in C_B(\mathbb{R}^n)$, we define

$$\begin{aligned}Q(Rm)(X,Y,Z,W) = \sum_{p,q} &-R(e_p,X,Y,e_q)R(e_pZ,W,e_q)\\ &+ R(e_p,X,Y,e_q)R(e_p,X,Q,e_q) - \cdots\end{aligned}$$

for an orthonormal basis $e_1,\dots,e_n$ of $\mathbb{R}^n$.

Lemma 4.4. *For $Rm \in C_B(\mathbb{R}^n)$, we have $Q(Rm) \in C_B(\mathbb{R}^n)$.*

Theorem 4.5. *Let $K \subseteq C_B(\mathbb{R}^n)$ be a closed, convex subset that is invariant under the natural $\mathrm{O}(n)$-action. Also assume that K is preserved by the ODE $\frac{d}{dt}Rm = Q(Rm)$. If g_t is a Ricci flow on a compact M and $Rm_{g_0} \in K$, then $Rm_{g_t} \in K$ as well.*

Here, $Rm_{g_0} \in K$ should be interpreted by taking a linear isometry between $\mathbb{R}^n$ and (T_pM, g_p).

4.2 Applications in the 3-dimensional case

Lemma 4.6. *In 3-dimensions, we can write*

$$R_{ijkl} = R_{il}g_{jk} - R_{jk}g_{jl} - R_{jl}g_{ik} + R_{jk}g_{il} - \frac{1}{2}(g_{il}g_{jk} - g_{ik}g_{jl}).$$

Proof. You can check that the difference has the symmetries of a curvature tensor, and you can check that it is traceless. Then you can use this condition 3 times to show that it is 0. □

Corollary 4.7. *The equation $\frac{d}{dt}Rm = Q(Rm)$ is equivalent to*

$$\frac{d}{dt}R_{ij} = -4R_{ij}^2 + 3RR_{ij} + 2|\mathrm{Ric}|^2\delta_{ij} - R^2\delta_{ij}.$$

Because the matrix R_{ij} is symmetric and we are working up to $\mathrm{O}(3)$, the three eigenvalues contain all the information. If $\alpha_1 \le \alpha_2 \le \alpha_3$ are the three eigenvalues, we get

$$\begin{aligned}\frac{d}{dt}\alpha_1 &= -4\alpha_1^2 + 3(\alpha_1+\alpha_2+\alpha_3)\alpha_1 + 2(\alpha_1^2+\alpha_2^2+\alpha_3^2) - (\alpha_1+\alpha_2+\alpha_3)^2\\ &= \alpha_2^2 + \alpha_3^2 + \alpha_1\alpha_2 + \alpha_1\alpha_3 - 2\alpha_2\alpha_3 = (\alpha_2-\alpha_3)^2 + \alpha_1(\alpha_2+\alpha_3).\end{aligned}$$

This becomes more pleasant if we make a change of variables

$$\lambda_1 = \frac{\alpha_2+\alpha_3}{2}, \quad \lambda_2 = \frac{\alpha_1+\alpha_3}{2}, \quad \lambda_3 = \frac{\alpha_1+\alpha_2}{2}.$$

Then we get

$$\frac{d}{dt}\lambda_1 = \lambda_1^2 + \lambda_2\lambda_3, \quad \frac{d}{dt}\lambda_2 = \lambda_2^2 + \lambda_1\lambda_3, \quad \frac{d}{dt}\lambda_3 = \lambda_3^2 + \lambda_1\lambda_2.$$

We want to find subsets of this eigenvalue space that is preserved under this system. Note that $\lambda_1 \le \lambda_2 \le \lambda_3$.

Example 4.8. The subset $\{\lambda_1, \lambda_2, \lambda_3 \geq 0\}$ is preserved. This is saying that nonnegative sectional curvature is preserved under the Ricci flow.

Example 4.9. For all $\delta \in [0,1]$, the subset $\{\lambda_1 + \lambda_2 \geq 2\delta\lambda_3\}$ is preserved. To see this, note that at $\lambda_1 + \lambda_2 - 2\delta\lambda_3 = 0$,

$$\frac{d}{dt}(\lambda_1 + \lambda_2 - 2\delta\lambda_3) = \cdots = \lambda_1^2 + \lambda_2^2 - 2\delta\lambda_1\lambda_2.$$

Example 4.10. For all $\delta \in [0,1]$ and $c > 0$, the intersection

$$\{\lambda_1 + \lambda_2 \geq 2\delta\lambda_3\} \cap \{(\lambda_3 - \lambda_1)^{1+\delta} \leq c(\lambda_1 + \lambda_2)\}$$

is preserved. To see this, we compute

$$\begin{aligned} \frac{d}{dt}\log(\lambda_3 - \lambda_1) &= \frac{\lambda_1^3 + \lambda_1\lambda_1 - \lambda_1^2 - \lambda_2\lambda_3}{\lambda_3 - \lambda_1} = \lambda_3 + \lambda_1 - \lambda_2 \leq \lambda_3, \\ \frac{d}{dt}\log(\lambda_1 + \lambda_2) &= \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 + \lambda_2} + \lambda_3 \geq \frac{\lambda_1 + \lambda_2}{2} \geq \lambda_3 \geq (1+\delta)\lambda_3. \end{aligned}$$

This is an extremely a good thing to have, because if λ_3 is blowing up, λ_2 also has to be blowing up at the same rate, and then λ_1 has to be very close to λ_3. Then after a rescaling, the metric is going to look more and more like a sphere.

The missing thing is that we need a uniform blowing up at all points. This isn't going to be too hard given what we have.

5 February 6, 2018

By the end of today, we should have our first main theorem, which is Hamilton's original theorem.

5.1 Hamilton's theorem

Definition 5.1. A subset $F \subseteq C_B(\mathbb{R}^n)$ of algebraic curvature tensors is a **pinching set** if it is

(1) closed, convex, $\mathrm{O}(n)$-invariant,

(2) invariant under $\frac{d}{dt}Rm = Q(Rm)$,

(3) for all $\delta \in (0,1)$, the set $\{Rm \in F Rm \text{ not weakly } \delta\text{-pinched}\}$ is bounded.

Definition 5.2. $Rm \in C_B(\mathbb{R}^n)$ is **weakly δ-pinched** if

$$0 \leq \delta K(\pi_1) \leq K(\pi_2)$$

for all 2-planes $\pi_1, \pi_2 \subseteq \mathbb{R}^n$.

So M is weakly 1-pinched is equivalent to M has constant curvature.

Theorem 5.3 (Hamilton, 1986)**.** *Let M be a compact manifold of dimension ≥ 3, and g_t be a Ricci flow for $t \in [0,T)$ (where T is maximally defined). Suppose in addition that g_0 has positive scalar curvature, and there exists some pinching set $F \subseteq C_B'(\mathbb{R}^n)$ such that $Rm_{g_0}(p) \in F$ for all $p \in M$. Then there exists a $c(t)$ such that*

$$c(t)g_t \to \textit{metric with constant sectional curvature}$$

in C^∞ as $t \to T$.

The condition $R > 0$ is preserved by the Ricci flow because

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2 > \Delta R$$

and then you can apply the maximal principle.

Proof. The third condition on the pinching set F shows that we have, for each $\delta \in (0,1)$,

$$K_{\min}(p,t) \geq \delta K_{\max}(p,t) - C_\delta$$

for some large enough C_δ. We know that the Riemann tensor should be blowing up somewhere, and this shows that

$$\limsup_{t \to T} K_{\max}(t) = \infty.$$

Now I want to say that $K_{\min}(t)/K_{\max}(t) \to 1$ as $t \to T$.

We necessarily need to compare sectional curvatures at different points of M. The idea is to have estimates of $|\nabla R| \leq (\cdots)$ and then integrate this estimate along geodesics. We calculate the evolution equation for the traceless Ricci tensors

$$\tfrac{\partial}{\partial t} \operatorname{Ric}^\circ = \Delta \operatorname{Ric}^\circ + Rm * \operatorname{Ric}^\circ .$$

This is going to give the Shi-type estimate

$$\sup_M |\nabla^{g(t)} \operatorname{Ric}^\circ_{g(t)}|^2 \leq C_n \Big(\sup_{M\times(0,t)} |Rm|_{g(t)} \Big) \sup_{M\times(0,t)} |\operatorname{Ric}^\circ_{g(t)}|.$$

Couple this with $|\nabla R| \leq \frac{2n}{n-2} |\nabla \operatorname{Ric}^\circ|$, and we get the estimate

$$\sup_M |\nabla R_{g(t)}|^2 \leq C \left(\sup_{M\times(0,t)} |Rm| \right) \left(\sup_{M\times(0,t)} |\operatorname{Ric}^\circ| \right)$$

We now have to do something about the terms on the right hand side. We note that the traceless Ric is going to be something about the differences between the eigenvalues of Ric. Then we get this bound in terms of the sectional curvatures. This, along with $K_{\min} \geq \delta K_{\max}(p,t) - C_\delta$, we get

$$\sup_M |\operatorname{Ric}^\circ|_{g(t)} \leq \delta K_{\max}(t) + C_\epsilon$$

for all t.

We need to choose a good curve to integrate this over. Choose (p_t, t_k) to approximately maximize curvature, that is,

$$K_{\max}(t_k) \geq \frac{1}{2} \sup_{M\times[0,t_k]} K_{\max}(t),$$

and the choose $p_k \in M$ such that $K_{\max}(p_k, t_k) = K_{\max}(t_k)$. Then we see that

$$\sup_{M\times(0,t_k)} |\operatorname{Ric}^\circ| \leq 2\epsilon K_{\max}(t_k) + C_\epsilon.$$

If we feed this into the scalar curvature estimate, we get

$$\sup_M |\nabla R_{g(t_k)}|^2 \leq C K_{\max}(t_k)(2\epsilon K_{\max}(t_k) + C_\epsilon)^2.$$

Integrating the estimation on $|\nabla R|$ along geodesics emanating from p_t gives

$$\inf\Big\{ R_{g(t_k)}(p) : p \in B^{g(t_k)}_{2\pi/\sqrt{K_{\max}(t_k)}}(p_k) \Big\} \geq R_{g(t_k)}(p_k) - 2\pi\sqrt{C}(2\epsilon K_{\max}(t_k) + C_\epsilon).$$

Because the scalar curvature is the sum of the sectional curvatures, we can replace

$$\inf\Big\{ K_{\max}(p,t_k) : p \in B^{g(t_k)}_{2\pi/\sqrt{K_{\max}(t_k)}}(p_k) \Big\} \geq K_{\min}(p_k,t_k) - 2\pi\sqrt{C}(2\epsilon K_{\max}(t_k) + C_\epsilon).$$

On the left hand side, we can apply the estimate

$$K_{\max}(p,t_k) \le \frac{K_{\min}(p,t_k)}{1-\epsilon} + \frac{c'}{1-\epsilon},$$

and on right hand side, we can apply

$$K_{\min}(p_k,t_k) \ge (1-\epsilon)K_{\max}(t_k) - C'.$$

Putting them together gives

$$\liminf_{K\to\infty} \frac{\inf\{K_{\min}(p,t_k) : p \in (\cdots)\}}{K_{\max}(t_k)} \ge (1-\epsilon)^2 - 2\pi\sqrt{C}2\epsilon(1-\epsilon).$$

As $\epsilon \to 0$, we get that the right hand side goes to 1.

Now let us improve what we have. We first claim that $\liminf_{k\to\infty} K_{\min}(t_k)/K_{\max}(t_k) = 1$. If not, we have just showed that the ball centered at p_t with radius $2\pi/\sqrt{K_{\max}(t_k)}$ is not the entire M. But we have

$$\inf_{\gamma_k} K_{\min}(-,t_k) \le \frac{\pi^2}{L(\gamma)^2} = \frac{1}{4}K_{\max}(t_k)$$

for a minimizing geodesic γ_k, and this is clearly false.

From this we say what we wanted to say. Suppose not, so that there exists a $\tau_k \to T$ such that

$$\liminf_{k\to\infty} \frac{K_{\min}(\tau_k)}{K_{\max}(\tau_k)} < 1.$$

Choose $t_k \in [0,\tau_k]$ to maximize curvature at $K_{\max}(t_k) = \sup_{[0,\tau_k]} K_{\max}(t)$. By the exactly same argument, we get

$$\liminf_{k\to\infty} \frac{K_{\min}(t_k)}{K_{\max}(t_k)} \ge 1.$$

But because R is increasing, we have $K_{\max}(\tau_k) \ge K_{\min}(t_k)$ and so

$$K_{\max}(\tau_k) \ge K_{\min}(t_k) \ge \frac{1}{2}K_{\max}(t_k) = \frac{1}{2}\sup_{[0,\tau_k]} K_{\max}(t).$$

This shows that the previous lemma is applicable to τ_k, and so

$$\liminf_{k\to\infty} \frac{K_{\min}(\tau_k)}{K_{\max}(\tau_k)} \ge 1.$$

Now the following lemma can use used to finish the proof. □

Lemma 5.4. *(1)* $(\tau - t)\sup_M R_{g(t)} \to \dfrac{n}{2}$.

(2) $(\tau - t)\inf_M R_{g(t)} \to \dfrac{n}{2}$.

(3) For a fixed $\alpha < \frac{1}{n-1}$*,*

$$\sup_M |\mathrm{Ric}^\circ_{g(t)}|^2 \leq C(\tau - t)^{2\alpha-2}$$

and then by the Shi estimates, for all $m \geq 1$*,*

$$\sup_M |\nabla^m \mathrm{Ric}^\circ_{g(t)}|^2 \leq C(\tau - t)^{2\alpha-m-2}.$$

(4) $\displaystyle\sup_M \left|\mathrm{Ric}_{g(t)} - \frac{1}{2(\tau - t)}\right|^2 \leq C(T-\tau)^{2\alpha-2}.$

Proof. These lemmas can be proved in a similar way, and is uninteresting. You can read Lemmas 5.18–23 in Brendle's book. □

Now using this lemma, we can mimic to proof of Hamilton's corollary to construct the limit

$$\frac{1}{2(n-1)(T-t)} g_t \to \tilde{g}_T.$$

This limit $\tilde{g}_t$ has constant scalar curvature $n(n-1)$ and (1) of the lemma, and constant curvature by the previous lemma. This finishes the proof of Hamilton's theorem.

Corollary 5.5. *A compact* 3*-manifold* M *with some metric* g_0 *of* $\mathrm{Ric} > 0$*, then the corresponding Ricci flow* g_t *has*

$$\frac{1}{4(T-t)} g_t \to \textit{constant curv. } 1$$

Proof. We claim that if $K \subseteq C_B(\mathbb{R}^3)$ is compact and is in the cone of $\mathrm{Ric} > 0$, then there exists a pinching set $F \subseteq C_B(\mathbb{R}^3)$ with $K \subseteq F$. This will prove the corollary.

This can be shown in the following way. We know that

$$K \subseteq \{Rm \in C_B(\mathbb{R}^3) : \lambda_1 + \lambda_2 \geq 0\}$$

where $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of $\frac{1}{2}Rg - \mathrm{Ric}$. Because K is compact, there exist δ, C such that

$$K \subseteq \{\lambda_1 + \lambda_2 > 0, \lambda_1 + \lambda_2 \geq 2\delta\lambda_3, (\lambda_3 - \lambda_1)^{1+\delta} \leq C(\lambda_1 + \lambda_2)\}.$$

You can check that this is a legitimate pinching set. □

Corollary 5.6. *If a* 3*-manifold has a metric with* $\mathrm{Ric} > 0$*, then also has metric with constant curvature.*

For instance, if $\pi_1 = 0$ then $M \cong S^3$. It would be interesting to construct a metric of $\mathrm{Ric} > 0$ just from $\pi_1 = 0$.

6 February 8, 2018

You might be interested in reading Hamilton's JDG 82 and 86 papers, Huisken's JDG 84, and Gage–Hamilton JDG 86, Grayson JDG 87 papers.

Proposition 6.1. *In dimension* 3*, nonegative sectional curvature is preserved by the Ricci flow.*

Proof. The relevant set is $\{\lambda_1 \geq 0\}$ where $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of $R\delta - 2\,\mathrm{Ric}$. The set is convex, obviously closed, O(3)-invariant, and preserved by the ODE $\frac{d\lambda_1}{dt} = \lambda_1^2 + \lambda_2\lambda_3 \geq 0$. $\square$

Here, the eigenvalues of $R\delta - 2\,\mathrm{Ric}$ is also the sectional curvatures in dimension 3.

6.1 The Bohm–Walking paper

Unfortunately, the Ricci flow does not preserve positive sectional curvature in higher dimensions. The Ricci flow cannot converge generally in this setting to constant curvature, since there are manifolds like $\mathbb{C}P^n$ with a positive sectional curvature metric but no constant curvature metric.

The reference is Bohm–Walking 2008 Annals paper. The difficulty is to produce sets in $C_B(\mathbb{R}^n)$ invariant under $\frac{d}{dt}Rm = Q(Rm)$, if we want to apply the pinching set criterion.

Definition 6.2. Define the **curvature operator** $Rm : \bigwedge^2 T_pM \to \bigwedge^2 T_pM$ given by

$$\langle Rm(X \wedge Y), Z \wedge W\rangle = 2g(Rm(X,Y)W,Z).$$

The curvature is well-defined, and is self-adjoint. Because this is self-adjoint, there exist eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$, we say that this is a positive curvature operator if $\lambda_1 > 0$.

Definition 6.3. For $s \in [0,1)$, consider $C(s)$ a continuous family of closed convex $\mathrm{O}(n)$-invariant cones in $C_B(\mathbb{R}^n)$. Call this a **pinching family** if

(1) any $Rm \in C(s) \setminus \{0\}$ has positive scalar cuvature,

(2) $\frac{d}{dt}Rm = Q(Rm)$ moves boundary points "inside" (into the interior)

(3) $C(s) \to \{\kappa(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\}$ as $s \to 1$ in the Hausdorff topology.

Here is the basic picture. If there exists a pinching family, and if g_0 of the Ricci flow has $Rm_{g_0}(p) \in C(0)$ for all o, then you can move $C(s)$ along with g_t. Then g_T sill have $Rm_{g_T}(p)$ a constant sectional curvature. By Schur's lemma, g_T sill have constant curvature.

Now the question is of how to construct a pinching family. Take numbers $a, b \geq 0$ and define a linear transformation $\ell_{a,b} : C_B(\mathbb{R}^n) \to C_B(\mathbb{R}^n)$ given by

$$\ell_{a,b}(Rm)_{ijkl} = R_{ijkl} + b(R_{il}\delta_{jk} - R_{ik}\delta_{jl} - R_{jl}\delta_{ik} + R_{jk}\delta_{il}) + \frac{a-b}{n}R(2\delta_{il}\delta_{jk} - 2\delta_{ik}\delta_{jl}).$$

This is a natural thing to say, because for any $Rm \in C_B$,

$$Rm = \frac{1}{n-2}(R^\circ_{ij}\delta_{jk} - R^\circ_{ik}\delta_{jl} - R^\circ_{jl}\delta_{ik} + R^\circ_{jk}\delta_{il}) \\ + \frac{R}{n(n-1)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + [Rm - \text{last two terms}]$$

then the inner product of any two of these three terms is zero. That is, this is the orthogonal decomposition of the tensor. (This is said to be related to the representation theory of $\mathrm{SO}(n)$.) So $\ell_{a,b}$ is just moves around 2 of the components.

Proposition 6.4 (Bohm–Walking)**.** $\ell^{-1}_{a,b} \circ Q \circ \ell_{ab}(Rm) = Q(Rm) + D_{ab}(Rm)$ *where* $D_{a,b}(Rm)$ *depends only on* Ric*.*

If we already have the preserved condition C and $\ell_{a,b}(C)$ to be preserved too, this is equivalent to saying that $\ell^{-1}_{a,b} \circ Q \circ \ell_{a,b}(Rm)$ points into C at any boundary point as well. Because we already know that Q points back into C, we just need $D_{a,b}(Rm)$ to point back in as well.

Proposition 6.5. *Let* $C \subseteq C_B(\mathbb{R}^n)$ *be some closed convex* $\mathrm{O}(n)$*-invariant,* Q*-invariant cone. Assume that nonnegative curvature operators are in* C *but* C *is contained in nonnegative sectional curvatures. For* $0 < b \leq \frac{1}{2}$ *and* $2a = \frac{2b+(n-2)b^2}{1+(n-2)b^2}$*, the set*

$$\{\ell_{a,b}(Rm) : Rm \in C, \mathrm{Ric} \geq \tfrac{\delta}{n} R\delta\}$$

is strictly preserved by the Q*-ODE.*

Proof. We need strict invariance at the boundary. So we first need to show that $Q(Rm) = D_{a,b}(Rm)$ points inside C and also

$$\mathrm{Ric}(Q(Rm) + D_{a,b}(Rm))(v,v) > \frac{\delta}{n} R(Q(Rm) + D_{a,b}(Rm)).$$

For the first part, take e_1 be an orthonormal eigenbasis of Ric. Then after scaling $R = n$, we can write Ric as a diagonal matrix with entries $1 + \lambda_i$, with $\sum_i \lambda_i = 0$. Then $e_i \wedge e_j$ becomes an eigenbasis of $D_{a,b}(Rm)$. The eigenvalues are

$$D_{a,b}(Rm)(e_i, e_j, e_j, e_i) = \cdots > \frac{\cdots}{\cdots} \sum_i \lambda_i$$

with the coefficient being positive. So $D_{a,b}(Rm)$ is strictly positive. Because C contains nonnegative curvature operators, the strictly positive $D_{a,b}(Rm)$ points strictly inwards.

For the second part, we use the same setup. Here

$$\mathrm{Ric}(Q(R)) = 2 \sum g(R(e_i, v)v, e_i) \,\mathrm{Ric}(e_i, e_i) \geq 2\delta \,\mathrm{Ric}(v,v) = 2\delta^2$$

and

$$R(Q(Rm)) = 2|Ric|^2 = 2n + 2|\mathrm{Ric}^\circ|^2$$

implies

$$\mathrm{Ric}(Q(R))(v,v) - \frac{\delta}{n}R(Q(Rm)) \geq -2\delta(1-\delta) - \frac{2}{n}\delta|\mathrm{Ric}^\circ|^2.$$

Then you can check using arithmetic. $\square$

Proposition 6.6. *Under the setting $b = \frac{1}{2}$ and $a > \frac{1}{2}$ and $\delta = 1 - \frac{4}{n-2+8a}$, we have the same conclusion.*

We need a good choice of closed convex $\mathrm{O}(n)$-invariant cone C. Then we are going to define

$$C(s) = \{\ell_{a(s),b(s)}(Rm) : Rm \in C, \mathrm{Ric} > \tfrac{\delta(s)}{n}R\delta\}$$

for a specific choice of $a(s)$, $b(s)$, and $\delta(s)$:

$$a(s) = \begin{cases} \frac{2s+(n-2)s^2}{2(1+(n-2)s^2)} & 0 < s \leq \frac{1}{2}, \\ s & s > \frac{1}{2}, \end{cases} \quad b(s) = \begin{cases} s & 0 < s \leq \frac{1}{2}, \\ \frac{1}{2} & s > \frac{1}{2}, \end{cases}$$

$$\delta(s) = \begin{cases} 1 - \frac{1}{1+(n-2)s^2} & 0 < s \leq \frac{1}{2}, \\ 1 - \frac{4}{n-2+8s} & s > \frac{1}{2}. \end{cases}$$

Then as $s \to \infty$, we would get

$$\lim_{s\to\infty} \frac{1}{a(s)}\ell_{a(s),b(s)}(Rm) = \frac{1}{n}R(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

So the difficulty is the first step of choosing C.

