Math 231br - Advanced Algebraic Topology

Taught by Alexander Kupers
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Spring 2018

This course was taught by Alexander Kupers in the spring of 2018, on Tuesdays and Thursdays from 10 to 11:30am. There were two large problem sets, and midterm and final papers. An official and much better set of notes can be found on the instructor's website `http://math.harvard.edu/~kupers/teaching/231br/index.html`.

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Lectures notes will be on the webpage `math.harvard.edu/~kupers/`.There will be biweekly homeworks, which will focus on topics we didn't have time to cover in class. You will also need to write two essays. Office hours are on Tuesdays from 3:30pm to 5:30pm.

1.1 Introduction

In this course, we will define some more algebraic invariant of spaces and compute them. Having more algebraic invariants helps us study topological spaces. For instance, if two spaces have different invariants, they are different. But we can also reverse this and study invariants using spaces. Invariants also allow us to answer geometric questions. The trick is to build invariants from geometry, then use our understanding of them to answer geometric questions. One of the invariants we will learn is topological K-theory. This is built from finite-dimensional vector bundles:

$$KU^0(X) = \mathbb{Z}\{\text{iso. classes of fin. dim. } \mathbb{C}\text{-vec. bund.}/X\}/([E]+[F]=[E\oplus F]).$$

The second invariant is built from compact smooth manifolds:

$$MO_n(X) = \{n\text{-dim. closed manifolds } M \text{ with map to } X\}/\text{cobordism}.$$

This idea of studying geometry using invariants were used to prove the following theorems.

Theorem 1.1 (Thom)**.** *Two closed smooth manifolds are cobordant if and only if they have the same Stiefel–Whitney numbers.*

Here, we say that M is **cobordant** to N if there exists a cobordism W between them. A **cobordism** between M and N is a manifold W such that $\partial W = M \amalg N$.

Theorem 1.2 (Hurwitz–Radon–Eckmann–Adams)**.** *For $n \in \mathbb{N}$, write $|n|_2 = c + 4d$ where $c \in \{0,1,2,3\}$, and define $\rho(n) = 2^c + 8d$. Then S^{n-1} admits $\rho(n) - 1$ linearly independent vector fields.*

The first part of the course will describe the class of invariants called generalized cohomology theories. We will also give methods to compute them. These are functors $E^*(-) : \mathsf{Top}^{\text{op}} \to \mathsf{GrAb}$ satisfying some axioms that aid in their computation. Concretely, this means that for each X and $i \in \mathbb{Z}$ there are abelian groups $E^i(X)$, and for each continuous $f : X \to Y$ and $i \in \mathbb{Z}$ there are homomorphisms $E^i(Y) \to E^i(X)$.

Example 1.3. Ordinary cohomology $H^*(-;A)$ with coefficients in an abelian group A are cohomology theories. KU^* and MO^* are also generalized cohomology theories.

One of the axioms is that $E^*(-)$ are homotopy invariant. So to study their values, we will learn how to break spaces into simpler pieces in the homotopy category Ho(Top). Using this, we will develop and important conceptual tool. We'll learn that $E^i(X)$ is given by mapping (de)suspensions of X into a spectrum E. A **spectrum** is an object of the stable homotopy category "Ho(Top)[Σ^{-1}]". Then we can study $E^*(-)$ by studying the spectrum E and applying the same techniques to E as we apply to space. For example, Thom's theorem I mentioned amounts to computing the homotopy groups $\pi_* MO$.

There is a third reason we want to study algebraic invariants. They teach us about the general homotopical machinery. The technical details, once property understood, can be used