7 February 13, 2018

The basic problem we faced at the very end was knowing whether we know a cone. What are the algebraic conditions on curvature preserved by the Ricci flow in dimension > 3? There are 3 basic answers.

- nonnegative curvature $Rm : \bigwedge^2 \to \bigwedge^2$
- 2-nonnegative $Rm : \bigwedge^2 \to \bigwedge^2$, meaning $\lambda_1 + \lambda_2 \geq 0$
- nonnegative isotropic curvature

The third one is due to Brendle–Schoen (JAMS 2009). The proof of each is analyzing the ODE $\frac{d}{dt} Rm = Q(Rm)$ on $C_B(\mathbb{R}^n)$.

7.1 Complexification

Definition 7.1. The **complexification** of the tangent space is $T_p^{\mathbb{C}} M = T_p M \otimes_{\mathbb{R}} \mathbb{C}$. There is a natural inner product

$$\langle v_1 + iv_2, u_1 + iu_2 \rangle = \langle v_1, u_1 \rangle - \langle v_2, u_2 \rangle + i(\langle v_1, u_2 \rangle + \langle v_2, u_1 \rangle).$$

(This is not nonnegative.) Call $z \in T_p^{\mathbb{C}}(M0$ **isotropic** if $\langle z, z \rangle = 0$. This is equivalent to, where $z = x + iy$,

$$\langle x, x \rangle = \langle y, y \rangle, \quad \langle x, y \rangle = 0.$$

Call a complex subspace $P \subseteq T_p^{\mathbb{C}} M$ **isotropic** if if all points in P are isotropic. For instance, a 2-dimensional space P is isotropic if it has basis $e_1 + ie_2, e_3 + ie_4$ with e_1, e_2, e_3, e_4 are orthonormal. We can extend $Rm : \bigwedge^2 T_p M \to \bigwedge^2 T_p M$ to $Rm : \bigwedge^2 T_p M \otimes_{\mathbb{R}} \mathbb{C} \to \bigwedge^2 T_p M \otimes_{\mathbb{R}} \mathbb{C}$ by

$$Rm(\omega_1 + i\omega_2) = Rm(\omega_1) + iRm(\omega_2).$$

Definition 7.2. Let $P \subseteq T_p^{\mathbb{C}} M$ be a complex subspace with $\dim_{\mathbb{C}} = 2$. We define **complex sectional curvature** of P as

$$\langle Rm(v_1 \wedge v_2), \overline{v_1 \wedge v_2} \rangle$$

for v_1, v_2 an orthonormal basis for P. Say M has **nonnegative complex sectional curvature** if complex sectional curvature of any 2-dimensional $P \subseteq T_p^{\mathbb{C}} M$ is nonnegative. Say M has **nonnegative isotropic sectional curvature** if complex sectional curvature of any isotropic 2-plane $P \subseteq T_p^{\mathbb{C}}(M)$ is nonnegative.

This really hasn't to much with the manifold.

Proposition 7.3. *Nonnegativity of isotropic curvature is equivalent to*

$$R_{1331} + R_{1441} + R_{2332} + R_{2442} \geq 2R_{1234}$$

for all orthonormal $e_1, e_2, e_3, e_4 \in T_p M$.

7.2 Construction of a Q-invariant cone

Theorem 7.4. *The ODE $\frac{d}{dt} = Q(Rm)$ on $C_B(\mathbb{R}^n)$ preserves*

(1) nonnegative complex sectional curvature and

(2) nonnegative isotropic curvature.

Let's just recall

$$Q(Rm)_{ijkl} = -2R_{pijq}R^p{}_{kl}{}^q + 2R_{pijq}R^p{}_{lk}{}^q - 2R_{pikq}R^p{}_{jl}{}^q + 2R_{pilq}R^p{}_{jk}{}^q.$$

Note that the first two terms combine to

$$R_{ij}{}^{pq}R_{pqkl} = Rm \circ Rm : \bigwedge^2 \to \bigwedge^2.$$

So this is nonnegative, so it is mostly sufficient to only look at the last 2 terms in Q.

Proof. (1) Suppose $Rm \in C_B(\mathbb{R}^n)$ has nonnegative complex sectional curvature with zero somewhere, i.e., $x, y \in \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ with

$$x^i y^j \bar{y}^k \bar{x}^l R_{ijkl} = 0.$$

Define $x_t = x + tw$ and $y_t = y + tz$ for some $w, z \in \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$. If we define

$$h(t) = x_t^i y_t^j \bar{y}_t^k \bar{x}_t^l R_{ijkl}$$

then the assumptions give $h(0) = 0$ and $h'(0) = 0$ and $h''(0) \geq 0$. If we do this, we get

$$0 \leq \frac{1}{2}h''(0) = (w^i y^j \bar{y}^k \bar{w}^l + (w^i \bar{x}^l + x^i \bar{w}^l)(z^j \bar{y}^k + y^j \bar{z}^k) + x^i z^j \bar{z}^k \bar{x}^l)R_{ijkl}.$$

Likewise, if you iw and iz instead of w and z, we get

$$0 \leq \frac{1}{2}h''(0) = (w^i y^j \bar{y}^k \bar{w}^l + (iw^i \bar{x}^l - ix^i \bar{w}^l)(iz^j \bar{y}^k - iy^j \bar{z}^k) + x^i z^j \bar{z}^k \bar{x}^l)R_{ijkl}.$$

If we add them, we get

$$0 \leq 2(w^i y^j \bar{y}^k \bar{w}^l + x^i z^j \bar{y}^k \bar{w}^l + w^i y^j \bar{z}^k \bar{x}^l + x^i z^j \bar{z}^k \bar{x}^l)R_{ijkl}.$$

We can view this inequality as having an nonnegative hermitian (x, y, R)-dependent inner product on (z, w). To apply the next lemma, we set

$$A_{pq} = x^i \bar{x}^l R_{ipql}, \quad C_{pq} = y^j \bar{y}^k R_{jpqk}, \quad B_{pq} = -x^j \bar{y}^k R_{ipqk}$$

and then we get

$$x^i \bar{x}^l \bar{y}^j y^k R_{ipql} R_j{}^{pq}{}_k = \operatorname{tr}(A\overline{C}) \geq \operatorname{tr}(B\overline{B}) = x^i \bar{y}^k \bar{x}^j y^l R_{ipqk} R_j{}^{pq}{}_l.$$

This just says that $x^i y^j \bar{y}^k \bar{x}^l$ times the last two terms of $Q(Rm)_{ijkl}$ is nonnegative.

(2) Suppose $Rm \in C_B(\mathbb{R}^n)$ has some nonnegative isotropic curvature, zero on some isotropic 2-plane. Take a basis $u = e_1 + \sqrt{-1}e_2$ and $v = e_3 + \sqrt{-1}e_4$ with e_1, e_2, e_3, e_4 orthonormal. Take a basis

$$u, \bar{u}, v, \bar{v}, e_5, \dots, e_n$$

of $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ that is orthonormal with the hermitian inner product. For $x, y \in \operatorname{span}_{p>4} e_p$, define the quadratic deformation

$$u_t = u + tx - \frac{t^2\langle x, x\rangle}{2}\bar{u} - \frac{t^2\langle x, y\rangle}{2}\bar{v}, \quad v_t = v + ty - \frac{t^2\langle x, y\rangle}{2}\bar{u} - \frac{t^2\langle y, y\rangle}{2}\bar{v}.$$

Then u_t, v_t stay isotropic as $t > 0$. As before, define $h(t)$ in the same way, and we will get

$$0 = h'(0) = 2\Re(Rm(x, v, \bar{v}, \bar{u}) + Rm(u, y, \bar{v}, \bar{u})).$$

But we can multiply them with i can get the same thing for imaginary parts, and so

$$Rm(x, v, \bar{v}, \bar{u}) = Rm(u, y, \bar{v}, \bar{u}) = 0.$$

We can also do the computation of $h''(0)$. Then

$$0 \le Rm(v, x, \bar{x}, \bar{v}) + Rm(u, y, \bar{y}, \bar{u}) + Rm(v, x, \bar{y}, \bar{u}) + Rm(u, y, \bar{x}, \bar{v}).$$

If we again apply the lemma, we get

$$\sum_{p,q>4} Rm(u, e_p, e_q, \bar{u})Rm(v, e_p, e_q, \bar{v}) - Rm(u, e_p, e_q, \bar{v})Rm(v, e_p, e_q, \bar{u}) \ge 0.$$

This almost what we need except for that we need $\sum_{p,q}$ instead of $\sum_{p,q>4}$. But all other terms vanish because of the vanishing condition $h(0) = h'(0) = 0$. At the end, you are left with

$$(Rm(u, \bar{u}, \bar{v}, \bar{u}) + Rm(v, \bar{v}, \bar{v}, \bar{u}))(Rm(u, v, \bar{v}, v) + Rm(u, v, \bar{u}, u)).$$

Go through the $h(t)$ again with $u_t = u + t\bar{v}$ and $v_t = v - t\bar{u}$ with $h'(0) = 0$ and with the i factor. Then the first factor is 0. $\square$

Lemma 7.5. *If $\left(\begin{smallmatrix} A & B \\ B^\dagger & C \end{smallmatrix}\right)$ is nonnegative Hermitian on $\mathbb{C}^{2n}$, then*

$$\operatorname{tr}(A\bar{C}) \ge \operatorname{tr}(B\bar{B}).$$

Definition 7.6. If $Rm \in C_B(\mathbb{R}^n)$, define $Rm \times \mathbb{R}^k \in C_B(\mathbb{R}^n \times \mathbb{R}^k)$ by the trivial extension to all of $\mathbb{R}^k$, i.e.,

$$(Rm \times \mathbb{R}^k)((v_1, u_1), \dots, (v_4, u_4)) = Rm(v_1, \dots, v_4).$$

This can be thought of as the Riemann tensor on $M \times \mathbb{R}^k$.

Corollary 7.7. *Fix $k \ge 0$. The ODE $\frac{d}{dt}Rm = Q(Rm)$ preserves*

$$\{Rm \in C_B(\mathbb{R}^n) : Rm \times \mathbb{R}^k \text{ has nonnegative isotropic curvature}\}.$$

Here is the crucial proposition:

(1) $Rm : \bigwedge^2 \to \bigwedge^2$ nonnegative implies $Rm \times \mathbb{R}^2$ has nonnegative isotropic curvature.

(2) $Rm \times \mathbb{R}^2$ has nonnegative isotropic curvature implies Rm has nonnegative sectional curvature.

This allows us to apply the Bohm–Walking theorem.

8 February 15, 2018

Definition 8.1. (M, g) has **nonnegative isotropic curvature** if

$$R_{1331} + R_{1441} + R_{2332} + R_{2442} \geq 2R_{1234}$$

for any orthonormal e_1, e_2, e_3, e_4.

Proposition 8.2. *The nonnegative isotropic curvature condition implies nonnegative scalar.*

Proof. Sum over all i, j, k, l and you get that $(n-3)R \geq 0$. □

Proposition 8.3. *$Rm \times \mathbb{R}$ nonnegative isotropic curvature implies* $\mathrm{Ric} \geq 0$. *Also, $Rm : \bigwedge^2 \to \bigwedge^2$ being 2-nonnegative implies $Rm \times \mathbb{R}$ nonnegative isotropic curvature.*

Proof. We use the fact that $Rm \times \mathbb{R}$ nonnegative isotropic curvature is equivalent to

$$R_{1331} + \lambda^2 R_{1441} + R_{2332} + \lambda^2 R_{2442} \geq 2\lambda R_{1234}$$

for all orthonormal $e_1, \ldots, e_4$ and $\lambda \in [0, 1]$. This is because we can use $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$, $(\lambda e_4, \sqrt{1-\lambda^2})$ in one direction. If $(v_1, x_1), \ldots, (v_4, x_4)$ are orthonormal on $\mathbb{R}^n \times \mathbb{R}$, you can show that $\varphi = v_1 \wedge v_3 + v_4 \wedge v_2$ and $\psi = v_1 \wedge v_4 + v_2 \wedge v_3$ can be represented as $\varphi = a_1 e_1 \wedge e_3 + a_2 e_4 \wedge e_2$ and $\psi = b_1 e_1 \wedge e_4 + b_2 e_2 \wedge e_3$.

If we take $\lambda = 0$ and sum over i, j, then we get $2(n-1)\mathrm{Ric}_{kk} \geq 0$. The 2-nonnegativity condition can be equivalently desribed as

$$\langle Rm(\varphi), \varphi \rangle + \langle Rm(\psi), \psi \rangle \geq 0$$

for $\varphi, \psi \in \bigwedge^2$ and $|\varphi|^2 = |\psi|^2$ and $\langle \varphi, \psi \rangle = 0$. We use

$$\varphi = e_1 \wedge e_3 + \lambda e_4 \wedge e_2, \quad \psi = \lambda e_1 \wedge e_4 + e_2 \wedge e_3$$

and get what we want. □

Similarly, we have the following.

Proposition 8.4. *$Rm \times \mathbb{R}^2$ having nonnegative isotropic curvature implies nonnegative sectional curvature. $Rm : \bigwedge^2 \to \bigwedge^2$ nonnegative implies $Rm \times \mathbb{R}^2$ nonnegative isotropic curvature.*

Proof. Here, the lemma is that $Rm \times \mathbb{R}^2$ nonnegative isotropic curvature is equivalent to

$$R_{1331} + \lambda^2 R_{1441} + \mu^2 R_{2332} + \lambda^2 \mu^2 R_{2442} \geq 2\lambda\mu R_{1234}$$

for orthonormal e_1, e_2, e_3, e_4 and $\lambda, \mu \in [0, 1]$. □

So here is the summary:

$$\begin{array}{ccccc}
\begin{array}{c} Rm : \bigwedge^2 \to \bigwedge^2 \\ \text{nonnegative} \end{array} & \Longrightarrow & \begin{array}{c} Rm : \bigwedge^2 \to \bigwedge^2 \\ \text{2-nonnegative} \end{array} & & \\
\Downarrow & & \Downarrow & & \\
Rm \times \mathbb{R}^2 \text{ NIC} & \Longrightarrow & Rm \times \mathbb{R} \text{ NIC} & \Longrightarrow & Rm \text{ NIC} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\sec \geq 0 & \Longrightarrow & \mathrm{Ric} \geq 0 & \Longrightarrow & R \geq 0
\end{array}$$

8.1 Existence of a pinching set

Lemma 8.5. *The set $\{Rm : Rm \times \mathbb{R}^2 \text{ NIC}\}$ is convex in $C_B(\mathbb{R}^n)$.*

Proof. This is obvious because it is a linear condition. ☐

Define

$$C(s) = \{\ell_{a(s),b(s)}(Rm) : Rm \times \mathbb{R}^2 \text{ NIC}, \mathrm{Ric} \geq \tfrac{\delta(s)}{n} R\delta\}$$

where $a(s)$ and $\delta(s)$ and $b(s)$ is as defined before.

Proposition 8.6. *$C(s) \subseteq C_B(\mathbb{R}^n)$ is closed, convex, $\mathrm{O}(n)$-invariant, and strictly invariant under the Q-ODE.*

The basic construction is

$$F = C(s_0) \cap \bigcap_{i=1}^{\infty} \{Rm : Rm + 2^i h(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \in C(s_i)\}$$

for some h and $s_i \to \infty$ we will choose. So you are pulling the cones back as the cones get thinner and thinner.

The hard thing is to verify ODE-invariance. Let K be a compact set in $\mathrm{int}(C(0))$. Take $s_0 > 0$ such that $K \subseteq C(s_0)$, and take

$$h = \max_K[\text{scalar curvature}].$$

We need to control the ODE under translations.

Proposition 8.7. *There exist $N(\bar{s}) \geq 1$ such that if $s \in [s_0, \bar{s}]$ and $Rm \in \partial C(s)$ and $\mathrm{scal}(Rm) \geq N(\bar{s})$ then $Q(\tilde{Rm})$ points into $C(s)$ at Rm as long as $|Rm - \tilde{Rm}| \leq 2|\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}|$.*

Proof. This is trivial by compactness, because $Q(Rm)$ points strictly into $C(s)$ and Q is homogeneous quadratic. ☐

Lemma 8.8. *There exists a decreasing $\delta(\bar{s})$ such that if $s \in [s_0, \bar{s}]$ and $Rm + (\delta * \delta) \in C(s)$ and $\mathrm{scal}(Rm) \leq N(\bar{s})$ then $Rm + 2(\delta * \delta) \in C(s + \delta(\bar{s}))$.*

Now we are going to set $s_j = s_{j+1} + \delta(s_{j-1})$. Define F_j to be the jth intersection in the definition of F.

Proposition 8.9. *F_j only changes for larger scalar curvature:*

$$F_{j+1} \cap \{R \leq 2^j N(s_j)h\} = F_j \cap \{R \leq 2^j N(s_j)h\}.$$

Proof. If Rm is in the right hand side then $Rm + 2^j h(\delta * \delta) \in C(s_j)$ and so $R \leq 2^j N(s_j)h$. Then $\frac{Rm}{2^j h} + \delta * \delta \in C(s_j)$ and $\frac{R}{2^j h} \leq N(s_j)$. The lemma implies that $\frac{Rm}{2^j h} + 2\delta * \delta \in (s_{j+1}$ and then rescaling gives that Rm is in the left hand side. ☐

By the definition of h, we have

$$K \subseteq C(s_0) \cap \{\text{scal} \leq h\} = F \cap \{\text{scal} \leq h\} \subseteq F.$$

Proposition 8.10. *F is invariant under the Q-ODE.*

Proof. We have $Rm \in \partial F$ implies $Rm \in \partial C(s_0)$ or $Rm \in \partial\{Rm : Rm + 2^i h(\delta * \delta) \in C(s_i)\}$. This necessarily implies $\text{scal} \geq 2^{i-1}N(s_i)h$. If we set

$$Rm' = \frac{Rm + 2^i h(\delta * \delta)}{2^{i-1}h}, \quad \tilde{Rm}' = \frac{Rm}{2^{i-1}h},$$

we can see that $Q(\tilde{Rm}')$ points into $C(s_i)$. So we can rescale. ☐

Proposition 8.11. *F is a pinching set.*

Proof. We claim that $Rm' \in C(s)$ and $s > \frac{1}{2}$ implies that Rm' is weakly $\frac{2s-1}{2s+n-1}$-pinched. This is enough.

If $Rm' \in C(s)$ then

$$Rm' = \ell_{a,\frac{1}{2}}(Rm) = Rm + \frac{1}{2}\,\text{Ric}(\wedge)\delta + \frac{1}{n}\Big(s - \frac{1}{2}R\delta(\wedge)\delta,$$

and so $Rm'(e_1, e_2, e_2, e_1) \geq \frac{1}{n}(s - \frac{1}{2})R$ by Rm having nonnegative sectional curvature. On the other hand,

$$\begin{aligned} Rm'(e_1, e_2, e_2, e_1) \leq{}& \frac{\text{Ric}(e_1, e_2) + \text{Ric}(e_2, e_2)}{2} + \frac{\text{Ric}(e_1, e_1) + \text{Ric}(e_2, e_2)}{2} \\ &+ \frac{1}{n}\Big(s - \frac{1}{2}\Big)R \leq R + \frac{1}{n}\Big(s - \frac{1}{2}\Big)R. \end{aligned}$$

This finishes the proof. ☐

Theorem 8.12 (Brendle–Schoen)**.** *If M is compact and $\dim \geq 4$, and g_t is a Ricci flow on $[0, T)$ and g_0 is such that $M \times \mathbb{R}^2$ has positive isotropic curvature, then*

$$\frac{1}{2(n-1)(T-t)} g_t \xrightarrow{C^\infty} \textit{metric of cons. curv.}$$

9 February 20, 2018

Theorem 9.1 (Brendle–Schoen, JAMS 2009)**.** *If M is compact with $\dim M \geq 4$, and g_t is a Ricci flow for $t \in [0, T)$ with g_0 such that $M \times \mathbb{R}^2$ having positive isotropic curvature, then*

$$\frac{1}{2(n-1)(T-t)} g_t \xrightarrow{C^\infty} \text{constant curvature } 1.$$

Now there is an observation by Berger that if $\underline{\kappa} \leq \kappa(P) \leq \overline{\kappa}$ for all 2-dimensional $P \subseteq T_pM$, then

$$R(e_1, e_2, e_3, e_4) \leq \frac{2}{3}(\overline{\kappa} - \underline{\kappa})$$

for all orthonormal e_1, e_2, e_3, e_4. This can be done by polarization, which expresses Rm in terms of the sectional curvatures.

Corollary 9.2. *If for all $p \in M$, $0 < \frac{1}{4}\kappa(P_1) < \kappa(P_2)$ for all 2-planes $P_1, P_2 \subseteq T_pM$, then $Rm_p \times \mathbb{R}^2$ has positive isotropic curvature for all p and so the theorem applies. That is, if (M, g) is "$\frac{1}{4}$-pinched" then M is diffeomorphic to a space form.*

The constant $\frac{1}{4}$ is optimal because $\mathbb{C}P^n$ with the Fubini–Study metric has $0 < \frac{1}{4}\kappa(P_1) \leq \kappa(P_2)$. Also, $\frac{1}{4}$-pinching is not preserved by the Ricci flow.

9.1 Brendle's theorem

Brendle proved a strengthening of this theorem, one year before.

Theorem 9.3 (Brendle, DMJ 2008)**.** *Everything is the same, except that $\dim M \geq 3$, and "$M \times \mathbb{R}$ has positive isotropic curvature" instead of "$M \times \mathbb{R}^2$ has positive isotropic curvature".*

In the 3-dimensional case, this reduces exactly to Hamilton's 1982 theorem. When proving this, we cannot directly apply $\ell_{a,b}$ Bohn–Walking, since $\{Rm : Rm \times \mathbb{R} \text{ NIC}\}$ is not contained in nonnegative sectional curvature. The way to get around this is a new condition.

Definition 9.4. For $Rm \in C_B(\mathbb{R}^n)$, define $Rm \times S^2 \in C_B(\mathbb{R}^n \times \mathbb{R}^2)$ by

$$Rm \times S^2((v_1, y_1), \ldots, (v_4, y_4)) = Rm(v_1, \ldots, v_4) + \langle y_1, y_4\rangle\langle y_2, y_3\rangle - \langle y_1, y_3\rangle\langle y_2, y_4\rangle.$$

Proposition 9.5. *$Rm \times S^2$ has nonnegative isotropic curvature if and only if*

$$R_{1331} + \lambda^2 R_{1441} + \mu^2 R_{2332} + \lambda^2\mu^2 R_{2442} - 2\lambda\mu + R_{1234} + (1-\lambda^2)(1-\mu^2) \geq 0.$$

for all orthonormal $e_1, \ldots, e_4$ and $\lambda, \mu \in [0, 1]$.

Corollary 9.6. *$Rm \times \mathbb{R}^2$ has nonnegative isotropic curvature implies $Rm \times S^2$ has nonnegative isotropic curvature implies $Rm \times \mathbb{R}$ has nonnegative isotropic curvature.*

Proposition 9.7. *$\{Rm : Rm \times S^2$ NIC$\}$ is preserved under $\frac{d}{dt}Rm = Q(Rm)$.*

Proposition 9.8. *If $0 < b \le \frac{\sqrt{4+2n(n-20}-2}{n(n-2)}$ and $a = b + \frac{n-2}{2}b^2$ then*

$$\{\ell_{a,b}(Rm) : Rm \times S^2 \text{ NIC}\}$$

is strictly preserved by $\frac{d}{dt}Rm = Q(Rm)$.

Proof. a and b defined ensure that the coefficients in $D_{a,b}(Rm)$ are either 0 or ≥ 0. We have that $Rm \times S^2$ nonnegative isotropic curvature implies $Rm \times \mathbb{R}$ nonnegative isotropic curvature implies $\mathrm{Ric} \ge 0$. So if we look at the eigenvectors $e_1, \dots, e_n$ of Ric, then $e_i \wedge e_j$ are eigenvectors of $D_{a,b}(Rm)$. It is then easy to check that eigenvalues are positive. Now use that $\{Rm \ge 0\} \subseteq \{Rm \times S^2 \text{ NIC}\}$.

We are trying to show that $\frac{d}{dt}Rm = Q(Rm) + D_{a,b}(Rm)$ preserves $\{Rm : Rm \times S^2 \text{ NIC}\}$. We know that Q is inward-pointing, and we just showed that $D_{a,b}$ strictly pointed inwards. □

Define

$$A(s) = \begin{cases} \frac{1-s}{s}\ell_{a(s),b(s)}\{Rm : Rm \times S^2 \text{ NIC}\} & 0 < s < 1 \\ \ell_{a(1),b(1)}\{Rm : Rm \times \mathbb{R}^2 \text{ NIC}\} & s = 1 \\ C(s-1) \cap \ell_{a(1),b(1)}\{Rm : Rm \times \mathbb{R}^2 \text{ NIC}\} & s > 1, \end{cases}$$

with $C(s)$ from last time. Then you can show that $A(s)$ is continuous in s,

$$A(s) \to \{Rm : Rm \times \mathbb{R}^2 \text{ NIC}\}$$

as $s \to 0$, and

$$A(s) \to \{\kappa(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) : \kappa > 0\}$$

as $s \to \infty$. The claim is now that $A(s)$ is strictly preserved by $\frac{d}{dt}Rm = Q(Rm)$. Then there exists a pinching set containing any choice of compact $K \subseteq \mathrm{int}(A(0))$. If $A(s)$ are not cones, we can just use $\bigcap_{\lambda > 0} \lambda A(s)$ as a strictly preserved cone.

Two references for what we have been doing are

- Brendle, chapters 1–3, 5–8,
- Andrews–Hopper, chapters 1–2, 4–8, 12–15.

9.2 Ricci flow without curvature restriction

What can expect if we have negativity of curvature?

Theorem 9.9 (Gromov–Thurston, 1987)**.** *For any $\delta > 0$ and any $\dim \ge 4$, there exist compact (M, g) with $-1 \le \sec \le -1 + \delta$ but M does not admit a metric of constant curvature.*

So you can't possibly hope for these sphere theorems in the negative curvature case. Even to this day, the Ricci flow is not well understood in the negative curvature case. There are two exceptions, in dim = 2 which we will talk about later, and in dim = 3 when the compact manifold of sec < 0.

Another related question is about pinching around zero curvature.

Theorem 9.10 (Gromov 1978). *Let (M, g) be a compact manifold with $-\kappa \leq \sec \leq \kappa$ and diameter D. If*

$$\kappa D^2 \leq \exp(-\exp(\exp(\exp(\cdots(\exp(n))))))$$

where there are 200 exponentials, then M is finitely covered by a nilmanifold.

Definition 9.11. A manifold is a **nilmanifold** if there exists a transitive action by a nilpotent Lie group. A nilpotent Lie group is a Lie group with a nilpotent Lie algebra.

Our new goal is to understand Ricci flow without curvature restrictions.

Theorem 9.12 (Hamilton–Ivey). *Let M be a compact manifold of $\dim = 3$. Let $\lambda \geq \mu \leq \nu$ be the eigenvalues of $Rm : \bigwedge^2 \to \bigwedge^2$. Let g_t be a Ricci flow, and suppose that $R|_{t=0} \geq -1$. Then*

$$\nu + f^{-1}(\lambda + \mu + \nu) \geq 0$$

for $f(x) = x \log x - x$.

Note that $R|_{t=0} \geq -1$ can be arranged by constant scaling, and also $R \geq -1$ is preserved by the Ricci flow because

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2.$$

Also, $R = \lambda + \mu + \nu$.

Theorem 9.13 (Hamilton compactness, 1995). *Let $p_k \in M_k$ be a sequence of manifolds (with same dimension) and a sequence of points. Let $g_k(t)$ be complete Ricci flows on M_k, with $t \in (T_1, T_2] \ni 0$. Assume that*

(1) for any radius $r > 0$, there exist $c(r), k(r) > 0$ such that

$$|Rm(g_k(t))|_{g_k(t)} \leq c(r)$$

for all $k \geq k(r)$ on $B_{p_k}^{g_k(0)}(r)$ and the entire time interval,

(2) the injectivity radii $\mathrm{inj}(M_k, p_k, g_k(0)) \gg 0$ are uniformly bounded below.

Then there exists a sequence $r_k \to \infty$ and a subsequence of $(B_{p_k}^{g_k(0)}(r_k), g_k(t), p_k)$ converging in C^∞_{loc} to $(M^\infty, g_\infty(t), p_\infty)$ for some $M^\infty \ni p_\infty$ some smooth manifold with $g_\infty(t)$ a complete Ricci flow on the entire time interval.

Here, locally C^∞-convergence means that there is a sequence $p_\infty \in U_1 \subseteq U_2 \subseteq \cdots$ with $\bigcup_{i=1}^\infty U_i = M_\infty$ and the diffeomorphisms are uniformly on compact subsets of $M_\infty \times (T_1, T_2]$. For (1), we often just use $|Rm|_k \le C$ everywhere.

So here is the basic picture. Let M^3 be compact, and let g_t be a Ricci flow on $[0, T)$. Suppose that T is maximal and assume $T < \infty$. We then know that $\limsup_{t\to T} \sup_M |Rm| = \infty$. Choose $x_k \in M$ and $t_k \nearrow T$ such that $|Rm(x_k, t_k)| \to \infty$. Then define

$$\tilde{g}^k(\tilde{t}) = \frac{g(t_k + \epsilon_k^2 \tilde{t})}{\epsilon_k^2}$$

for $\tilde{t} \in [-\frac{t_k}{\epsilon_k^2}, \frac{T - t_k}{\epsilon_k^2})$ for $\epsilon_k = \frac{1}{\sqrt{|Rm(x_k, t_k)|}}$. You can check that $\tilde{g}^k$ are Ricci flows. If we apply the Compactness theorem (we will be more careful later), we get a limit $(M_\infty, g_\infty(t), p_\infty)$. Here, changing $g \to \tilde{g}^k$ changes

$$(Rm : \textstyle\bigwedge^2 \to \bigwedge^2) \quad \longrightarrow \quad \frac{Rm}{|Rm(x_k, t_k)|}.$$

Because $|Rm| \to \infty$, either $\lambda \to \infty$ or $\nu \to -\infty$. But Hamilton–Ivey tells us that if $\nu \to -\infty$ then $\lambda \to \infty$. We can then rearrange the inequality to

$$-\nu \le \frac{\lambda + \mu}{\log(-\nu)}$$

and then

$$-\frac{\nu}{\sqrt{\lambda^2 + \mu^2 + \nu^2}} \le \frac{\lambda + \mu}{\sqrt{\lambda^2 + \mu^2 + \nu^2}} \frac{1}{\log(-\nu)}.$$

In both case $\nu \to -\infty$ and $\nu \not\to -\infty$, we get the left hand side going to 0. Either way, we see that g_∞ has nonnegative $Rm : \bigwedge^2 \to \bigwedge^2$.

10 February 22, 2018

Today we are going to make a short digression and talk about Yau's estimates.

Theorem 10.1 (Laplacian comparison theorem)**.** *(M^n, g) be a complete,* Ric $\geq$ $-(n-1)kg$ *and let $p \in M$ be $\rho(q) = d_g(p,q)$. If ρ is smooth at q, then*

$$\delta^g \rho \leq \frac{n-1}{\rho}(1 + \sqrt{k}\rho).$$

Let us take this for granted.

10.1 Yau's estimate

Theorem 10.2 (Yau, CPAM, 1975)**.** *Let (M^n, g) be a complete Riemannian metric with* Ric $\geq -(n-1)kg$, *and let $u : B_a(p) \to (0, \infty)$ such that $\Delta u = 0$. Then*

$$\frac{|\nabla u|}{u} \leq C_n \Big(\frac{1 + a\sqrt{k}}{a} \Big)$$

on $B_{n/2}(p)$.

Proof. We can compute

$$\begin{aligned} \Delta |\nabla u|^2 &= 2|\nabla \nabla u|^2 + 2\langle \nabla u, \Delta \nabla u \rangle \\ &= 2|\nabla\nabla u|^2 + 2\langle \nabla u, \nabla \Delta u \rangle + 2\operatorname{Ric}(\nabla u, \nabla u). \end{aligned}$$

Because

$$\Delta |\nabla u|^2 = 2|\nabla u| \delta |\nabla u| + 2|\nabla |\nabla u||^2,$$

we can also rewrite this in terms of $|\nabla u|$ as

$$|\nabla u| \Delta |\nabla u| = |\nabla \nabla u|^2 - |\nabla \nabla u|^2 + \operatorname{Ric}(\nabla u, \nabla u).$$

Then we can write this in terms of $\frac{|\nabla u|}{u}$ as

$$\Delta \frac{|\nabla u|}{u} = \frac{|\nabla\nabla u|^2 - |\nabla|\nabla u||^2}{u|\nabla u|} + \frac{\operatorname{Ric}(\nabla u, \nabla u)}{u|\nabla u|} - 2\frac{\nabla u}{u} \cdot \nabla \frac{|\nabla u|}{u}.$$

The first numerator is nonnegative by the Schwartz inequality (this is also called the "Kato inequality"). But the key observation is that

$$|\nabla\nabla u|^2 - |\nabla|\nabla u||^2 \geq \frac{1}{n-1} |\nabla |\nabla u||^2$$

for a harmonic function u. If we use this, and the inequality

$$\frac{\nabla|\nabla u| \cdot \nabla u}{u^2} \leq \frac{|\nabla|\nabla u|||\nabla u|}{u^2} \leq \frac{1}{2}\Big(\frac{|\nabla|\nabla u||^2}{u|\nabla u|} + \frac{|\nabla u|^3}{u^3} \Big),$$

then we get

$$\Delta\varphi \geq -(n-1)k\varphi - \Big(2 - \frac{2}{n-1}\Big)\frac{\nabla u}{u} \cdot \nabla\varphi + \frac{\varphi^3}{n-1}$$

for $\varphi = \frac{|\nabla u|}{u}$.

Now define on $B_a(p)$,

$$F(x) = (a^2 - \varphi(x)^2)\varphi(x)$$

so that F has interior maximum point on x_0, so that $\nabla F(x_0) = 0$ and $\Delta F(x_0) \leq 0$. Then the conditions can be written as

$$(a^2 - \rho^2)\nabla\varphi = \varphi\nabla\rho^2, \quad (a^2 - \rho^2)\Delta\varphi - \frac{2|\nabla\rho^2|\varphi}{a - \rho^2} - \varphi\Delta\rho^2 \leq 0.$$

If we use $|\nabla\rho| = 1$ and

$$\Delta\rho^2 = 2\rho\Delta\rho + 2|\nabla\rho|^2 \leq 2(n-1)(1 + \sqrt{k}\rho) + 2,$$

then we get

$$0 \geq -(n-1)k - \Big(2 - \frac{2}{n-1}\Big)\frac{\nabla u}{u} \cdot \frac{\nabla\varphi}{\varphi} + \frac{\varphi^2}{n-1} - \frac{C_n(1 + \sqrt{k}\rho)}{a^2 - \rho^2} - \frac{8\rho^2}{(a^2 - \rho^2)^2}.$$

Also,

$$\Big(2 - \frac{2}{n-1}\Big)\frac{\nabla u}{u} \cdot \frac{|\nabla\varphi|}{\varphi} = \Big(2 - \frac{2}{n-1}\Big)\frac{\nabla u}{u} \cdot \frac{\nabla\rho^2}{a^{-}\rho^2} \leq 2\varphi\frac{\rho}{a^2 - \rho^2}.$$

Replacing φ by $\frac{F}{a^2 - \rho^2}$, and using $\rho < \alpha$, we get

$$0 \geq \frac{F^2}{n-1} - 2\Big(2 - \frac{2}{n-1}Fa - C_n' a^2(1 + \sqrt{k}a\Big)^2.$$

The quadratic formula then shows that $F \leq C_n'' a(1 + \sqrt{k}a)$. Then use $\sup_{B_{a/2}} \leq F(x_0)$ to get the theorem. ☐

Proposition 10.3. *For $\Delta u = 0$, we have*

$$|\nabla\nabla u|^2 - |\nabla|\nabla u||^2 \geq \frac{1}{n-1}|\nabla|\nabla u||^2.$$

Proof. Take $p \operatorname{im} M$ and normal coordinates at p, so that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$, and also assume $\nabla_1 u(p) = |\nabla u|(p)$ and $\nabla_2 u(p) = \cdots = \nabla_n u(p) = 0$. We also locally have

$$|\nabla u| = \sqrt{g^{ij}\nabla_i u \nabla_j u}.$$

So we can take the derivative

$$\nabla_k|\nabla u| = \frac{\partial_k g^{ij}\nabla_i u \nabla_j u + 2g^{ij}\partial_k(\nabla_i u)\nabla_j u}{2|\nabla u|} = \nabla_k\nabla_1 u$$

by our choice of coordinates.

Now the left hand side is

$$\begin{aligned}\sum_{i,j}(\nabla_i\nabla_j u(p))^2 - \sum_k(\nabla_k\nabla_i u(p))^2 &\geq \sum_{i\neq 1}(\nabla_1\nabla_i u(p))^2 + \sum_{i\neq 1}(\nabla_i\nabla_i u(p))^2\\ &\geq \frac{1}{n-1}\sum_{i\neq 1}(\nabla_1\nabla_i u(p))^2 + \frac{1}{n-1}\sum_{i\neq 1}(\nabla_i\nabla_i u(p))^2\\ &= \frac{1}{n-1}\sum_{i=1}^n(\nabla_1\nabla_i u(p))^2.\end{aligned}$$

So we get the desired inequality. □

Yau proves this theorem using an "approximate maximum principle".

Corollary 10.4. *If (M, g) is complete with* Ric ≥ 0*, then every positive harmonic function is constant.*

Corollary 10.5 (Harnack inequality)**.** *If (M, g) is complete and* Ric $\geq -(n-1)kg$*, then $u : B_a(p) \to (0, \infty)$ is harmonic, then*

$$\sup_{B_{a/2}} u \leq C_{n,a,k} \inf_{B_{a/2}} u.$$

Proof. The theorem gives $\frac{|\nabla u|}{u} \leq C_{n,a,k}$ in B_a. Now take $p, q \in B_{a/2}$ such that $\inf_{B_{a/2}} u$ and $u(q) = \sup_{B_{a/2}}$. Take the minimal geodesic from p to q, and we get

$$\log\frac{u(q)}{u(p)} \leq \int_\gamma \frac{|\nabla u|}{u} \leq Ca.$$

□

This gives a new proof of the Harnack inequalities by a gradient inequality proved by the maximum principle. This is somewhat weaker version of the De Giorgi–Nash–Moser Harnack inequality for quasilinear scalar elliptic equations.

10.2 Minimal surface in $\mathbb{R}^{n,1}$

Theorem 10.6 (Cheng–Yau, Annals, 1976)**.** *Any complete closed hypersurface in $\mathbb{R}^{n,1}$ with mean curvature 0 (assuming the induced metric is Riemannian) is a linear plane.*

Actually, completeness is redundant. Also, this is not true with $\mathbb{R}^{n,1}$ replaced by $\mathbb{R}^n$.

Proof. The Gauss equation gives

$$R_{ijkl} = -h_{il}h_{jk} + h_{ik}h_{jl},$$

and then $R_{jk} = h^i{}_k h_{ij}$ is nonnegative. Also, we have

$$\frac{1}{2}\Delta|h|^2 = |\nabla h|^2 + \langle h, \nabla\nabla H\rangle + |h|^4 - Hh_i{}^j h_j{}^k h_k{}^i = |\nabla h|^2 + |h|^4.$$

If we define

$$g(x) = \frac{1}{u(x)}(a^2 - \rho(x)^2)^{-\alpha}$$

on $B_a(p)$, g has some interior minimum x_0. Exactly as before, we have $\nabla g = 0$ and $\Delta g \geq 0$. Then

$$-\frac{\nabla u}{u} + \frac{2\rho\nabla\rho}{a^2-\rho^2} = 0, \quad -\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} + \frac{2\alpha(1+\rho\Delta\rho)}{a^2-\rho^2} + \frac{4\alpha\rho^2}{(a^2-\rho^2)^2} \geq 0.$$

So

$$\frac{\Delta u}{u} \leq \frac{4\alpha(\alpha+1)}{(a^2-\rho^2)^2} + \frac{2\alpha(1+\rho\Delta r)}{a^2-\rho^2}$$

and the Laplacian comparison equation gives $\rho\Delta\rho \leq n-1$. Also, the Simons equation gives $\frac{1}{2}\Delta|h|^2 \geq |h|^4$, and so

$$(a^2-\rho^2)^2\frac{u}{2} \leq (4\alpha^2+4\alpha)\rho^2 = 2\alpha(a^2-\rho^2)n.$$

Take $\alpha = 2$ and we get

$$\max_{B_a}(a^2-\rho^2)^2\frac{u}{2} \leq 24\rho^2 + 4a^2n^2.$$

As $a \to \infty$ we get $u \equiv 0$. The second fundamental form vanishes, so it should be a plane. $\square$

11 February 27, 2018

Today we are going to look at parabolic estimates. Recall

Theorem 11.1 (Cheng–Yau, 1975)**.** *If (M^n, g) is complete with* $\mathrm{Ric} \geq -(n-1)kg$*, and if $u : B_a(p) \to (0, \infty)$ is harmonic, then*

$$\frac{|\nabla u|}{u} \leq c_n \Big(\frac{1 + a\sqrt{k}}{a} \Big)$$

on $B_{a/2}(p)$.

11.1 Parabolic estimate

Theorem 11.2 (Li–Yau, Acta 1986)**.** *If (M^n, g) is complete and* $\mathrm{Ric} \geq -kg$*, and if $u > 0$ has $\frac{\partial u}{\partial t} = \Delta u$ on some $B_{2R}(p)$, then*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{c_n \alpha^2}{R^2} \Big(\frac{\alpha^2}{\alpha^2 - 1} + R\sqrt{k} \Big) + \frac{n\alpha^2 k}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}$$

on $B_R(p)$, for all $\alpha > 1$.

The proof is similar to Cheng–Yau we saw last time.

Proof. First write the PDE for $F = t(|\nabla f|^2 - \alpha f_t)$ for $f = \log u$. Then we have

$$\Big(\frac{\partial}{\partial t} - \Delta \Big) F \leq 2\langle \nabla f, \nabla F \rangle + \frac{F}{t} + 2kt|\nabla f|^2 - \frac{2t}{n}(|\nabla f|^2 - f_t)^2.$$

In the simple case $\alpha = 1$, we will have something like

$$\Big(\frac{\partial}{\partial t} - \Delta \Big) F = \frac{F}{t} - 2t|\nabla\nabla f|^2 - 2t\langle \nabla f, \Delta \nabla f \rangle.$$

The trace inequality gives $-2t|\nabla\nabla f| \leq \frac{2t}{n}(\Delta f)^2$. Then we have

$$-2t(\nabla f, \Delta \nabla f) = -2t(\nabla f, \nabla \Delta f) - 2t \,\mathrm{Ric}(\nabla f, \nabla f)$$

and then we can use the estimate on Ricci.

Now we take $\psi \in C_c^\infty(\mathbb{R})$ a cutoff function so that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $[0, 1]$ and $\psi \equiv 0$ on $[2, \infty)$, $|\psi'| \leq 0$, $\psi'' \geq -c_1$, $\frac{|\psi'|^2}{\psi} \leq c_2$. Take $\varphi = \psi \circ \frac{\rho}{R}$ for ρ a distance function to p.

Then φF has a maximum on $B_{2R}(p) \times [0, T]$. If this is nonpositive, we have a 0 on the right hand side of the theorem, and we are done. If it is positive, then at an interior point we have $0 = \nabla(\varphi F)$ and $\Delta(\varphi F) \leq 0$ and $\frac{\partial}{\partial t}(\varphi F) \geq 0$. The last two implies that $(\frac{\partial}{\partial t} - \Delta)(\varphi F) \geq 0$. Use $\nabla(\varphi F) = 0$ to replace all ∇F terms by $F, \varphi, \nabla\varphi$. Put the PDE into $\frac{\partial}{\partial t}(\varphi F) \geq 0$ and use the Laplacian comparison theorem for the $\Delta\varphi$ term. As for Cheng–Yau, we want extra terms as quadratics for F. If we look at the PDE, we have

$$2\langle \nabla f, \nabla F \rangle + 2kt|\nabla f|^2 - \frac{2t}{n}(|\nabla f|^2 - f_t)^2$$

as quadratic in $|\nabla f|^2 - \alpha f_t$. Again use $0 = \nabla(\varphi F)$ to replace ∇F and $\frac{|\nabla\varphi|}{\sqrt{\varphi}} \le \frac{\sqrt{c_2}}{R}$ to get an estimate. $\square$

Here is the simpler more advertisable theorem.

Theorem 11.3. *Let M be a compact (or complete) manifold with $\operatorname{Ric} \ge 0$ and $u > 0$ and $\frac{\partial u}{\partial t} = \Delta u$. Then*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le \frac{n}{2t}.$$

Proof. Take $R \to \infty$ and $\alpha \to 1$. Or we can apply the maximal principle to $0 \le \frac{F}{t} - \frac{2t}{n}\frac{F^2}{t^2} = \frac{2F}{nt}(\frac{n}{2} - F)$. $\square$

Corollary 11.4 (Harnack)**.** *Suppose (M, g) has $\operatorname{Ric} \ge 0$ and $u > 0$ solve $\frac{\partial u}{\partial t} = \Delta u$, then*

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \ge \Big(\frac{t_2}{t_1}\Big)^{-\frac{n}{2}} \exp\Big(\frac{-d(x_1, x_2)^2}{4(t_2 - t_1)}\Big).$$

Proof. Choose a geodesic γ from x_1 to x_2, and then we write

$$\begin{aligned}
\log \frac{u(x_2, t_2)}{u(x_1, t_1)} &= \int_{t_1}^{t_2} \frac{d}{dt} \log u(\gamma(t), t) dt = \int_{t_1}^{t_2} \frac{\partial}{\partial t} \log u + \nabla \log u \frac{d\gamma}{dt} \\
&\ge \int_{t_1}^{t_2} |\nabla \log u|^2 - \frac{n}{2t} + \nabla \log u \frac{d\gamma}{dt} \ge \int_{t_1}^{t_2} -\frac{n}{2t} - \frac{1}{4}\Big|\frac{d\gamma}{dt}\Big|^2 \\
&= -\frac{n}{2} \log \frac{t_2}{t_1} - \frac{1}{4}\frac{d(x_1, x_2)^2}{t_2 - t_1}.
\end{aligned}$$

$\square$

Of course, there is a more complicated version for $\operatorname{Ric} \ge -kg$. Also, note that the basic solution of $\frac{\partial u}{\partial t} = \Delta u$ on $\mathbb{R}^n$ is $(4\pi t)^{-n/2} e^{-|x|^2/4t}$.

Proposition 11.5. *Take a compact 2-dimensional Ricci flow with positive curvature. (Here $R_{ij} = \frac{1}{2} R g_{ij}$ and $\frac{\partial R}{\partial t} = \Delta R + R^2$.) Define*

$$Q = \frac{\partial}{\partial t} \log R - |\nabla \log R|^2 + \frac{1}{t}.$$

Then $Q = \Delta \log R + R + \frac{1}{t}$ and

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) Q \ge 2\langle \nabla \log R, \nabla \log Q\rangle + Q\Big(Q - \frac{2}{t}\Big).$$

Then the maximal principle gives $Q \ge 0$, and integrating gives

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \ge \frac{t_1}{t_2} \exp\bigg(-\frac{1}{4}\int_\gamma \Big|\frac{d\gamma}{dt}\Big|^2_{g_t}\bigg) dt.$$

Proof. Exercise. Hamilton did this in two hours. $\square$

We need $R > 0$, and this implies that distances are decreasing. So in the exponential term, we can just take the distance with $d_{g(t_1)}$.

Theorem 11.6 (Hamilton, JDG 1993)**.** *Let M^n be compact and g_t be a Ricci flow with $Rm : \bigwedge^2 \to \bigwedge^2$ nonnegative. Define*

$$P_{kij} = \nabla_k R_{ij} - \nabla_i R_{jk}, \quad M_{ij} = \Delta R_{ij} - \frac{1}{2}\nabla_i \nabla_j R + 2R_{pijq}R^{pq} - R_i{}^p R_{pj} + \frac{R_{ij}}{2t}.$$

For any choice of a 2-form U and a 1-form W, define

$$Z(U, W) = M_{ij}W^i W^j + 2P_{kij}U^{ki}W^j + R_{pqij}U^{pq}U^{ji}.$$

Then $Z(U, W) \geq 0$ for all U, W.

Note that Z can be considered as a quadratic form on $\bigwedge^1 \oplus \bigwedge^2$. This holds even if M is noncompact but g_t has bounded nonnegative curavature.

Theorem 11.7 (Brendle, JDG 2009)**.** *This holds if $Rm \times \mathbb{R}^2$ has NIC.*

Corollary 11.8. $\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_v R + 2\operatorname{Ric}(v, v) \geq 0$ *for every vector field v.*

Proof. Take $U_{ij} = \frac{1}{2}(v_i w_j - v_j w_i)$ and trace over w. Then the M term is going to be $\frac{1}{2}\frac{\partial R}{\partial t} + \frac{R}{2t}$, the P term is $\nabla_v R$, and the Rm term is $\operatorname{Ric}(v, v)$. □

We can also integrate this.

Corollary 11.9. *Under the same hypothesis,*

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq \frac{t_2}{t_1}\exp\Big(\frac{-d_{g(t_1)}(x_1, x_2)^2}{2(t_2 - t_2)}\Big)$$

for all x_1, x_2 and $0 < t_1 < t_2$.

Proof. We can use $Rm \geq 0$ implies $\operatorname{Ric} \geq 0$ implies distances nonincreasing. Also, $\operatorname{Ric} \geq 0$ implies $\operatorname{Ric} \leq Rg$. Then

$$\begin{aligned} 0 &\leq \frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_v R + 2R|v|^2 \\ &= R\Big(\frac{\partial}{\partial t}\log(tR) - \frac{1}{2}|\nabla \log(tR)|^2 + 2\Big|v + \frac{1}{2}\nabla \log R\Big|^2\Big). \end{aligned}$$

So we can choose $v = -\frac{1}{2}\nabla \log R$ and then integrate as before. □

Corollary 11.10. *Under the same hypotheses, tR is nondecreasing at every point.*

Proof. Take $v = 0$, and then $0 \leq \frac{\partial R}{\partial t} + \frac{R}{t} = \frac{1}{t}\frac{\partial}{\partial t}(tR)$. □

12 March 1, 2018

Recall that

$$\begin{aligned} P_{kij} &= \nabla_k R_{ij} - \nabla_i R_{jk}, \\ M_{ij} &= \Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2R_{pijq}R^{pq} - R_i{}^q R_{pj} + \frac{R_{ij}}{2t}, \\ Z(U,W) &= M_{ij}W^iW^j + 2P_{kij}U^{ki}W^j + R_{pqij}U^{pq}U^{ji} \end{aligned}$$

for $U \in \bigwedge^2$ and $W \in \bigwedge$.

Theorem 12.1. *Let (M, g_t) be a compact or complete with bounded curvature.*

(1) (Hamilton) If $Rm : \bigwedge^2 \to \bigwedge^2$ is positive, then $Z(U,W) \geq 0$ for all U, W.

(2) If $Rm \times \mathbb{R}^2$ is NIC, then

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_v R + 2\operatorname{Ric}(v,v) \geq 0$$

for all vector field v.

Corollary 12.2. *tR is nondecreasing everywhere.*

Corollary 12.3. *If the Ricci flow is on $t \in (-\infty, 0)$ then R is nondecreasing everywhere.*

Proof. If we take $v = 0$, then tR is nondecreasing. Also, if we take $v = 0$ and $t = 0$ arbitrarily large (after translating time) then we get R nondecreasing. □

Definition 12.4. A Ricci flow on $t \in (-\infty, 0)$ is called **ancient**.

Let (M^3, g_t) be a compact Ricci flow on $t \in [0, T)$. Define

$$\tilde{g}^{(k)}(t) = |Rm(x_k, t_k)| g\Big(t_k + \frac{\tilde{t}}{|Rm(x_k, t_k)|}\Big)$$

for $\tilde{t} \in [-|Rm(x_k, t_k)|t_k, |Rm(x_k, t_k)|(T - t_k))$. Then for some $t_k \nearrow T$ and x_k such that $|Rm(x_k, t_k)| \to \infty$, we can see that the $\tilde{g}^{(k)}$ is a Ricci flow, with left endpoint of $\tilde{t}$-interval going to $-\infty$.

But what happens to the right endpoint? Also, how can we apply compactness? Can we get bounded curvature of the limit? If all is good, then there exists a limit with $\sec \geq 0$ by Hamilton–Iveys and $\frac{\partial R}{\partial t} \geq 0$.

12.1 Chow–Chu construction

Here is Hamilton's crucial observation: $\bigwedge^2 T_pM$ has a natural Lie algebra structure. There is a map

$$\bigwedge^2 T_pM \to \mathfrak{so}(T_pM)$$

sending $e_i \wedge e_j$ to the linear map that projects to $e_i \wedge e_j$ and then rotates $90°$ in the 2-plane. This is an isomorphism. Then $\bigwedge^2 T_pM \oplus T_pM$ has a Lie algebra structure given by

$$[(u_1, v_1), (u_2, v_2)] = ([u_1, u_2], u_1(v_2, -) - u_2(v_1, -)).$$

Definition 12.5. If V is a Lie algebra which has inner product $\langle -,-\rangle$, define for $T : V \to V$ the **Lie algebra square** $T^\# : V \to V$ by

$$\langle T^\#(v), v\rangle = \frac{1}{2}\sum_{\alpha,\beta}\langle [T(e_\alpha), T(e_\beta)], v\rangle\langle [e_\alpha, e_\beta], v\rangle$$

for e_α orthonormal.

Lemma 12.6. $\Big(\dfrac{\partial}{\partial t} - \Delta\Big) Rm = Rm^2 + Rm^\#$ *after the Uhlenbeck trick.*

Lemma 12.7. $\Big(\dfrac{\partial}{\partial t} - \Delta\Big) Z = Z^2 + Z^\#$ *viewing* $Z : \wedge^2 \oplus \wedge^1 \to \wedge^2 \oplus \wedge^1$.

Here, the inner product on $\wedge^2 \oplus \wedge^1$ is

$$\langle (u_1, v_1), (u_2, v_2)\rangle = \langle u_1, u_2\rangle.$$

Definition 12.8 (Chow–Chu, MRL 1995)**.** Let (M^n, g_t) be a Ricci flow on $t \in I$, and take $\tilde{M} = I \times M$. Define $\tilde{g}^{ab}$ on $\tilde{M}$ by

$$\tilde{g}^{ij}_{(t,p)} = g^{ij}_p(t), \quad \tilde{g}^{0j} = \tilde{g}^{i0} = \tilde{g}^{00} = 0.$$

(We are going to denote the t component by 0, and $1 \le i, j, \dots \le n$ and $0 \le a, b, \dots \le n$.) We can define a connection on $T\tilde{M}$ by

$$A^k_{ij} = \Gamma^k_{ij}, \quad A^k_{i0} = A^k_{0i} = -R_i{}^k, \quad A^k_{00} = -\frac{1}{2}\nabla^k R, \quad A^0_{00} = -\frac{1}{2t},$$

and all other components zero. Let us write $\tilde{\nabla}_{\partial_a}\partial_b = A^c_{ab}\partial_c$.

Lemma 12.9. $\tilde{\nabla}\tilde{g} = 0$.

Proof. You can compute this. You are going to need $A^k_{ij} = \Gamma^k_{ij}$ and $A^k_{i0} = A^k_{0i} = -R_i{}^k$, and vanishing of some components. $\square$

We can formally define $\tilde{Rm}$ as a $(3,1)$-tensor formally, and also define $\tilde{R}_{ij}$ formally.

Proposition 12.10. *We have*

$$\frac{\partial}{\partial t}\tilde{g}^{ab} = 2\tilde{g}^{ac}\tilde{g}^{bd}\tilde{R}_{cd}$$

and

$$\frac{\partial}{\partial t}A^c_{ab} = -\tilde{g}^{cd}(\tilde{\nabla}_a\tilde{R}_{bd} + \tilde{\nabla}_b\tilde{R}_{ad} - \tilde{\nabla}_d\tilde{R}_{ab}.$$

That is, $\tilde{g}, \tilde{\nabla}$ *formally satisfies the Ricci flow.*

Proof. We can just compute

$$\mathrm{Ric} = \begin{pmatrix} \frac{1}{2}(\frac{\partial R}{\partial t} + \frac{R}{t}) & \frac{1}{2}\nabla_k R \\ \frac{1}{2}\nabla_k R & R_{ij} \end{pmatrix}. \qquad \square$$

For $W \in \wedge^1$ and $U \in \wedge^2$ (considered as $U_i{}^j$), consider $T = \left(\begin{smallmatrix} 0 & W \\ -W & U \end{smallmatrix}\right)$. Then

$$\tilde{g}^{ab}\tilde{R}^l_{bcd}T_a{}^cT_l{}^d = Z(U, W),$$

and $\mathrm{Ric}(v + \frac{\partial}{\partial t}, v + \frac{\partial}{\partial t})$ is the trance Harnack.

12.2 Perelman's construction

Definition 12.11 (Perelman, 2002)**.** Let (M^n, g_t) be a Ricci flow with $t \in I \subseteq (-\infty, 0)$. On $M \times I \times S^N$, take the metric

$$g_N = \begin{pmatrix} g & 0 & 0 \\ 0 & R(g) - \frac{N}{2t} & 0 \\ 0 & 0 & tg^{S_N} \end{pmatrix}.$$

Proposition 12.12. *If f is a function constant on each $\{p\} \times \{t\} \times S^N$, then*

$${}^N\Delta f = \Delta f - \Big[1 + \frac{\frac{N}{t^2} + 2\frac{\partial R}{\partial t}}{(R - \frac{N}{2t})^2}\Big]\frac{\partial f}{\partial t} + \frac{1}{R - \frac{N}{2t}}\frac{\partial^2 f}{\partial t^2} + \frac{\frac{1}{2}\langle \nabla R, \nabla f\rangle}{R - \frac{N}{2t}}.$$

Note that $\lim_{N\to\infty} {}^N\Delta f = \Delta f - \frac{\partial f}{\partial t}$ and this is a meaningful limit since f is constant on S^n fibers. So $\frac{\partial f}{\partial t} = \Delta f$ on M is saying something like ${}^N\Delta f = 0$ modulo $\frac{1}{N}$. The first equation is parabolic while the second equation is elliptic.

Proposition 12.13. $\lim\limits_{n\to\infty} (-t)^{\frac{N-1}{2}}\, {}^N\Delta((-t)^{-\frac{N-1}{2}} f) = \Delta f + \dfrac{\partial f}{\partial t} - R$. *Here, this $\Delta + \frac{\partial}{\partial t} - R$ is the conjugate heat operator:*

$$\iint \Big(\frac{\partial f}{\partial t} - \Delta f\Big) g d\mu_t dt = \iint f\Big(-\frac{\partial g}{\partial t} - \Delta g + Rg\Big) d\mu_t dt.$$

Let us write $i, j, \ldots$ for the M coordinates, and $\alpha, \beta, \ldots$ for the S^N coordinates.

Lemma 12.14. $\lim\limits_{N\to\infty} \begin{cases} {}^N R_{ijkl} = R_{ijkl} \\ {}^N R_{i0kl} = P_{kli} \\ {}^N R_{i00l} = M_{il} = 2R_i{}^p R_{pl} \\ {}^N R_{\alpha\beta\gamma\delta} = R^{(S^N)}_{\alpha\beta\gamma\delta} \end{cases}$ *and others goes to* 0.

Lemma 12.15. ${}^N\operatorname{Ric} = \dfrac{1}{R - \frac{N}{2t}}(-) + \dfrac{1}{2(R - \frac{N}{2t})^2}(-)$.

Corollary 12.16. $\lim_{N\to\infty} {}^N\operatorname{Ric} = 0$.

As $N \to \infty$, the heat equation becomes something like the Laplace equation, and the Ricci flow becomes $\operatorname{Ric} = 0$. The moral is that a parabolic equation can be thought of as an infinite-dimensional elliptic problem.

Theorem 12.17 (Bishop–Gromov volume comparison)**.** *Let (M^n, g) be complete, and $\operatorname{Ric} \geq (n-1)kg$, and let $v_k(R)$ be the volume of $B_R(x)$ in the simply connected manifold of constant curvature k. Then for all $p \in M$,*

$$\frac{\operatorname{vol} B_R(p)}{v_k(R)}$$

decreases as R increases.

Perelman's idea is to use this.

13 March 6, 2018

Recall that we had (M, g_t) a Ricci flow with $t \in I \subset (-\infty, 0)$ and we defined

$$^N\tilde{g}_{(p,q,t)} = \begin{pmatrix} g_p & & \\ & tg_q^{S_N} & \\ & & R_p(g(t)) - \frac{N}{2t} \end{pmatrix}.$$

Then on $M \times S^N \times I$, we had $\lim_{N\to\infty} \mathrm{Ric}(^N\tilde{g}) = 0$.

13.1 Volumes in $^N\tilde{g}$

Given a geodesic ball around $(p, s, 0)$, the length of $(\gamma(t), \sigma(t), t)$ is

$$\begin{aligned} \text{length} &= \int_{-T}^{0} \sqrt{|\dot\gamma(t)|^2_{g(t)} + t^2|\dot\sigma(t)|^2_{g^{S_N}} + R - \tfrac{N}{2t}}dt \\ &= \int_{-T}^{0} \sqrt{\frac{N}{-2t}}\sqrt{1 - \frac{2t}{N}(R + |\dot\gamma(t)|^2 + t|\dot\sigma(t)|^2)}dt \\ &\approx \int_{-T}^{0} \sqrt{\frac{N}{-2t}}\Big(1 - \frac{t}{N}(R + |\dot\gamma(t)|^2 + t|\dot\sigma(t)|^2) + O(N^{-2})\Big)dt \\ &= \sqrt{2NT} + \frac{1}{\sqrt{2N}}\int_{-T}^{0} \sqrt{-t}(R + |\dot\gamma(t)|^2 + t|\dot\sigma(t)|^2)dt + O(N^{-3/2}). \end{aligned}$$

Now define

$$L(g, s', T) = \inf \int_{-T}^{0} \sqrt{t}(\cdots)dt$$

for all $(\gamma(t), \sigma(t), t)$ from $(p, s, 0)$ to (q, s', T).

If (q, s', T) is on the boundary of radius $\sqrt{2N\bar{T}}$, then we would get

$$\sqrt{\bar{T}} = \sqrt{T} + \frac{1}{2N}L(q, s', T) + O(N^{-2}).$$

So as $N \to \infty$ the boundary almost looks like $\{T = \bar{T}\}$. Then we can heuristically say that

$$\begin{aligned} \text{vol(boundary)} &\approx \int_M \mathrm{vol}(t_R g^{S_N})d\mu_{g(\bar{T})}(p) \\ &= C_N N^{N/2}\int_M t_p^{N/2}d\mu_{g(\bar{T})}(p) \\ &\approx C_N N^{N/2}\int_M (\sqrt{\bar{T}} - \tfrac{1}{2N}L(p, -, t_p) + O(N^{-2}))^{N/2}d\mu_{g(\bar{T})}(p). \end{aligned}$$

for $(p, -, t_p)$ on the boundary. On the other hand, we have

$$\text{vol(boundary of } B_{\sqrt{2N\bar{T}}} \text{ in } \mathbb{R}^{n+N+1}) = (2N\bar{T})^{\frac{n+N}{2}}C_{n+N}.$$

So by the Bishop–Gromov comparison, we will get

$$\frac{\text{vol(boundary)}}{\text{vol(boudary} \subseteq \mathbb{R}^{n+N+1}} \approx C_N \int_M \bar{T}^{-n/2} \exp\Big(\frac{-L(q,-,\bar{T})}{2\sqrt{\bar{T}}}\Big) d\mu_{g(\bar{T})}.$$

So let us actually make the definitions.

Definition 13.1. Let (M, g_τ) be a backwards Ricci flow ($\frac{\partial g_\tau}{\partial \tau} = 2\operatorname{Ric}$) for $\tau \in [0, T)$. For $\gamma : [\tau_1, \tau_2] \to M$, define length

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R_{g(\tau)}(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2_{g(\tau)})d\tau.$$

Definition 13.2. In the same context, fix $p \in M$. Define $L : M \times [0, T) \to \mathbb{R}$ by $L(g, \tau) = \inf \mathcal{L}(\gamma)$ over all $\gamma : [0, \tau] \to M$ from p to q.

Also define

$$\ell(q,\tau) = \frac{L(q,\tau)}{2\sqrt{\tau}}, \quad V(\tau) = \int_M \frac{1}{(4\pi\tau)^{n/2}} e^{-\ell(q,\tau)} d\mu_{g(\tau)}(q).$$

Theorem 13.3 (Perelman, 2002)**.** *If (M, g_τ) is a backwards Ricci flow, with M compact or complete with bounded curvature, then $V(\tau)$ is nonincreasing in τ.*

13.2 Perelman's Li–Yau inequality

Let (M, g_τ) be a backwards Ricci flow. Define

$$E(f) = 2\Delta f - |\nabla f|^2 + R + \frac{f}{\tau} - \frac{n}{\tau}$$

for $f \in C^\infty(M)$, and denote

$$\Phi = \frac{1}{(4\pi\tau)^{n/2}} e^{-f} d\mu_{g(\tau)}.$$

Proposition 13.4 (Li–Yau type)**.** *If $\frac{\partial}{\partial \tau}\Phi = \Delta\Phi$ then*

$$\frac{\partial}{\partial \tau}(\tau E(f)\Phi) = \Delta(\tau E(f)\Phi) - 2\tau\Big|\operatorname{Ric}^{g(\tau)} + \operatorname{Hess}^{g(\tau)} f - \frac{g}{2\tau}\Big|^2\Phi.$$

Proof. Exercise. ☐

Corollary 13.5. *If $\tau_1 \le \tau_2$ and $E(f) \le 0$ at $t \le \tau_1$ and $\frac{\partial \Phi}{\partial \tau} = \Delta\Phi$, then $E(f) \le 0$ at $t = \tau_2$.*

Proof. This is the maximal principle. ☐

If $E(f) \leq 0$, then we can subtract $\frac{1}{2}E(f)$ off of the previous identity. Then we have

$$\frac{\partial f}{\partial \tau} \leq -\frac{1}{2}|\nabla f|^2 + \frac{R}{2} - \frac{f}{2\tau}.$$

Then if we integrate along $\gamma(\tau)$, we have

$$\frac{d}{d\tau} f(\gamma(\tau), \tau) = \frac{\partial f}{\partial \tau} + \nabla f \cdot \frac{d\gamma}{d\tau}$$

and so

$$\begin{aligned} \frac{d}{d\tau}(2\sqrt{\tau} f(\gamma(\tau), \tau)) &\leq \Big(2\nabla f \cdot \frac{d\gamma}{dt} - |\nabla f|^2 + R\Big)\sqrt{\tau} \\ &= \Big(-\Big|\nabla f - \frac{d\gamma}{dt}\Big|^2 + \Big|\frac{d\gamma}{dt}\Big|^2 + R\Big)\sqrt{\tau}. \end{aligned}$$

Corollary 13.6. *With the same setup, for any $\gamma : [\tau_1, \tau_2] \to M$ from p_1 to p_2,*

$$2\sqrt{\tau_2} f(p_2, \tau_2) - 2\sqrt{\tau_1} f(p_1, \tau_1) \leq \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2_{g(\tau)}) d\tau.$$

Note that $\mathcal{L}$ was motivated as length in the space whose Rm recovered Hamilton's Harnack expression. Also, $\mathcal{L}$ is the Li–Yau distance for its own Li–Yau inequality.

By the "standard theory" for parabolic equations, for any $Q \in M$ there exists a unique solution to $\frac{\partial \Phi}{\partial \tau} = \Delta \Phi$ such that $\Phi(\tau) \to \delta_Q$ as $\tau \to 0$. This is called the "fundamental solution".

Theorem 13.7. *Let $q \in M$ and Φ_q be the fundamental solution based at q. Then the corresponding f_q has $E(f_q) \leq 0$.*

Proof. See Lei Ni's paper in *Comm. Anal. Geom.* 2016. □

Corollary 13.8. *$f_q(p, \tau) \leq \ell(p, \tau)$ for ℓ defined relative to q.*

Theorem 13.9 (Li–Yau, 1986)**.** *Let (M^n, g) be complete with $\mathrm{Ric} \geq 0$. For $q \in C^2(M)$, assume that $\Delta q \leq \theta$. The fundamental solution H of $(\frac{\partial}{\partial t} - \Delta + q)(u) = 0$ has*

$$H(x, y, t) \geq \frac{1}{(4\pi t)^{n/2}} \exp\Big(-t\sqrt{\frac{n\theta}{2}} - \rho(x, y, t)\Big),$$

where

$$\rho(x, y, t) = \inf_\gamma \Big(\frac{1}{4t}\int_0^1 |\dot{\gamma}| + t\int_0^1 q(\gamma(s)) ds\Big)$$

is the infimum over all $\gamma : [0, 1] \to M$ from x to y.

Proof. The idea is to redo Riemannian geometry for ρ. The first variational formula gives a geodesic equation

$$\nabla_X X = 2t^2 \nabla q.$$

Along a geodesic, we are going to have

$$\frac{d}{dt}|\dot\gamma|^2 = 2\langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 2t^2\langle\dot\gamma, \nabla q\rangle$$

and so $|\dot\gamma(s)|^2 - 4t^2 q(\gamma(s))$ is constant in s. If we pick a path $\sigma(\tau)$ through y and minimizing geodesics γ'_τ from x to $\sigma(\tau)$, then we get

$$\nabla_Y \rho(x,y,t) = \frac{\dot\gamma(1)}{2t}.$$

If we take derivative with t, we also get

$$\frac{\partial}{\partial t}\rho(x,y,t) = -\frac{1}{4t^2}|\dot\gamma(1)|^2 + q(y).$$

In particular,

$$\frac{\partial\rho}{\partial t} + |\nabla_y \rho|^2 = q(y).$$

If we look at the second variation formula, we get

$$\frac{\partial^2\rho}{\partial t^2} = \frac{1}{2t}\Big(\int_0^1 \langle R(x,v)x,v\rangle + \langle\nabla_v v, x\rangle|_0 + \int_0^1 |\nabla_x v|^2\Big) + t\int_0^1 \operatorname{Hess} q(v,v)dt$$

for v the variation field. The index form $I(v,v)$ is going to be the same thing without $\langle\nabla_v v, x\rangle|_0$. The Jacobi field equation is

$$\nabla_v\nabla_x x = 2t^2\nabla_v(\nabla q).$$

Also, you can compute the Hessian as

$$\operatorname{Hess}_y \rho_{(x,y,t)}(v,v) = I(\tilde v,\tilde v),$$

for $\tilde v$ Jacobi fields along the minimal geodesic $\tilde v(0) = 0$ and $\tilde v(1) = v_y$. There is also going to be an index form lemma $I(v,v) \le I(w,w)$ for all vector fields v,w along a minimal geodesic, with $v(0) = w(0)$ and $v(1) = w(1)$ with v a Jacobi field.

If $e_1,\dots,e_n$ are orthonormal at $\gamma(1)$, we can extend it along γ by parallel transport. If we write $w_i(s) = s^\alpha e_i(s)$, then

$$\begin{aligned}\Delta^y\rho(x,y,t) &\le \sum_{i=1}^n I(w_i,w_i) = \frac{1}{2t}\Big(\int_0^1 -s^{2\alpha}\operatorname{Ric}(x,x) + \int_0^1 n\alpha^2 s^{2\alpha-2}\Big) + t\int_0^1 s^{2\alpha}\Delta q \\ &\le \frac{n\alpha^2}{2t(2\alpha-1)} + \frac{\theta t}{2\alpha+1}.\end{aligned}$$

Choosing α to minimize the right hand side, we get

$$\Delta^y\rho(x,y,t) \le \frac{n}{2t} + \sqrt{\frac{n\theta}{2}}.$$

Now we compute

$$\Big(\frac{\partial}{\partial t} - \Delta + q\Big)\Big[\frac{1}{(4\pi t)^{n/2}}\exp\Big(-t\sqrt{\frac{n\theta}{2}} - \rho\Big)\Big] = -\frac{\exp(-)}{(4\pi t)^{n/2}}\Big(-\Delta\rho + \frac{n}{2t} + \sqrt{\frac{n\theta}{2}} + |\nabla\rho|^2 - q + \rho_t\Big) \le 0.$$

Then we can use the maximal principle. $\square$

14 March 8, 2018

Let (M, g_t) be a backwards Ricci flow with $\tau \in [0, T)$. For $\gamma : [\tau_1, \tau_2] \to M$, we defined

$$\mathcal{L}[\gamma] = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R^{g(\tau)}_{(\gamma(\tau))} + |\dot{\gamma}(\tau)|^2_{g(\tau)})d\tau.$$

For fixed $p \in M$, we had $L : M \times [0, T) \to \mathbb{R}$ by

$$L(q, \tau) = \inf \mathcal{L}[\gamma].$$

over all paths $\gamma : [0, \tau] \to M$ from p to q. Then we define reduced length and volume as

$$\ell(q, \tau) = \frac{L(q, \tau)}{2\sqrt{\tau}}, \quad V(\tau) = \int_M \frac{1}{(4\pi\tau)^{n/2}} e^{-\ell(-,\tau)} d\mu_{g(\tau)}.$$

14.1 Perelman's monotonicity

Theorem 14.1 (Perelman, §9)**.** *Let Φ be the fundamental solution of $\frac{\partial \Phi}{\partial \tau} = \Delta \Phi$. Let us write*

$$\Phi = \frac{1}{(4\pi\tau)^{n/2}} e^{-f} d\mu_{g(\tau)}.$$

Then

$$f(p_2, \tau_2) \le f(p_1, \tau_2) + \frac{L(\gamma)}{2\sqrt{\tau_2}}$$

for all $\tau_1 < \tau_2$ and any $[\tau_1, \tau_2] \to M$ from p_1 to p_2.

Theorem 14.2 (Perelman, §7)**.** *$V(\tau)$ is nonincreasing in τ, if M is compact or g_τ is complete with bounded curvature.*

To show this, we need to understand the integrand changes in τ.

Definition 14.3. Define $\mathcal{L}\exp(\tau) : T_pM \to M$ is the map $v \mapsto \gamma(\overline{\tau})$ where $\gamma : [0, T] \to M$ is the $\mathcal{L}$-geodesic with $\lim_{\tau \to 0} \sqrt{\tau}\dot{\gamma}(\tau) = v$.

If we change variables to $s = \sqrt{\tau}$, then we can write

$$\mathscr{L}[\gamma] = 2\int_{s_1}^{s_2} \frac{1}{4}\left|\frac{d\gamma}{ds}\right|^2 + s^2 R(\gamma(s))ds$$

with Euler–Lagrange equation

$$\nabla_{\hat{X}}\hat{X} - 2s^2\nabla R + 4s\,\mathrm{Ric}(\hat{X}, -) = 0$$

where $\hat{X} = \frac{d\gamma}{ds} = 2sX$.

Pulling everything into T_pM, we get

$$V(\tau) = \int_{T_pM} \tau^{-n/2} \exp(\ell(\mathcal{L}\exp_v(\tau), \tau))J(v, \tau)\chi_\tau(v)d\tau$$

where $J(v, \tau) = \det d(\mathcal{L}(\exp(\tau)))_v$ is the change of variables foctor, and χ_τ is the characteristic function to make $\mathcal{L}\exp$ a diffeomorphism.

Proposition 14.4. *The integrand is pointwise nonincreasing.*

Proof. For $x_1, \dots, x_n \in T_pM$ linearly independent, we have

$$\tau \mapsto \mathcal{L}\exp_{r+sx_i}(\tau)$$

a $\mathcal{L}$-geodesic for any s. Then $Y_i(\tau) = (\mathcal{L}\exp_v(\tau))(x_i)$ is the $\mathcal{L}$-Jacobi field. Then

$$J(x,\tau)^2 = \frac{\det\langle Y_i, Y_j\rangle(\tau)}{\det\langle v_i, v_j\rangle}$$

and we can compute

$$\frac{d}{d\tau}\log J(v,\tau) = \frac{1}{2}\frac{d}{d\tau}\log J^2 = \frac{1}{2}\frac{\frac{d}{d\tau}(J^2)}{J^2} = \frac{1}{2}\operatorname{tr}\Big(S^{-1}\frac{dS}{d\tau}\Big)$$

where $S_{ij} = \langle Y_i, Y_j\rangle$. For some τ, we choose x_1, x_n such that $S_{ij}(\overline{\tau}) = I_n$. Then

$$\frac{d}{d\tau}\Big|_{\tau=\overline{\tau}} J = \frac{1}{2}\sum_{i=1}^{n}\frac{d|Y_i|^2}{d\tau}\Big|_{\tau=\overline{\tau}}.$$

For any i, we have

$$\frac{d|Y_i|^2}{d\tau} = 2\operatorname{Ric}(Y_i, Y_i) + 2\langle\nabla_X Y_i, Y_i\rangle \le \frac{1}{\overline{\tau}} - \frac{1}{\sqrt{\overline{\tau}}}\int_0^{\overline{\tau}}\sqrt{\tau}H(X, e_i)d\tau$$

by the Hessian comparison of the handout. Here, e_i is the extension $\nabla e_i = -\operatorname{Ric}(e_i, -) + \frac{1}{2\tau}e_i$. Then we get

$$\frac{d}{d\tau}\Big|_{\tau=\overline{\tau}}\log J \le \frac{n}{2\overline{\tau}} - \frac{1}{\overline{\tau}^{3/2}}\int_0^{\overline{\tau}}\tau^{3/2}H(x)d\tau.$$

On the other hand, we have

$$\frac{d\ell}{d\tau}\Big|_{\tau=\overline{\tau}} = -\frac{2}{\overline{\tau}}\ell + \frac{1}{2}(R(x(\overline{\tau})) + |\dot{\gamma}(\tau)|^2).$$

We combine this with the equation

$$\frac{d}{d\tau}(R + |X|^2) = -H(X) - \frac{1}{\tau}(R + |X|^2)$$

in the handout. Then we get

$$R + |X|^2 = \frac{1}{2\overline{\tau}^{3/2}}L(\gamma(\overline{\tau}), \overline{\tau}) - \frac{1}{\overline{\tau}^{3/2}}\int_0^{\overline{\tau}}\tau^{3/2}H(x)d\tau.$$

If we add them all together, we get that $\frac{d}{d\tau}$ of the log of the integrand is nonpositive. $\square$

Let us look at a corollary.

Definition 14.5. A Ricci flow g_t on $t \in [0,T)$ is **κ-noncollapsed on scale** ρ if for all $r < \rho$, if (x_0, t_0) has $t_0 \geq r^2$ and we have control $|Rm(x,t)| < \frac{1}{r^2}$ on $x \in B_r(x_0)$ relative to g_0 and $t \in [t_0 - r^2, t_0]$, then

$$\operatorname{vol}^{g(t_0)} B_r^{g(t_0)}(x_0) \geq \kappa r^n.$$

Theorem 14.6 (Perelman)**.** *Consider ρ, K, c constants. Let (M^n, g_t) be a Ricci flow on $[0,T)$ for $T < \infty$. Suppose $|Rm|$ is uniformly bounded on any compact $[0,T']$, and also supposed that g_0 has $|Rm| \leq K$ and injectivity radius $\operatorname{inj} g_\bullet \geq c$. Then the Ricci flow is κ-noncollapsed on the scale of ρ, where $\kappa = \kappa(\rho, K, c, n, T) > 0$.*

Because I have more time, let me talk about the Cheeger–Gromoll theorem.

Theorem 14.7 (Cheeger–Gromoll, JDG 1972)**.** *Let (M,g) be complete and $\operatorname{Ric} \geq 0$. If there exists a line, then $M = \tilde{M} \times \mathbb{R}$ isometrically.*

Definition 14.8. A curve $\gamma : \mathbb{R} \to M$ is a **line** if $\gamma|_{[t_1,t_2]}$ is a minimizing geodesic for all t_1, t_2.

Proof. Let us define $B_t(x) = d(x, \gamma(t)) - t$ and $B^+(x) = \lim_{t\to\infty} B_t(x)$. These will give level sets orthogonal to the geodesics. $\square$

15 March 22, 2018

Recall that we had (M, g_τ) a backwards Ricci flow, and for $\gamma : [\tau_1, \tau_2] \to M$ defined length

$$\mathcal{L}[\gamma] = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(R^{g(\tau)}(\gamma(\tau)) + |\dot\gamma(\tau)|^2_{g(\tau)})d\tau$$

and for fixed $p \in M$ distance $L(q, \tau) = \inf \mathcal{L}[\gamma]$ over $\gamma : [0, \tau] \to M$ from p to q. We defined $\ell(q, \tau) = \frac{L(q,\tau)}{2\sqrt{\tau}}$ and "reduced volume"

$$V(\tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-\ell(q,\tau)} d\mu_{g(\tau)}(q).$$

Theorem 15.1. *$V(\tau)$ decreases as τ increases.*

Also recall the technical geodesic statements

1. $\dfrac{\partial L}{\partial \tau}(q, \tau) = \sqrt{\tau}(R^{g(\tau)} - |\chi(\tau)|^2_{g(\tau)})$ for $\gamma : [0, \tau] \to M$ the minimal $\mathcal{L}$-geodesic from p to q.
2. The Laplacian comparison theorem:

$$\Delta^{g(\tau)} L(q, \tau) \le \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau} R^{g(\tau)}(q) - \frac{1}{\tau}\int_0^T \bar\tau^{3/2} H(x) d\bar\tau$$

where $H(X)$ is Hamilton's trace Harnack.

We also have

$$\frac{d}{d\tau}(R + |X|^2) = -H(X) - \frac{1}{\tau}(R + |X|^2).$$

If we multiply by $\tau^{3/2}$ and integrate, we get

$$\tau^{3/2}(R + |X|^2) = -\int_0^\tau \bar\tau^{3/2} H(x) d\bar\tau + \frac{L(q, \tau)}{2}.$$

Plugging this in the $\frac{\partial L}{\partial \tau}$ computation, and then adding to the Laplacian comparison, we get

$$\frac{\partial L}{\partial \tau} + \Delta L \le \frac{n}{\sqrt{\tau}} - \frac{L}{2\tau}.$$

We can change variables from L to ℓ, and the rearrange:

$$\frac{\partial}{\partial \tau}\Big(\tau\ell(q, \tau) - \frac{n\tau}{2}\Big) + \Delta\Big(\tau\ell(q, \tau) - \frac{n\tau}{2}\Big) \le 0.$$

Now we are in a situation where we can apply the maximal principle. Then we get that

$$\min_{q\in M}\Big(\tau\ell(q, \tau) - \frac{n\tau}{2}\Big)$$

decreases as τ increases to 0.

Corollary 15.2. $\min_{q\in M} \ell(q, \tau) \le \frac{n}{2}$ *for all τ.*

15.1 Noncollapsing

Let (M, g_t) be a compact Ricci flow, and take

$$\Omega = B_r^{g(t')}(p) \times [t' - r^2, t'].$$

We do the $\mathcal{L}$-geometry with $\tau = t' - t$ with p as basepoint. Consider the following two statements:

Proposition 15.3. *1. $V(\tau = t') \gg 0$.*

2. If $|Rm| \le r^{-2}$ on ω and if $\mathrm{vol}^{g(t')} B_r^{g(t')}(p) \ll r^n$ then $V(\tau = \epsilon) \ll 1$ contradicts V decreasing as τ increases.

Proof. (1) Roughly this means that $e^{-\ell}$ is bounded away from zero, which means that ℓ is bounded above. Fix some $\bar{t} < t'$. By the corollary above, there exists $q \in M$ such that $\ell(q, \bar{t}) \le \frac{n}{2}$. For any $Q \in M$, define $\gamma_Q : [0, t'] \to M$ from p to Q as

$$\gamma_Q = \begin{cases} \min \mathcal{L}\text{-geodesic } p \to q & \tau \in [0, \bar{t}], \\ \min g(t=0)\text{-geodesic } q \to Q & \tau \in [\bar{t}, t']. \end{cases}$$

Then we have control on both parts, and we get

$$\mathcal{L}[\gamma_Q] = \int_0^{t'} [\dots] \le n\sqrt{\bar{t}} + \int_{\bar{t}}^{t'} [\dots] \le C$$

because the geometry on $\tau \in [\bar{t}, t']$ is uniformly bounded on the compact interval $t \in [0, t' - \bar{t}]$.

(2) This is more technical. The control on $|Rm|$ implies, by the Shi estimates, control on $|\nabla Rm|$. Then a $\mathcal{L}$-geodesic $\gamma : [0, \tau] \to M$ from p with $\lim_{\tau \to 0} \sqrt{\tau}\dot{\gamma}(\tau) = v$ and $|v| \le \frac{1}{4j\frac{r}{\sqrt{\epsilon}}}$ has $\gamma(\epsilon) \subseteq B_r^{t=t'}(p)$.

Now we can split the integral of the reduced volume to the two regions

$$V(\tau) = \int_M \cdots d\mu_{g(\tau)} = \int_{T_pM} (\cdots) J(v, \tau) \chi dv = \int_{|v| \le \frac{1}{4}\frac{r}{\sqrt{\epsilon}}} + \int_{|v| \ge \frac{1}{4}\frac{r}{\sqrt{\epsilon}}}.$$

The first term is

$$\int_{|v| \le \frac{1}{4}\frac{r}{\sqrt{\epsilon}}} \le \int_{B_r^{t=t'}(p)} \frac{1}{(4\pi\epsilon)^{n/2}} e^{-\ell(q,\epsilon)} d\mu^{g(t=t'-\epsilon)}.$$

Here, we can estimate ℓ by

$$L(q, \epsilon) = \int_0^\epsilon \sqrt{\tau}(R + |X|^2) d\tau \ge -\int_0^\epsilon \sqrt{\tau} n(n-1) r^{-2} d\tau = -c_n r^{-2} e^{3/2}$$

and so $\ell(q, \epsilon) \ge -c_n r^{-2}\epsilon$. Plugging this in, we get

$$\int_{|v| \le \frac{1}{4}\frac{r}{\sqrt{\epsilon}}} \le \frac{c_n}{\epsilon^{3/2}} e^{c_n r^{-2}\epsilon} \,\mathrm{vol}(B_r^{t=t'}(p) \text{ rel. to } g(t = t' - \epsilon)).$$

This volume is almost going to be equal to $\mathrm{vol}(B_r^{t=t'}(p)$ rel. $g(t=t'))$.

Now for the other term, we can integrate monotonicity and get

$$\int_{|v|\geq \frac{1}{4}\frac{r}{\sqrt{\epsilon}}} (\cdots)J(v,\tau)|_{\tau=\epsilon}\chi dv \leq \int_{|v|\geq \frac{1}{4}\frac{r}{\sqrt{\epsilon}}} (\cdots)J(v,\tau)|_{\tau=0} dv = \frac{1}{(4\pi)^{n/2}}\int_{|v|\geq \frac{1}{4}\frac{r}{\sqrt{\epsilon}}} e^{-|v|^2}dv \leq \frac{\epsilon^{n/2}}{r^n}.$$

If we take $\epsilon = (\frac{\mathrm{vol}(B)}{r^n})^{1/n} r^2$ then $\frac{\mathrm{vol}}{\epsilon^{n/2}} = \sqrt{\frac{\mathrm{vol}}{r^n}} \ll 1$ and also $\frac{\epsilon^{n/2}}{r^n} = \sqrt{\frac{\mathrm{vol}}{r^n}} \ll 1$. $\square$

Theorem 15.4. *Let (M, g_t) be a compact Ricci flow for $t \in [0,T)$. Then there exist k, ρ_0 such that for all $\Omega = B_r^{g(t')}(p)\times[t'-r^2, t']$ with $r < \rho_0$ and $|Rm| \leq r^{-2}$ on Ω then*

$$\mathrm{vol}_{t=t'} B_r^{t=t'}(p,t') \geq kr^n.$$

This means that (M, g_t) is k-noncollapsed on scale ρ_0.

15.2 κ-solutions

The four key Ricci flow estimates are:

1. Hamilton–Ivey: on a 3-manifold the scalar curvature is large compared to the most negative eigenvalue of $Rm : \Lambda^2 \to \Lambda^2$.
2. Hamiton's Harnack: if (M^n, g_t) is a Ricci flow with $Rm : \Lambda^2 \to \Lambda^2$ is nonnegative, then

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_v R + 2\,\mathrm{Ric}(v,v) \geq 0$$

   for all $v \in TM$.
3. Shi estimates: there exist θ, C_k, depending on n such that if a Ricci flow is on $[0, \frac{\theta}{Rm}]$ and $|Rm| \leq M$ on $B^{g(0)}(p,r)$ then

$$|\nabla^k Rm(p,t)|^2 \leq C_k M^2\Big(\frac{1}{r^{2k}} + \frac{1}{t^k} + M^k\Big).$$

4. Perelman's noncollapsing.

Finite-time Ricci flow singularity and rescalings and compactness suggest the following.

Definition 15.5. A "**κ-solution**" is a Ricci flow (M^n, g_t) on $t \in (-\infty, 1)$ such that each g_t is complete with bounded curvature and $Rm : \Lambda^2 \to \Lambda^2$ is nonnegative, such that (M, g_t) is κ-noncollapsed on all scales.

Theorem 15.6. *If (M, g_t) is a non-flat κ-solution, then there exist $p_k, t_k \to -\infty$ such that*

$$g^k(t) = -\frac{1}{t_k} g(t_k - t_k t)$$

for $t \in (-\infty, 0)$ converges smoothly to a non-flat gradient shrinking soliton.

Definition 15.7. (M, g) is a **gradient-shrinking soliton** if there exist $f \in C^\infty(M)$ and a constant $\lambda > 0$ such that $\mathrm{Ric}(g) + \mathrm{Hess}\, f = \lambda g$.

If (M, g, f) is a gradient-shrinking soliton, let φ_t be the 1-parameter family of diffeomorphisms generated by ∇f. Then $(1 - \lambda t)\varphi_t^* g$ is a Ricci flow.

Theorem 15.8 (Hamilton). *If (M^2, g) is a complete gradient-shrinking soliton, then (M^2, g) is either S^2 is the standard metric or $\mathbb{R}P^2$ with the standard metric.*

Theorem 15.9 (Perelman). *Let (M^3, g) be a complete gradient-shrinking soliton. Suppose $|Rm|$ is uniformly bounded with $\sec \geq 0$. Also assume κ-noncollapsed on all scales. Then (M^3, g) is either (S^3, g_{stan}) or S^3/G for some $G \in \mathrm{SO}(4)$, or $(S^2 \times \mathbb{R}, g_{\text{stan}})$ or its $\mathbb{Z}/2$-quotient.*

Theorem 15.10 (Perelman). *If g_k is a sequence of 3-dimensional κ-solutions, with $p_k \in M$ so that $R(p_k, 0) = 1$, there exists a convergent subsequence to κ-solutions.*

This is some kind of a compactness of κ-solutions modulo scaling.

16 March 27, 2018

16.1 Estimates on solitons

Definition 16.1. Recall that (M, g, f) is a **gradient-shrinking Ricci soliton** if (M, g) is a Riemannian manifold with $f \in C^\infty(M)$ such that $\operatorname{Ric}(g) + \operatorname{Hess}^g f = \frac{1}{2} g$.

To do some analysis with this, we need some computations. Taking the trace first gives

$$R + \Delta f = \frac{n}{2}.$$

Then if we take ∇_i and commute order, we get

$$\nabla_i R_{jk} - \nabla_j R_{ik} - R_{ijkl} \nabla^l f = 0.$$

Taking the jk-trace gives

$$\frac{1}{2} \nabla_i R = R_{il} \nabla^l f.$$

We can also write this as $\nabla^i(R_{ij} e^{-f}) = 0$. So we get

$$\frac{1}{2} \nabla_i (R + |\nabla f|^2) = R_{il} \nabla^l f + \nabla_i \nabla_l f \nabla^l f = \frac{1}{2} \nabla_i f,$$

so $R + |\nabla f|^2 - f$ is a constant. Then we can add a constant to f to make the right hand side 0.

Theorem 16.2 (Cao–Zhou, JDG 2010)**.** *Suppose that $R \geq 0$. Then after fixing $p \in M$, we have*

$$\frac{1}{4}(d(x,p) - c)^2 \leq f(x) \leq \frac{1}{4}(d(x,p) + c)^2$$

and $\operatorname{vol} B_r(p) \leq C r^n$ *for* $r > 0$.

Note that a basic example if $(M^n, g) = (\mathbb{R}^n, \delta)$ and $f(x) = \frac{1}{4}|x|^2$.

Proof. Because $|f|^2 = f - R \leq f$, we have $|\nabla \sqrt{f}| \leq \frac{1}{2}$ and we already have the upper bound on f. For the lower bound for f, take a minimizing geodesic from p to q, and take the variation fields φ_{e_i}, orthonormal along γ, parallely transported. Then second variation gives

$$\int_0^L \varphi^2 \operatorname{Ric}(X, X) \leq (n-1) \int_1^L |\dot{\varphi}|^2.$$

If we chose φ to be a function with $0 \leq \varphi \leq 1$ with $\varphi(0) = \varphi(L) = 0$ and $\varphi|_{[1, L-1]} = 1$, and linearly interpolating in between, then

$$\int_0^L \operatorname{Ric}(X, X) ds \leq (n-1) \int_0^L (\dot{\varphi})^2 + \max_{B_1(p)} |\operatorname{Ric}| + \max_{B_1(q)} |\operatorname{Ric}|.$$

On the other hand,

$$\int_0^L \operatorname{Ric}(X,X)ds = \frac{0}{L}\frac{1}{2} - \nabla_X\nabla_X f ds = \frac{L}{2} - (f^\circ(L) - f^\circ(0)).$$

But we don't have control on the Ricci. So instead, we consider

$$\frac{L-2}{2} - (f^\circ(L-1) - f^\circ(1)) = \int_1^{L-1} \operatorname{Ric}(X,X) = \int_1^{L-1} \varphi^2 \operatorname{Ric}(X,X)$$
$$= 2(n-1) - \int_0^1 [\cdots] - \int_{L-1}^L [\cdots].$$

The second term is bounded by $\max_{B_1(p)}|\mathrm{Ric}|$. Integration by parts will give

$$\int_{L-1}^L \varphi^2 \operatorname{Ric}(X,X) = \int_{L-1}^L \frac{1}{2}\varphi^2 - \int_{L-1}^L \varphi^2\nabla_X\nabla_X f = \frac{1}{6} - f^\circ(L-1) - 2\int_{L-1}^L \varphi\nabla_X f.$$

So substituting gives

$$2\int_{L-1}^L \varphi f^\circ \geq \frac{L}{2} - 2n + \frac{7}{6} + f^\circ(1) - \max_{B_1(p)}|\mathrm{Ric}|.$$

The upper bound on f gives $|f^\circ| \leq \sqrt{f} \leq \sqrt{f(q)} + \frac{1}{2}$. So the left hand side bounded above by $\sqrt{f(q)} + \frac{1}{2}$.

For the volume estimation, consider the "distance" $\rho = 2\sqrt{f}$ and $D_r = \{\rho < r\}$. Also consider $V_r = \mathrm{vol}(D_r)$ and $\chi_r = \int_{D_r} R$. Then we ahve

$$V_r = \int_0^r \int_{\partial D_s} \frac{1}{|\nabla\rho|}, \quad \chi_r = \int_0^r \int_{\partial D_r} \frac{R}{|\nabla\varphi|}.$$

Then we have

$$\frac{dV_r}{dr} = \frac{r}{2}\int_{\partial D_r} \frac{1}{|\nabla f|}, \quad \frac{d\chi_r}{dr} = \frac{r}{2}\int_{\partial D_r} \frac{R}{|\nabla f|}.$$

Because we have $R + \Delta f = \frac{n}{2}$, we have

$$nV_r - 2\chi_R = 2\int_{D_r} \Delta f = 2\int_{D_r} |\nabla f| = rV_r - \frac{4}{r}\chi_r.$$

If we integrate this, we get

$$\frac{V_r}{r^n} - \frac{V_{r_0}}{r_0^n} \leq \frac{4(\chi_r - \chi_{r_0})}{r^{n+2}} + \frac{2\chi_{r_0}}{n}\Big(\frac{1}{r^n} - \frac{1}{r_0^n}\Big) \leq \frac{4\chi_r}{r^{n+2}} \leq \frac{2nV_r}{r^{n+2}}$$

if we use $r_0 = \sqrt{2n+2}$. So $V_r/r^n \leq C(r_0, n)$. □

Theorem 16.3 (Munteanu–Sesum)**.** *If $R \geq 0$, then*

$$\int_M |\mathrm{Ric}|^2 e^{-\lambda f} < \infty$$

for all $\lambda > 0$. (Here, $e^{-\lambda f}$ looks like some Gaussian.)

Proof. If we consider a large bump function φ, we have

$$\int_M |\mathrm{Ric}|^2 e^{-\lambda f}\varphi^2 = \frac{1}{2}\int R e^{-\lambda f}\varphi^2 + (1-\lambda)\int_M \mathrm{Ric}(\nabla f, \nabla f) e^{-\lambda f}\varphi^2 + \int_M \mathrm{Ric}(\nabla f, \nabla(\varphi^2)) e^{-\lambda f}\varphi^2.$$

The first term can be taken care of using $R = f - |\nabla f|^2 \leq f$. For the second term, we use

$$\leq \frac{1}{4}\int |\mathrm{Ric}|^2 e^{-\lambda f}\varphi^2 + (1-\lambda)^2 \int |\nabla f|^4 e^{-\lambda f}\varphi^2.$$

The former is absorbed, and the latter is $|\nabla f|^2 = f - R \leq f$. For the third term we do the same thing

$$\leq \frac{1}{4}\int |\mathrm{Ric}|^2 e^{-\lambda f}\varphi^2 + 4\int |\nabla f|^2 e^{-\lambda f} |\nabla\varphi|^2.$$

So we get the estimate. □

Theorem 16.4 (Muntaeanu–Sesmum)**.** *If $R \geq 0$ and*

$$R_{ijkl}\nabla^l f = \frac{1}{n-1}(R_{il}g_{jk} - R_{jl}g_{ik})\nabla^l f,$$

then $\int_M |\nabla \mathrm{Ric}|^2 e^{-f} = \int_M |\mathrm{div}\, Rm|^2 e^{-f}$ and both are finite.

Proof. We contract the second Bianchi identity, and then get

$$(\mathrm{div}\, Rm)_{ijk} - \nabla_i R_{jk} + \nabla_j R_{ik} = 0.$$

Now if we plug in the soliton for R_{jk} and R_{ik}, then we get

$$(\mathrm{div}\, Rm)_{ijk} = R_{ijkl}\nabla^l f.$$

So $|\mathrm{div}\, Rm|^2 \leq C_n |\mathrm{Ric}|^2 |\nabla f|^2 e^{-f}$. Integrability comes from this estimate $|\nabla f|^2 = f - R \leq f \leq Ce^{f/10}$. For equality, we note that

$$\int_M |\nabla \mathrm{Ric}|^2 e^{-f}\varphi^2 - \int |\mathrm{div}\, Rm|^2 e^{-f}\varphi^2 = \int 2R_{ijkl}R^{il}\nabla^k f e^{-f}\nabla^l(\varphi^2) - \int \nabla_k R_{ij}R^{ij}e^{-f}\nabla^k(\varphi^2)$$

goes to 0 where φ is a cutoff. □

Theorem 16.5. *If $\dim \geq 4$ and $R \geq 0$, then $\mathrm{div}\, W = 0$ if and only if (M, g) is a finite quotient of $N \times \mathbb{R}^k$ for some Einstein N with $f = \frac{1}{4}|x|^2$ on the $\mathbb{R}^k$ factor.*

Here, recall that the Weyl tensor is

$$W = Rm - \frac{\mathrm{Ric}(\wedge)g}{n-2} - \frac{R}{(n-1)(n-2)}g(\wedge)g.$$

Proof. If $\operatorname{div} W = 0$, then we have

$$\nabla_i R_{jk} - \nabla_j R_{ik} = \frac{g_{jk}\nabla_i R - g_{ik}\nabla_j R}{2(n-1)}.$$

If we put the soliton equation for Ric, and use $\frac{1}{2}\nabla_i R = R_i{}^j \nabla_j f$ on the right hand side, then we get the assumption from the previous theorem. Now we get

$$\int |\operatorname{div} Rm|^2 e^{-f} = \int |\nabla \mathrm{Ric}|^2 e^{-f}$$

where $|\operatorname{div} Rm|^2 = \frac{|\nabla R|^2}{2(n-1)}$ and $|\nabla \mathrm{Ric}|^2 \geq \frac{|\nabla R|^2}{n}$. This shows that R is constant, and because $\frac{1}{2}\nabla_i R = R_i{}^j\nabla_j f$, we get $\mathrm{Ric}(\nabla f, -) = 0$. So $Rm(-,-,-,\nabla f) = 0$. Then we can use the next proposition. □

Proposition 16.6 (Petersen–Wylie 2009)**.** *If R is constant and $Rm(\nabla f, X, X, \nabla f) = 0$, then we get the conclusion.*

Proof. We have

$$0 = \nabla^j f(-R_{ijk}{}^l \nabla_l f) = (\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f)\nabla^j f.$$

Here, this is

$$\nabla_i \nabla_j \nabla_k f \nabla^j f = -\nabla_i R_{jk} \nabla^j f = R_{jk} \nabla_i \nabla^j f = \Big(\frac{1}{2} g_{jk} - \nabla_j \nabla_k f\Big) \nabla_i \nabla^j f.$$

So we get

$$\nabla_{\nabla f} \nabla_i \nabla_k f = \Big(\frac{1}{2} g_{jk} - \nabla_j \nabla_k f\Big) \nabla_i \nabla^j f.$$

Recall that $R + |\nabla f|^2 - f$ is constant. If we renormalize f so that the constnat is R, then $f = |\nabla f|^2$. Then $N = f^{-1}(0)$ is the minimal points of f. If we plug in this in that formula above, we see that the eigenvalues of $\nabla^2 f$ are 0 or $\frac{1}{2}$ on N. So all eigenvalues are ≥ 0, and so f is convex. That is, N is the minimal points is a totally convex subset. Moreover, the multiplicities should be constant on N, and so $\nabla^2 f$ has constant rank on N. This shows that N is a submanifold, and the tangent spaces are $\ker \nabla^2 f$. □

Theorem 16.7 (Chen, JDG 2009)**.** *Any complete ancient Ricci flow has $R \geq 0$. In 3-dimensions,* $\sec \geq 0$.

17 March 29, 2018

Theorem 17.1 (Munteanu–Sesum)**.** *Let (M, g, f) be a complete gradient-shrinking soliton, and if* $\dim \geq 4$*, then* $\operatorname{div} W = 0$ *if and only if $(M, g) = N \times \mathbb{R}^k$ for N Einstein with $f = \frac{1}{4}|x|^2$ on $\mathbb{R}^k$.*

For $\dim = 2$, both sides are automatically true. For $\dim = 3$, the left hand side still automatically works but the right hand side works if and only if (M, g) is locally conformally flat.

17.1 2-dimensional solitons

In $\dim = 2$, the soliton equation becomes

$$2 \operatorname{Hess}(f) = (1 - R)g.$$

Theorem 17.2 (Hamilton)**.** *The only 2-dimensional complete gradient-shrinking soliton is the round S^2 with $f = 0$ and the flat $\mathbb{R}^2$ with $f(x) = \frac{1}{4}|x|^2$.*

Proof. The proof goes like this.

1. (Chen, JDG 2009) A complete gradient-shrinking soliton has $R \geq 0$.
2. (Ni, 2005) If a complete gradient-shrinking soliton has $\operatorname{Ric} \geq 0$, then (M, g) is either flat or $\inf R > 0$.
3. (Cheeger–Colding, 1996) For a compact soliton equation in $\dim = 2$, it is rotationally symmetric on S^2.
4. (Chen–Lu–Tian, 2006) If a soliton is rotationally symmetric on S^2, then it is a round metric with $f = 0$.

For 4, we have $g = dr^2 + h(r)^2 d\theta^2$ on $r \in (0, 1)$ and $\theta \in (0, 2\pi)$. Then $h(0) = h(1) = 0$ and $h'(0) = 1$ and $h'(1) = -1$. If we put this into the soliton equation, we get

$$-\frac{h''}{h} = 1 + f'', \quad -\frac{h''}{h} = 1 + \frac{h' f'}{h}.$$

Then you get $f' = (\text{const})h$. Then if we put this in an integrate, then

$$-\frac{(h')^2}{2}\bigg|_0^1 = \frac{h^2}{2}\bigg|_0^1 + (\text{const}) \int_0^1 h(h')^2.$$

Then the constant has to be 0 by the boundary condition, and then $f' = 0$ and $f'' = 0$. So $-h'' = h$ and so we get $h = \sin$.

For 3, we claim the following. On (M^n, g), if there exist functions f, k such that $\operatorname{Hess} f = \kappa g$, then $g = dr^2 + h(r)^2 g_0$ for some g_0 and $h(r)$. For globality, this is iffy, so we are instead going to follow Chen–Lu–Tian. Let $J : T_p\Sigma^2 \to T_p\Sigma^2$ be the $90°$ rotation. Then $J\nabla f$ is a Killing field, because you can compute

$$(\nabla(Jdf))(X, Y) = \frac{1}{2}(1 - R)d\mu(X, Y).$$

Then because Σ is compact, there exists a p such that $\nabla f(p) = 0$, and $J\nabla f$ generates a 1-parameter of isometries $\varphi_t : \Sigma \to \Sigma$. Because $\dim = 2$, there exists a $t > 0$ such that $d\varphi_t(p) = d\varphi_0|_p$. Then the isometry is determined by $\varphi|_p$, so $\varphi_t = \varphi_0$. That is, we get an isometric S^1-action.

For 2, take a minimal geodesic from p to q of length L. We use the same thing from last time,

$$\int_0^{L-r} \operatorname{Ric}(X,X) \le C(M) + \frac{n-1}{r} - \int_{L-r}^{L} \Big(\frac{L-s}{r}\Big)^2 \operatorname{Ric}(X,X)ds \le C(M) + \frac{n-1}{r}.$$

Then if $L > A(M)$ and $R(q) \le 1$, you will be able to show

$$\int_0^L \operatorname{Ric}(X,X) \le \frac{L}{4} + C(M).$$

To see this, we use $|\nabla f|^2 \le \lambda d^2$. Then $\nabla_i R = 2R_{ij}\nabla^j f$ shows that $|\nabla R|^2 \le 4\lambda d^2 R^2$. Integrating $|\nabla \log R| \le 2\sqrt{\lambda} d$ along the geodesic,

$$R(\gamma(s)) \le R(\gamma(L)) \exp(2\sqrt{\lambda} L(L-s))$$

for $s \le L$. Then

$$\int_{L-r}^{L} \operatorname{Ric}(X,X) \le \int_{L-r}^{L} R \le R(\gamma(L)) r \exp(2\sqrt{\lambda} Lr) \le r\exp(2\sqrt{\lambda} Lr).$$

If you chose $A(M) = 20(n-1)$ and $r = 10(n-1)/L$, and if you add this to the equation above, we get the result.

By the soliton equation, we get

$$\langle \nabla f, X\rangle|_{s=0}^{L} = \frac{1}{2}L - \int_0^L \operatorname{Ric}(X,X) \ge \frac{L}{4} + C(M).$$

Now take $\sigma(\eta)$ the integral curve of ∇f at q. Along this curve,

$$\frac{dR}{d\eta} = \langle \nabla R, \nabla f\rangle = 2\operatorname{Ric}(\nabla f, \nabla f) \ge 0,$$

so R increases, and as long as $d(p,\sigma(\eta)) > A(M)$ and $d(p,\sigma(\eta)) \ge 8(C(M) + |\nabla f|(p))$,

$$\frac{d}{d\eta} d(p,\sigma(\eta)) = \langle X, \nabla f\rangle(q) \ge \frac{L}{4} - C(M) - \langle \nabla f, X\rangle|_{s=0} \ge \frac{L}{4} - C(M) - |\nabla f|(p)$$

and so $d(p,\sigma(\eta))$ also increases. Follow σ backwards until either $d(p,\sigma(\eta)) = A(M)$ or $8(C(M) + |\nabla f|(p))$. In this compact region, $R(q)$ is bounded above 0. $\square$

17.2 3-dimensional solitons

Theorem 17.3 ('Perelman')**.** *Then only complete gradient–shrinking solitons in* dim $= 3$ *are*

- *round* S^3 *with* $f = 0$,
- *round* $S^2 \times \mathbb{R}$ *with* $f(p, x) = \frac{1}{4}x^2$,
- *flat* $\mathbb{R}^3$ *with* $f(x) = \frac{1}{4}|x|^2$,

or finite isometric quotients.

Proof. Here are the steps.

1. (Chen 2009) Any complete gradient-shrinking soliton in dim 3 has sec ≥ 0.
2. (Ni–Wallach, 2008) In this case, we have the result.

For 2, we are using the following Hamilton's 1982 computation:

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\frac{|\mathrm{Ric}|^2}{R^2} = -\frac{2}{R^4}|R\nabla_k R_{ij} - R_{ij}\nabla_k R|^2 - \frac{P}{R^3} + \Big|\nabla\frac{|\mathrm{Ric}|^2}{R^2}, \nabla \log R^2\Big|$$

where

$$P = \frac{1}{2}((\mu+\nu-\lambda)^2(\nu-\lambda)^2 + (\nu+\mu-\lambda)^2(\mu-\lambda)^2 + (\lambda+\nu-\mu)^2(\nu-\mu)^2)$$

for λ, μ, ν eigenvalues of Ric. We multiply the equation by $|\mathrm{Ric}|^2 e^{-f}$, and integrate by parts. (To justify this, we observe that R and Ric grows at most quadratically, and then using the local Shi estimates, $|\nabla^k Rm|$ grows at most $(1+r)^{k+2}$, while f grows at least quadratically. Also, $\inf R > 0$.) Then we get

$$0 = -\int\Big|\nabla\frac{|\mathrm{Ric}|^2}{R^2}\Big|^2 R^2 e^{-f} - \int \frac{2|\mathrm{Ric}|^2}{R^4}|R\nabla_k R_{ij} - R_{ij}\nabla_k R|^2 - \int \frac{P}{R^3}|\mathrm{Ric}|^2 e^{-f}.$$

So we get vanishing of all the integrands. If we assume sec > 0, then

$$R\nabla_k R_{ij} = R_{ij}\nabla_k R$$

for all i, j, k. Then for $i \neq j$, we have $R\nabla_k R_{ij} = 0$, so $\nabla_k R_{ij} = 0$. The Bianchi identity tells us that $R_{kk} = \frac{1}{2}R$, so $R = \frac{3}{2}R$. So $\nabla \mathrm{Ric} = 0$, so (M, g) is a locally symmetric space. These are classified. $\square$

Note that for higher dimensions, there are nontrivial examples. For instance (Feldman–Imanen–Knopf) there exists a gradient-shrinking soliton on $\mathbb{C}^2$ blown up at a point, that does not have Ric ≥ 0.

Theorem 17.4 (Naber)**.** *For a* 4*-dimensional complete gradient-shrinking solution, if* $Rm \geq 0$ *and is bounded, then these are* $\mathbb{R}^4$ *or* $S^3 \times \mathbb{R}$ *or* $S^2 \times \mathbb{R}^2$.

18 April 3, 2018

Let's recall the main discussion after digression. Recall that a κ-solution is a complete Ricci flow g_t on $t \in (-\infty, 1)$ with $|Rm|$ bounded, $Rm : \wedge^2 \to \wedge^2$ is nonnnegative, and κ-noncollapsed on all scales. Also recall that κ-noncollapsed on all scales means that if $|Rm| \leq \frac{1}{r^2}$ on $B_r^{t_0}(p) \times [t_0 - r^2, t_0]$ then $\mathrm{vol}^{t_0} B_r^{t_0}(p) \geq \kappa r^n$. This implies that if $|Rm| \leq \kappa$ on $\Omega \subseteq M \times (-\infty, 1$ then $\mathrm{vol}^{t_0} B_r^{t_0}(p) \geq \kappa r^n$ for all r, p such that $r \leq 1/\sqrt{\kappa}$ and $B_r^{t_0}(p) \times [t_0 - r^2, t_0] \subseteq \Omega$.

18.1 Limit of a κ-solution

The imprecise idea is that a κ-solution is supposed to be a blowup limit of a finite-time singularity of a Ricci flow on the compact manifold. This could be topologically/geometrically complicated.

Theorem 18.1. *Let (M, g_t) be a κ-solution. Then there exists a sequence of points q_λ such that the rescaled manifolds $(M, \frac{1}{\lambda} g(\lambda t), q_\lambda)$ smooth convergence to $(M^\infty, g^\infty(t), g^\infty)$ subsequentially, and the limit is a complete gradient shrinking soliton.*

Recall that a gradient-shrinking solution is the same thing as an ancient Ricci flow such that there is a 1-parameter family $\varphi_t : M^\infty \to M^\infty$ of diffeomorphisms such that $g(t) = (-t)\varphi_t^* g(-1)$. Here, φ_t is generated by $\nabla^{g(-1)} f$ for some $f : M^\infty \to \mathbb{R}$. Then this is equivalent to the original condition

$$\mathrm{Ric} + \mathrm{Hess}(f) = \frac{1}{2} g$$

for $g = g(-1)$.

Proof. We first want to construct $(M^\infty, g^\infty(t))$. Here, we use Hamilton's compactness theorem. To apply this, we need on compact subsets, upper bounds on $|Rm|$, and injectivity radii bounded below from 1. We are going to recognize f as the limit of reduced length functions. Then we need Arzela–Ascoli for l and ∇l.

Let the basepoint for l be $(p, 0)$. Let q_λ such that $\ell(q_\lambda, \lambda) \leq \frac{n}{2}$. (Here, recall that $\inf \ell(-, \tau) \leq \frac{n}{2}$ for all τ.) Recall that

$$\frac{\partial \ell}{\partial \tau} = -\frac{\ell}{\tau} + R + \frac{\kappa}{2\tau^{3/2}}, \quad |\nabla \ell|^2 = -R + \frac{\ell}{\tau} - \frac{\kappa}{\tau^{3/2}}, \quad \nabla \ell \leq -R + \frac{n}{2\tau} - \frac{\kappa}{2\tau^{3/2}},$$

where $\kappa = \int_0^\tau g^{3/2} H(X) ds$ and $H(X)$ is the Hamilton trace Harnack. If we have a κ-solution, and $Rm \geq 0$, then $H(X) \geq -R/\tau$ and then

$$\kappa \geq -\int_0^\tau R\sqrt{-s} ds \geq -L = -2\sqrt{\tau}\ell.$$

then $R + |\nabla \ell|^2 \leq 3\ell/\tau$.

To get control, we fix $\overline{\tau}$. Then we have $\ell(q_{\overline{\tau}}, \overline{\tau}) \leq \frac{n}{2}$ and integrate out to a neighborhood of $q_{\overline{\tau}}$ using $|\nabla \ell|^2 \leq 3\ell/\tau$. This gives uniform control on R from $R \leq 3\ell/\tau$. Now using $\frac{\partial R}{\partial \tau} \leq 0$ for ancient solutions (coming from Harnack), we get uniform control over R in a spacetime neighborhood. Now we have

$$\frac{\partial \ell}{\partial \tau} + \frac{1}{2}|\nabla \ell|^2 = \frac{1}{2}R - \frac{\ell}{2\tau}$$

which is an ODE in ℓ, with uniform control on R. Then we have uniform control of $\nabla \ell$ on a spacetime neighborhood. Now apply Hamilton's compactness and Arzela–Ascoli. (Bounded away injective radii follows from κ-noncollapsing.)

So we have our basic objects. We can get $\tilde{\ell}$ as a Lipschitz limit. If we take linear combinations of the equations we have, we get elliptic and parabolic inequalities

$$2\nabla \ell - |\nabla \ell|^2 \leq -R + \frac{n-\ell}{\tau}, \quad \frac{\partial \ell}{\partial \tau} - \Delta \ell + |\nabla \ell|^2 \geq R - \frac{n}{2\tau}.$$

Rearranging them gives

$$(4\Delta - R)e^{-\ell/2} \geq \frac{\ell - n}{2} e^{-\ell/2}, \quad \Big(\frac{\partial}{\partial \tau} - \Delta + R\Big)((4\pi\tau)^{-n/2} e^{-\ell}) \leq 0.$$

Then give a weak form, we can extend in to $\tilde{\ell}$.

Now where does the soliton equation come from? We compute

$$\ell_{\frac{1}{\lambda}g(\lambda t)}(q, \tau) = \ell_{g(t)}(q, \lambda\tau), \quad V_{\frac{1}{\lambda}g(\lambda t)}(\tau) = V_{g(t)}(\lambda\tau).$$

So we have, by monotonicity, $V_{g(t)}(\lambda\tau) \to V_0$ as $\lambda \to \infty$. Then $V_{\frac{1}{\lambda}g(\lambda t)}(B) - V_{\frac{1}{\lambda}g(\lambda t)}(A) \to 0$ as $\lambda \to \infty$, for any fixed A, B.

But the left hand side is

$$LHS = \int_A^B \frac{d}{d\tau} V(\tau) d\tau = \int_A^B \int_M \Big(\frac{\partial}{\partial \tau} - \Delta + R\Big)\Big(\frac{e^{-\ell(-,\tau)}}{(4\pi\tau)^{n/2}}\Big) d\mu_{g_\tau} d\tau.$$

(We inserted the Laplacian term because that's fine by Stokes.) We know that the integrand is nonpositive, and have weak convergence to 0. Therefore

$$\Big(\frac{\partial}{\partial \tau} - \Delta^\infty + R^\infty\Big)\Big(\frac{e^{-\tilde{\ell}}}{(4\pi\tau)^{n/2}}\Big) = 0.$$

Then we have equality in the equations, so $2\nabla \ell - |\nabla \ell|^2 + R - \frac{n-\ell}{\tau} = 0$. This shows that ℓ is smooth.

Now it follows that

$$-2\tau \Big|\mathrm{Ric} + \mathrm{Hess}\,\ell - \frac{g}{2\tau}\Big|^2 \frac{e^{-\ell}}{(4\pi\tau)^{n/2}} = 0.$$

So this is a gradient-shrinking soliton. $\square$

Proposition 18.2. *If a κ-solution is nonflat, then so is the gradient-shrinking solution.*

Proof. If the soliton is flat, then $\operatorname{Hess} \tilde{\ell} = \frac{g}{2\tau}$ and $\Delta \tilde{\ell} = \frac{n}{2\tau}$. Then $|\nabla \tilde{\ell}|^2 = \tilde{\ell}/\tau$ and $|\nabla \sqrt{4\pi \tilde{\ell}}| = 1$. So $\sqrt{4\tau \tilde{\ell}}$ is a smooth distance function and (M^∞, g^∞) is just Euclidean space. Then $\lim_{\tau \to 0} V_{g(t)}(\tau) = 1$.

But we also have $\lim_{\tau \to \infty} V_{g(t)}(\tau) = 1$. So monotonicity of V implies that we have equality. Then (M, g_t) is a gradient-shrinking soliton, and (M, g_t) is its "asymptotic" gradient shrinking soliton.

□

Theorem 18.3 (universal κ)**.** *In dimension 3, there exists a $\kappa_0 > 0$ such that any non-flat 3-dimensional κ-solution is either κ_0-noncollapsed or isometric to S^3/Γ with the round metric.*

Note that S^3/Γ is only κ-noncollapsed on small scales.

Proof. Because it is nonflat, its asymptotic gradient-shrinking soliton is either S^3/Γ or $S^2 \times \mathbb{R}/\Gamma$. Now if it is S^3/Γ then $g(-\lambda_k)$ is arbitrarily round for sufficiently large λ_k. By Bonnet–Myers, this is compact. Also, by the Hamilton–Iveys estimate, the metric only becomes rounder and so $g(-\lambda_k + t)$ is arbitrarily round for $t > 0$. Then $g(t)$ is round for all t. (Here, "roundness" can be taken to mean $|\mathrm{Ric}^\circ|^2/R^2$.)

If the gradient-shrinking soliton is $(S^2 \times \mathbb{R})/\Gamma$, then it is κ_0-noncollapsed on some scale φ. Here, we have that $\frac{1}{\lambda_k} g(\lambda_k t) \to (S^2 \times \mathbb{R})/\Gamma$ and $g(\lambda_k t)$ is κ_0-noncollapsed on scale $\lambda_k \varphi$. □

Theorem 18.4. *The only 2-dimensional κ-solutions are S^2 or $\mathbb{R}P^2$ with the round metric.*

Proof. The proof is the same. The hard thing is the almost round S^2 only get rounder under the Ricci flow. □

19 April 10, 2018

Recall that a κ-solution is an ancient complete Ricci flow, with bounded $Rm \geq 0$, that is κ-noncollapsed at every scale. Last time we say that

1. If (M, g_t) is a κ-solution for all $\lambda > 0$, then there exists a $g_\lambda \in M$ such that there is subsequential C^∞_{loc} convergence
$$(M, \tfrac{1}{\lambda} g(\lambda t), g_\lambda) \to (M^\infty, g^\infty(t), p^\infty).$$
The limit is going to be a gradient shrinking soliton.
2. If (M, g) is non-flat, then a gradient shrinking soliton is non-flat.
3. The only 2-dimensional κ-solutions are round S^2 or $\mathbb{R}P^2$.
4. There exists a $\kappa_0 > 0$ such that any 3-dimensional κ-solution is either constant 0 curvature or constant positive curvature or κ_0-solution.

19.1 Volume controls curvature

Theorem 19.1. *A κ-solution is either flat or*
$$\lim_{r\to\infty} \frac{\operatorname{vol}^{g(t)}(B_p^{g(t)}(r))}{r^n} = 0$$
for all p, t. In 2 and 3 dimensions, we have
$$\lim_{q\to\infty} R(q) d(p,q)^2 = \infty$$
for all t.

Let's see the consequences of this.

Theorem 19.2. *Consider (M, g_t) a Ricci flow on $t \in [0, Tr^2]$. If we have $Rm \geq -r^2$ on $B_r^{g(t)}(p)$ and $\operatorname{vol}^{g(t)} B_r^{g(t)}(p) \geq wr^n$ for all t, then*
$$|Rm| \leq \frac{C(w)}{r^2} + \frac{B(w)}{t}$$
on $B_{r/4}^{g(t)}(p)$, for all t.

Lemma 19.3. *Let (M, g) be a complete non-compact manifold with unbounded $|Rm|$. Then there exist $p_j \in M$ such that $|Rm(p_j)| \geq j$ and $|Rm| \leq 4|Rm(p_j)|$ on $B_{j/\sqrt{|Rm(p_j)|}}(p_j)$.*

Proof. For fixed j, we inductively construct y_k so that $|Rm(y_0)| \geq j$. If y_k are not already good, then we choose y_{k+1} such that
$$|Rm(y_{k+1})| > 4|Rm(y_k)|, \quad d(y_k, y_{k+1}) < \frac{j}{\sqrt{|Rm(y_k)|}}.$$
Then $Rm(y_k)$ grows exponentially, but $d(y_k, y_{k+1})$ decreases exponentially. So y_k stays in a finite region. □

Now there is a Ricci flow version.

Lemma 19.4. *For all $B > 4$ and $C > 1000$, there exists an $A < \min(A/4, B/1000)$ (with $A \to \infty$ as $B, C \to \infty$) such that for any $p \in M$ and (M, g_t) a complete Ricci flow on $t \in [0, T)$, if (q, s) is such that*

$$|Rm^{g(s)}(\epsilon)| > C + \frac{B}{s}, \quad d_{g(s)}(p, q) < \frac{1}{4},$$

then there exists (q', s') such that $d_{g(s')}(p, q') < \frac{1}{3}$ and $|Rm^{g(s')}(\epsilon')| > C + \frac{B}{s'}$ and $|Rm| \le 4|Rm^{g(s')}(\epsilon')|$ on

$$B^{g(t)}_{\frac{1}{10}\sqrt{\frac{A}{Q}}(p)} \text{ for } t \in \left(s' - \frac{A}{Q}, s'\right]$$

The proof is the same except for that the distance changes.For a minimizing geodesic $\gamma : [0, 1] \to (M, g_k)$ from p to q, we can compare

$$\frac{d}{dt} d_t(p, q) = -\int_0^L \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) ds$$

against the second variation formula. If $\mathrm{Ric} \le \kappa g_t$ on $B^{g(t)}_r(p) \cup B^{g(t)}_r(q)$ then we have

$$\frac{d}{dt} d_t(p, q) \ge -\frac{2(n-1)}{r} - \frac{4}{3}\kappa r.$$

Proof of Theorem 19.2. If this estimate is false, there exists a sequence of contradicting points. Then the point-picking lemma improve sthe contradicting sequences to some local control of curvature. Rescale the Ricci flow by $|Rm(x_k, t_k)|$ along (x_k, t_k). Then local control becomes uniform control of curvature. Volume control gives uniform control of the injectivity radius as well. By Hamilton's compactness theorem, there exists a smooth limit of rescalings, and it is nonflat since $|\tilde{Rm}(x_\infty, t_\infty)| = 1$. The limit has $AVR \ge w$ and this contradicts the previous theorem. □

There is an improvement.

Theorem 19.5. *The conditions are equal to those Theorem 19.2 except for that we only assume the volume bound at the finial time. Then there exists a $\tau_0 = \tau_0(w)$ such that*

$$|Rm| \le \frac{C(w)}{r^2} + \frac{B(w)}{t - (T - \tau_0)r^2}$$

on $B^{g(t)}_{r/4}(p)$ and $t \ge (T - \tau_0)r^2$

Proof. Use the distance distortion to propagate the volume estimate backwards . □

Theorem 19.6. *There exists a positive increasing function $w : [0, \infty) \to [0, \infty)$ such that for all 3-dimensional κ-solution, for all t,*

$$R^{g(t)}(p) \le R^{g(t)}(q) w(R^{g(t)}(q) d^{g(t)}(p, q)^2)$$

for all p, q.

This is a universal uniform curvature estimate.

Proof. Fix $t = 0$ and $y \in M$. By the $ASCR = \infty$ in 2, 3-dimensions, there exists a $z \in M$ such that $R(zd(y, z)^2 \ge 1$. Choose z closest to y satisfying this condition, so that $R(z)d(y, z)^2 = 1$ and $Rad = d(y, z)$. Then for $x \in B$, we have $R(x)d(x, y)^2 \le 1$ and so

$$R(x) \le \frac{4}{Rad^2}$$

for all $x \in B$. Hamilton Harnack extends this estimate to $\le R$ at every prior time. Because of κ_0-noncollapsing, we have

$$\operatorname{vol} B_{8Rad}(z) \ge \operatorname{vol} B \ge \kappa_0 \Big(\frac{Rad}{4}\Big)^3 = \frac{\kappa_0}{2^{15}}(8Rad)^3.$$

Then the previous theorem implies that $|Rm| \le c(\kappa_0)R(z)$ on $B_{8Rad}(z)$.

Now this extends back in time on B by Hamilton–Harnack. Applying the local Shi estimates, we get $\frac{\partial R}{\partial t}(z) \le c(\kappa_0)R(z)^2$. Then $R(z)$ is controlled by its value in prior time. If we use Hamilton's Li–Yau estimate, the value at prior time is controlled by $R(y)$. So

$$|Rm| \le (\text{const}(\kappa_0))\tilde{A}R(y).$$

Then noncollapsing gives $\operatorname{vol} B_{r_0}(y) \ge \kappa_0 y_0^3$ for some $r_0(\kappa_0)$ and so $\operatorname{vol} B_{R_0}(y) \ge \kappa_0(\frac{r_0}{R_0})^3 R_0^3$ for $R_0 \ge r_0$. Then apply the previous theorem. □

Theorem 19.7. *If (M_k, g_k^t) are 3-dimensional nonflat κ-solutions, and there exist $p_k \in M_k$ such that $R^{g_k^0}(p_k) = 1$, then it subsequentailly C^∞_{loc} converges to a κ-solution.*

Proof. We only need to check that the limiting solution has bounded curvature. Suppose not, that $g^\infty(0)$ has unbounded curvature. Apply point-picking in the Riemannian setting. Local control around points extends backwards by Hamilton–Harnack, and rescaling around points, and appeal to Hamilton compactness. Then we get a smooth limit.

Arrange the points so that $d(p, p_{k+1}) \gg d(p, p_k)$. Also, assume that

$$(\dot\gamma_k(p), \dot\gamma_{k+1}(p)) < \frac{1}{k}$$

where γ_k is a minimizing geodesic from p to p_k. Now the claims is that γ_k converge to a geodesic line in $(N, g^N(0))$. (Toponogov's theorem) The splitting theorem implies that (N, g^0) splits as a metric product, and then (N, g^t) is either $S^2 \times \mathbb{R}$ or $\mathbb{R}P^2 \times \mathbb{R}$ by the classification of 2-dimensional κ-solutions. □

20 April 12, 2018

20.1 ϵ-neck

Definition 20.1. Let (M^n, g) be complete. We say that a $N \subseteq M$ is an **ϵ-neck of radius** r if $(N, r^{-2}g)$ is ϵ-close to $C^{1/\epsilon}_{\text{loc}}$ to $S^{n-1} \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$.

Proposition 20.2. *There exists $\epsilon_0 = \epsilon_0(n)$ such that if (M, g) has $\sec \geq 0$ then there does not exist a sequence of ϵ-necks with $\epsilon < \epsilon_0(n)$ and radius going to 0.*

This finishes the proof compactness for κ-solutions, because

1. if the limit as $\sup|Rm| = \infty$ then we can rescale the offending slice by blowup sequence for curvature,
2. we get a limit, and splits off a line, and the limit is metrically $(S^2 \times \mathbb{R})/\Gamma$ by the 2-dimensional κ-solution classification,
3. the limit from (1) has arbitrarily small necks with arbitrarily small radius.

In the second point, basic Toponogov's theorem gives this spitting. For a geodesic triangle in (M, g) with $\sec \geq 0$, if we take a triangle in $\mathbb{R}^n$, the angle is greater than what we would have for $\mathbb{R}^n$. So as the angle $p_n p p_{n+1}$ becomes small and $d(p, p_{n+1}) \gg d(p, p_n)$, we will get that the angle $p p_n p_{n+1}$ goes to π.

Definition 20.3. Let (M, g_t) be a Ricci flow. We say that (p, t_0) is a **center of a ϵ-neck** for $Q = R^{g(t_0)}(p)$, the subset $B^{g(t_0)}_{\frac{1}{\epsilon}Q^{-1/2}}(p) \times (t_0 - \frac{1}{\epsilon^2 Q}, t_0)$ rescaled by Q is ϵ-close in $C^{1/\epsilon}_{\text{loc}}$ to $S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ evolving by the Ricci flow on $t \in (-1, 0)$ with $R = 1$ at $t = 0$.

Definition 20.4. We say that $B \subseteq M \times \{b\}$ is a **final slice** of a strong ϵ-neck if there exists a such that $B \times [a, b]$ can be rescaled to be ϵ-close in $C^{1/\epsilon}$ to $S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ and the Ricci flow on $t \in [-1, 0]$ is with $R = 1$ at $t = 0$.

Definition 20.5. We say that $B \subseteq (M, g)$ is an **ϵ-neck** if it can be rescaled to be ϵ-close in $C^{1/\epsilon}$ to $S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$.

Definition 20.6. A metric on $S^3 \setminus \overline{B}^3$ and $\mathbb{R}P^3 \setminus \overline{B}^3$ is a **ϵ-cap** if there exists a compact subset such that every point of the complement is contained in an ϵ-neck.

Theorem 20.7. *Let (M, g_t) be a noncompact 3-dimensional κ-solution. For arbitrary $\epsilon > 0$, consider M_ϵ the points at $t = 0$ which are not centers of ϵ-necks.*

1. *Then M_ϵ is compact with boundary.*
2. $\operatorname{diam} M_\epsilon \leq C_\epsilon Q^{-1/2}$.
3. *$C_\epsilon^{-1} Q \leq R \leq C_\epsilon Q$ on M_ϵ for $Q = R^{g(0)}(x)$ for some $x \in \partial M_\epsilon$.*

Proof. If there exists $x_k \in M_\epsilon$ diverging to ∞, then we had a universal estimate

$$R(x_0) \le R(x_k)\omega(R(x_k)d(x_0, x_k)^2).$$

Then because $d(x_0, x_k) \to \infty$, we have $R(x_k)d(x_0, x_k)^2 \to \infty$. Now rescale to normalize $R(x_k) = 1$. Then by the κ-compactness theorem, we get a smooth limit. This limit splits off a line by the same Toponogov argument. Now we know all the 2-dimensional κ-solutions, so the limit is $S^2 \times \mathbb{R}$. This means that for $k \gg 1$ the point x_k is a center of a ϵ-neck. This contradicts $x_k \in M_\epsilon$, and so M_ϵ should be compact.

For (2) and (3), suppose that there exists a sequence of κ-solutions M^i with $x_i \,\mathrm{im}\, M^i_\epsilon$ such that we have any one of

$$d^{g(0)}(x_i, y_i)^2 R^{g(0)}(y_i) \ge i, \quad R^{g(0)}(y_i) \ge iR^{g(0)}(x_i), \quad R^{g(0)}(x_i) \ge iR^{g(0)}(y_i)$$

for all $y_i \in \partial M^i_\epsilon$. Again, we rescale to normalize $R^{g(0)}(x_i)$ and κ-compactness gives a smooth limit. For $y_\infty \in \partial(M^\infty_\epsilon)$ consider $y_i \to y_\infty$ with $y_i \in \partial M^i_\epsilon$. Then at the limit, we have either

$$d^{g(0)}(x_\infty, y_\infty)R^{g(0)}(y_\infty) = \infty, \quad R^{g(0)}(y_\infty) = \infty, \quad R^{g(0)}(y_\infty) = 0.$$

The third one is a contradiction by Hamilton's Harnack inequality. □

Theorem 20.8. *For $\epsilon \ll 1$, if (M, g_t) is a 3-dimensional κ-solution, then for every (p, t) there exists a*

$$r \in \epsilon[R^{g(t)}(p)^{-1/2}, C_\epsilon R^{g(t)}(p)^{-1/2}]$$

and a neighborhood $B^{g(t)}_r$ such that one of following holds:

1. *$B \subseteq M \times \{t\}$ is a final slices of a strong ϵ-neck.*
2. *B is an ϵ-cap with one complement.*
3. *B is closed without boundary, and has positive sectional curvature.*

20.2 Canonical neighborhoods theorem

Definition 20.9. Let $\phi \in C^\infty(\mathbb{R})$ be positive increasing such that $\phi(s)/s$ is decreasing with limit 0. We say that a Ricci flow (M, g_t) has ϕ-almost nonnegative curvature if $Rm \ge -\phi(R)$ everywhere.

We are trying to do something like Hamilton–Iveys estimate.

Theorem 20.10 (canonical neighborhood theorem)**.** *For all ϵ, k, ρ, ϕ, there exists a $r_0 > 0$ such that the following holds. Suppose (M, g_t) (for $T \ge 1$) is a compact 3-dimensional Ricci flow with ϕ-almost nonnegative curvature, κ-noncollapsed on scales $\le \sigma$. Then at every point (p, t_0) with $t_0 \ge 1$ and $R^{g(t_0)}(p) \ge r_0^{-2}$, for $Q = R^{g(t_0)}(p)$ the region $R^{g(t)}_{1/\sqrt{\epsilon Q}}(p) \times [t_0 - \frac{1}{\epsilon Q}, t_0]$ is ϵ-close to a subset of a κ-solution after rescaling by Q.*

Corollary 20.11. *Let (M^3, g_t) be a compact Ricci flow. For all $x_i \in M$ with $t_i \to T < \infty$ and such that $Q_i = R^{g(t_i)}(x_i) \to \infty$,*

$$\tilde{g}_i(t) = Q_i g(t_i + \tfrac{t}{Q})$$

converges subsequentially in C^∞_{loc} to a κ-solution.

Proof. This is an immediate corollary of the canonical neighborhood theorem and the compactness for κ-solutions. $\square$

We want to do surgery and glue in the "standard solutions". Let $(\mathbb{R}^3, g)$ have $R \geq 1$, rotationally symmetric, and $\mathbb{R}^3 \setminus B_1(0)$ is isometric to the round cylinder $S^2 \times [1, \infty)$ with $R = 1$.

Definition 20.12. We call (M, g_t) a **standard solution** if g_0 as above, with uniformly bounded $|Rm|$ on every compact time interval.

Note that given any compact time interval, there is a short-time exists by Shi, and also uniqueness by Chen–Zhu 2006. This is not immediate because M is not compact. Then automatically

1. g_t has nonnegative curvature, after applying the maximal principle in some way,
2. $\limsup_{t \to T} \sup_{\mathbb{R}^3} |Rm| = \infty$,
3. $(\mathbb{R}^3, g_t)_{t \leq 2}$ is κ-noncollapsed on scales ≤ 1,
4. $(\mathbb{R}^3, g_t)$ satisfies the canonical neighborhood theorem.

The harder claim is that maximal time existence is $T = 1$. Also, $(\mathbb{R}^3, g_t)$ is still rotationally symmetric.

We now need to choose g_0. On the complement of the ball, take cooreinates $(-B, \infty) \times S^2$ and $g_0 = e^{2F(z)}(dz^2 + g_{S^2})$ so that

1. $F = 0$ on $[0, \infty)$,
2. F vanishes to infinite order at 0,
3. $f < 0$ and $f' > 0$ and $f'' < 0$ on $(-A, 0)$,
4. $\max(|f|, |f'|) \leq \epsilon f''$ on $(-A, 0)$.

We can choose moreover so that g_0 has sec ≥ 0, and we can smoothly glue in a ball of constant positive curvature.

21 April 17, 2018

Here are the informal statements for surgery.

21.1 Analysis of blowup regions

Theorem 21.1. *For all $\epsilon \ll 1$ and a 3-dimensional κ-solution, every $(p,t) \in M \times (-\infty, 0)$ has a spatial neighborhood B, one of the following of which is true:*

1. *is a final slice of a parabolic ϵ-neck,*
2. *is a ϵ-cap, with one corresponding ϵ-neck,*
3. *B is a closed manifold with* $\sec > 0$.

The size of B is going to be comparable to $R^{g(t)}(p)^{-1/2}$. Also $R^{g(t)}(p)$ controls R on $B \times \{t\}$ and $R^{g(t)}(p)$ controls $\mathrm{vol}^{g(t)}(B)$ from below. In the third case, the sectional curvature is controlled from below by the scalar curvature.

Theorem 21.2. *If (M^3, g_t) is a compact Ricci flow, for every $\epsilon > 0$ there exists a $r_0 > 0$ such that if $Q = R^{g(t)}(p) \geq r_0^{-2}$ then the $\frac{1}{\sqrt{\epsilon Q}}$-parabolic neighborhood of (p,t) is rescaled by Q to be ϵ-close to some region of a κ-solution.*

Let (M^3, g_t) be a compact Ricci flow with finite maximal interval of existence, $[0,T)$ where $T < \infty$. By the Shi estimates, we have

$$\lim_{t \to T} \sup\nolimits_M |Rm| = \infty.$$

Define

$$\Omega = \{p \in M : \sup\nolimits_{t<T} |Rm^{g(t)}(p)|_{g(t)} < \infty\}.$$

We know that there exists point $q \in M \setminus \Omega$.

In the case of κ-solutions, we have the universal curvature bounds. This and the local Shi estimates show that there exists a universal η such that

$$|\nabla R| \leq \eta R^{3/2}, \quad \left|\frac{\partial R}{\partial t}\right| \leq \eta R^2$$

everywhere on the κ-solution.

Lemma 21.3. *$p \notin \Omega$ if and only if* $\lim_{t \to T} R^{g(t)}(p) = \infty$.

Proof. The backwards direction is obvious. For the forward direction, we have $|Rm^{g(t_i)}(p)| \to \infty$. Then by Hamilton–Ivey, we have $R^{g(t_i)}(p) \to \infty$. Then by the gradient estimates, we have $R^{g(t)}(p) \to \infty$. ☐

Lemma 21.4. *$\Omega \subseteq M$ is open.*

Corollary 21.5. *Every connected component of Ω is noncompact, because $q \in M \setminus \Omega$.*

Lemma 21.6. *$\Omega \neq \emptyset$ implies that M is diffeomorphic to either S^3/Γ or $S^2 \times S^1/\Gamma$.*

Proof. M is covered by ϵ-necks or ϵ-caps. ☐

Assume that $\emptyset \subsetneq \Omega \subsetneq M$. From the Shi estimates and uniform local control of $|Rm|$ on Ω, we get uniform local control of $|\nabla^p Rm|$. Then there exists a smooth limit $\bar{g} = \lim_{t\to T} g$ on Ω.

Lemma 21.7. *$(\Omega, \bar{g})$ has finite volume.*

Proof. We have

$$\frac{d}{dt}\operatorname{vol}(M, g_t) = -\int_M R^{g(t)} d\mu_{g(t)}.$$

Because $\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{3}R^2$, the maximal principle shows that $\operatorname{vol}(M, g_t)$ can grow at at most polynomial order. ☐

Define

$$\Omega_\rho = \{x \in \Omega : \bar{R}(x) \leq \rho^{-2}\}.$$

Then $x \notin \Omega_\rho$ for $\rho < r_0/2$, and so satisfies the canonical neighborhood theorem conclusion.

Lemma 21.8. *$\Omega_\rho \subseteq M$ is compact.*

Suppose C is a connected component of Ω, which does not intersect Ω_ρ. Then $\bar{R} > \rho^{-2}$ on C. Now the canonical neighborhood theorem applies to any point of C and so every point of C has a neighborhood B_x, which is either an ϵ-neck or an ϵ-cap.

Lemma 21.9. *If B_x is a ϵ-neck for all $x \in C$ then C is a double ϵ-horn. If B_x is a ϵ-cap for some $x \in C$ then C is a capped ϵ-horn.*

Definition 21.10. A **double ϵ-horn** is a metric on $S^2 \times I$ such that every point has a ϵ-neck neighborhood and scalar curvature goes to ∞ as the interval coordinate $z \to 0, 1$. A **capped ϵ-horn** is a ϵ-cap on $S^3 \setminus \overline{B}^3$ or $\mathbb{R}P^3 \setminus \overline{B}^3$ such that scalar curvature goes to ∞ on the end.

Proof. You just glue the local pictures together. ☐

If C is a connected component of Ω which does intersect Ω_ρ, then C is open and Ω_ρ compact. So there exists a $x \in C \setminus \Omega_\rho$.

Lemma 21.11. *Every connected component of $C \setminus (C \cap \Omega_\rho)$ is either*

1. *an ϵ-tube with boundary components in Ω_ρ,*
2. *an ϵ-cap with boundary in Ω_ρ,*
3. *an ϵ-horn with boundary in Ω_ρ.*

21.2 Surgery on the limiting metric

The idea of surgery is to throw out the connected components of Ω which do not intersect Ω_ρ, because we know exactly what double ϵ-horns look like. In the remaining horns, we cut out the tip and gluing in the "standard solution". Then we restart the Ricci flow on the manifold.

But in context, we need to locate exactly how much we cut and where we glue in. So we need to think quantitatively how we are going to do this. Say the Ricci flow with surgery satisfies the **ϵ-a priori assumptions** if there exists a nonincreasing $r : [0, \infty) \to (0, \infty)$ such that

1. $R \geq (-\nu)(-\log(-\nu) + \log(1+t) - 3)$ anywhere $\nu < 0$ (where $\nu \leq \mu \leq \lambda$ are the eigenvalues of $Rm : \bigwedge^2 \to \bigwedge^2$)
2. If $R^{g(t)}(p) \geq r(t)^{-2}$ then there exists a neighborhood $B \ni p$ such that it is one of ϵ-neck, ϵ-cap, closed with sec < 0, and $B_\sigma^{g(t)}(p) \subseteq B \subseteq B_{2\sigma}^{g(t)}(p)$ for some $\sigma < C_1(\epsilon) R^{g(t)}(p)^{-1/2}$.

So the question is whether we can choose $r(t)$ such that the Ricci flow with surgery satisfies the a priori assumptions. The answer is yes, but this is really subtle.

Theorem 21.12. *If (M^3, g) is with $R \geq 0$, then either M is flat or diffeomorphic to the connected sum of S^3/Γ and $S^2 \times S^1$.*

Proof. If $R > 0$ then the maximal principle to

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{3} R^2.$$

Then R blows up in finite time. Here, doing surgery only changes the manifold in large R regions. So the estimate survives surgery. This means that the Ricci flow with surgery becomes extinct in finite time. ☐

Theorem 21.13 (Poincaré conjecture)**.** *If (M^3, g) is compact with $\pi_1(M) = 0$, then it is diffeomorphic to S^3.*

This is hard, but if $\pi_1(M) = 0$ then the Ricci flow with surgery is extinct in finite time. Perelman's argument is not totally trustworthy, but there is a replacement argument by Colding–Minicozi 2006.

22 April 19, 2018

We have two goals: the canonical neighborhoods theorem, and a priori assumptions for the Ricci flow surgeries. Roughly, the canonical neighborhoods theorem is

Theorem 22.1 (Hamiton–Iveys pinching)**.** *Given a 3-manifold and a Ricci flow, for all $\epsilon > 0$ there exists an $r_0 > 0$ such that if $t \geq 1$ and $R^{g(t)}(p) \geq r_0^{-2}$ then a $(\epsilon R^{g(t)}(p))^{-1}$-parabolic neighborhood round (p, t) is ϵ-close to a region of a κ-solution after parabolic rescaling by $R^{g(t)}(p)$.*

22.1 Proof of the canonical neighborhoods theorem

Assume that the claim does not hold. Then there exist Ricci flows $(M^k, g^k(t))$ all on $t \in [0, \geq 1)$, all κ-noncollapsed on scales $\leq \sigma$, but there exist $r_k \to 0$ and $x_k \in M^k$ and $t_k \geq 1$ such that $R^{g(t_k)}(x_k) \geq r_k^{-2}$ but the ϵR^k-neighborhood of (x_k, t_k) is not scaled by R^k to be ϵ-close to a κ-solution.

Now the claim is that we can suppose that the theorem holds for all

$$(x, t) \in M^k \times \left[t_k - \frac{r_k^{-2}}{4R^{g(t_k)}(x_k)}, t_k\right]$$

such that $R^{g(t)}(x) \geq 2R^{g(t_k)}(x_k)$. This is because if this doesn't hold, we can inductively replace (x_k, t_k) by (x_k^l, t_k^l) and have $R(x_k^l, t_k^l)$ exponentially growing on a compact regions of spacetime. If $R^{g(t_k)}(x_k)$ locally controls curvature, then we can apply Hamilton compactness and get a κ-solution in the limit.

For any $(\bar{x}, \bar{t})$ with

$$\bar{t} \in \left[t_k - \frac{1}{8}\frac{r_k^{-2}}{8R^{g(t_k)}(x_k)}, t_k\right],$$

we claim that we have $R \leq 4(R^{g(t_k)}(x_k) + |R^{g(\bar{t})}(\bar{x})|)$ on a $\sqrt{C(k)}/\sqrt{R^{g(t_k)}(x_k) + |R^{g(\bar{t})}(\bar{x})|}$-parabolic neighborhood around $(\bar{x}, \bar{t})$. If $R^{g(t)}(x) \leq 2R^{g(t_k)}(x_k)$ then this is obvious. If $R^{g(t)}(x) > 2R^{g(t_k)}(x_k)$ then take a piecewise smooth curve γ that connects (x, t) and $(\bar{x}, \bar{t})$ though $(x, \bar{t})$ which is a constant point path on one side and a geodesic on the other side. Then for some subinterval of γ, we have $R \geq 2R^{g(t_k)}(x_k)$. On this subinterval, the picture is close to a κ-solution so we inherit gradient estimates $|\nabla R| \leq \eta R^{3/2}$ and $|\partial_t R| \leq \eta R^2$. Integrate along subintervals, and we can control either $R^{g(t)}(x)$ by $R^{g(\bar{t})}(\bar{x})$ (if the subinterval is the entire γ) or by $2R^{g(t_k)}(x_k)$ if it is a strictly subinterval.

Now we rescale (M^k, g_t^k) by $R(x_k, t_k)$. The claim is that for all $\rho > 0$, the rescale R is unirforly bounded on $R_\rho^{g(t_k)}(x_k)$. The previous estimates show that we can extend backwards in time a little bit, and Hamilton–Ivey gives the Shi estimates and so the bounds on $|\nabla^p Rm|$. The amount of backwards extension only depends on the t_k distance, so any derivative $\nabla^p Rm$ is uniformly bounded on compact sets. Hamilton compactness implies the existence of a limiting metric $(M^\infty, g^\infty, x^\infty)$ as a limit of $(M^k, \tilde{g}(t_k), x_k)$.

Hamilton–Iveys shows that $Rm^\infty \geq 0$. This Rm^∞ is also bounded. This bound is independent of the distance, so the extension back in time is uniform in the distance. This implies convergence on nontrivial time intervals.

Let t' be the minimal by which we can extend backwards $(t', t_0]$ on which there are uniform bounds on curvature. The claim is that $t' = -\infty$. If not $\sup_M R$ should blow up as $t \searrow t'$. Hamilton arnack shows that $\frac{\partial R}{\partial t} + \frac{R}{t-t'} \geq 0$ on M^∞, and so

$$R(-, t) \leq Q \frac{t_0 - t'}{t - t'}$$

for $Q = \sup_M R(-, t_0)$. The distance distorsion can be estimated as

$$|d_{g(t)}(x, y) - d_{g(t_0)}(x, y)| \leq C$$

for all $t \in (t', t_0]$.

Because $\min_M R$ is nondecreasing, there exists a y_∞ such that $R^{g(t' + \frac{c}{10})}(y_\infty) < \frac{3}{2}$. Then there exists a sequence $y_k \to y_\infty$ and this implies $R^{g(t)}(y_k) \leq 10$ for $t \in [t' - \frac{c}{10}, t' + \frac{c}{10}]$. The distance distortion estimate shows that the same holds for balls centered at x_k. Then we can use Hamilton compactness to extend the limit back to $t' - \frac{c}{10}$. This shows that the solution can be extended to an ancient solution.

This is arguably the most important qualitative result of Perelman.

Lemma 22.2. *For all $\epsilon < \frac{1}{100}$, $\delta < \epsilon$, and $T > 0$, there exists a $h < \delta^2 r(T)$ such that if (M^3, g_t) is a Ricci flow with surgery on $[0, T)$ where T is the singular time with ϵ-a priori assumptions with $r(t)$, then at T, for x in an ϵ-horn with boundary in $\Omega_{\delta r(t)}$ such that $\bar{R}(x) \geq h^{-2}$ then the set $B^{g(T)}_{\delta^{-1}\bar{R}(x)^{-1/2}}(x)$ is the final slice of a parabolic δ-neck.*

23 April 24, 2018

Brendle 2018 does Ricci flow with surgery on compact manifolds with positive isotropic curvature and $n \geq 12$.

Corollary 23.1. *$M^{n\geq 12}$ compact has metric with positive isotropic curvature, and contains no nontrivial incompressible space forms, then M is diffeomorphic to a connect sum of S^n/Γ and $(S^{n-1} \times \mathbb{R})/\Gamma$.*

23.1 Brendle's Ricci flow with surgery in high dimensions

If Rm has positive isotropic curvature, then $R > 0$ and then there is finite time blowup of Ricci flow because

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{r} R^2.$$

The main contribution is applying Hamilton–Iveys to this context.

Theorem 23.2. *Let $n \geq 12$ and $K \subseteq C_B(\mathbb{R}^n)$ be compact with $K \supseteq \mathrm{PIC}$. For all $T > 0$, there exist $\theta, N > 0$ and $f(x)/x \to 0$ as $x \to \infty$ and a closed $\mathrm{O}(n)$-invariant $\{F_t\}_{0\leq t\leq T}$ continuous in t set with $K \subseteq F_0$ and F_t invariant under the Q-ODE, such that*

$$F_t \subseteq \left\{ Rm : \begin{matrix} Rm - \theta R g(\wedge) g \in \mathrm{PIC},\ R_{11} + R_{22} - \theta R + N \geq 0, \\ Rm = f(R) g(\wedge) g \in 2\mathrm{PIC} \end{matrix} \right\}$$

for all orthonormal e_1, e_2. Here, we say that $Rm \in 2\mathrm{PIC}$ if $Rm \times \mathbb{R}^2 \in \mathrm{PIC}$.

The moral is that the blowup limit is 2NIC and uniformly PIC. (Uniformly means that $Rm - \theta g(\wedge) g \in \mathrm{NIC}$ for some $\theta > 0$.)

Proof. The proof is 35 pages. It uses the Böhm–Walking construction. $\square$

This motivates the following definition.

Definition 23.3. A **κ-solution** is an ancient complete Ricci flow, bounded curvature with 2NIC and κ-noncollapsed on all scales.

Theorem 23.4 (Brendle 2012)**.** *Hamilton's Li–Yau inequality holds on complete bounded Ricci flows with* 2NIC.

Recall that 2NIC is contained in sec ≥ 0, so Toponogov's theorem holds and we have splitting theorems. But the difference between this and the 3-dimensional case is that we don't have classification of gradient-shrinking solitons.

Theorem 23.5 (Brendle–Huisken–Sinesterai)**.** *If a complete ancient Ricci flow with bounded Ricci curvature is uniformly* 1PIC*, then it has constant curvature.*

This mimics $S^3, S^2 \times \mathbb{R}, \mathbb{R}^3$ with the splitting theorem. The proof of the universal curvature bound goes as in 3-dimension, and universality depends only on κ. So we have compactness theorem for the space of κ-solutions.

Let us first show that the limit has bounded cuvature. Recall that Perelman did this by contradiction. If you have a blowup, then there is a splitting of limit, and classification of 2-dimensional solitons gives a limit of $S^2 \times \mathbb{R}$. Then the limit has ϵ-necks that are arbitrarily small, and this contradicts sec ≥ 0.

Proof. The Harnack inequality

$$\frac{\partial R}{\partial z} + 2\nabla_v R + 2\operatorname{Ric}(v,v) \geq 0$$

for all v carries over to the limit.

Now if the curvature is unbounded, pick points that are blowup sequences, so that $Rm \to \infty$. Then Harnack gives local control of curvature extending back in time. The Shi estimates give local control of $\nabla^p Rm$, and Hamilton compactness implies that there is a C^∞_{loc} limit that splits a line. Because the limit is uniformly PIC, the $(n-1)$-dimensional factor is uniformly 1PIC. If we use Brendle–Huisken–Sinestrai on this factor, the original unbounded slices has arbitrarily small ϵ-necks, and this is impossible when sec ≥ 0. $\square$

The canonical neighborhoods theorem can be essentially proven as in the 3-dimensional case. We need nonexistence of nontrivial incompressible space forms to rule out necks $(S^{n-1} \times \mathbb{R})/\Gamma$ for nontrivial Γ. Surgery also works the same. We can directly check this with the choice of a cap. Brendle's Hamiton–Iveys estimates is preserved by surgery. We call **Ricci flow with surgery with parameters** ϵ, r, δ, h if

1. the canonical neighborhoods theorem is satisfied where $R \geq r^{-2}$ and accuracy 4ϵ, and
2. surgery on δ-necks, around points with $R \geq h^{-2}$ can be done.

Proposition 23.6. *For all $\epsilon > 0$ there exist $\kappa, \tilde{\delta}(-)$ such that any Ricci flow with parameters $\epsilon, r, \delta < \tilde{\delta}(r), h$ is κ-noncollapsed on scales $\leq \epsilon$.*

Proof. This is a careful case-by-case analysis depending upon R at the center point. $\square$

Proposition 23.7. *For all $\epsilon > 0$, there exist $\hat{r}, \hat{\delta} > 0$ such that if there is a Ricci flow with parameters $\epsilon, \hat{r}, \hat{\delta}, h$ on $[0, T)$, then the canonical neighborhoods theorem is satisfied where $R \geq (2\hat{r})^{-2}$ with accuracy 2ϵ.*

Proof. Replay the proof of the canonical neighborhoods theorem. The limiting κ-solution is covered by 2ϵ-necks and 2ϵ-caps. $\square$

Choose any $\epsilon > 0$ and take $\hat{\delta}, \hat{r}$ as the proposition.

Proposition 23.8. *There exist $h < \hat{\delta}\hat{r}$ such that for every Ricci flow with surgery with parameters $\epsilon, \hat{r}, \hat{\delta}, h$, singular at $T < \infty$, if x is in a 4ϵ-horn at g_T with $R^{g(T)}(x) = h^{-2}$ then the parabolic neighborhood of size $h/\hat{\delta}$ around (sx, T) is surgery-free.*

Proof. We do proof by contradiction. We show that there is a uniform curvature estimates. We look at a blowup and apply Brendle–Huisken–Sinestrai. □

Theorem 23.9. *For all $\epsilon > 0$, one can choose $\hat{\delta}, \hat{r}, h$ as in the above proposition such that for all $g_0 \in \mathrm{PIC}$ there exists a Ricci flow with surgery with parameter $\epsilon, \hat{\delta}, \hat{r}, h$.*

Proof. We induct on the surgery time. Volume drops by h^n at each surgery, and so the surgery times cannot accumulate. □

23.2 Further topics

For arbitrary (M^3, g) such that Ricci flow with surgery exists for all time, we need to study long-time behavior. This takes extra work.

Kleiner–Lott 2017 passes the surgery parameters to 0 so that we get a limiting "Ricci flow with surgery". Bamler–Kleiner has the application that for every 3-dimensional space form,

$$\mathrm{Isom}(M) \hookrightarrow \mathrm{Diff}(M)$$

is a homotopy equivalence. This is called Smale's conjecture.

What does Ricci flow with surgery on 4-manifolds look like? Mean curvature flow with surgery has been defined for 2-convex surfaces in $\mathbb{R}^{n+1}$, by Huisken–Sinestrai and Brendle–Huisken. Another question is whether this is possible for general surfaces. It is not obvious, but any S^4 homeomorphic to S^4_{std} can be smoothly embedded in $\mathbb{R}^5$. Then the hope is to do mean curvature flow with surgery on this.

There are some other questions of whether there are modifications of Ricci flow for manifolds of negative curvature. You might also want to define Ricci flow with boundary. There are other newer flows people are looking into, e.g., Laplacian G_2-flow or Donaldson's Yang–Mills flow which contributed to the Kobayashi–Hitchin correspondence. You can also try to look at deformations of complex structures.

So these are the essential points in the theory:

1. Hamilton's compactness theorem
2. Hamiton–Ivey estimates
3. Li–Yau inequalities (Hamilton's version of Rm under the Ricci flow, and Perelman's $\frac{\partial w}{\partial t} + \Delta w = 0$ under the Ricci flow)
4. Geodesic geometry of Li–Yau length
5. Formal arguments

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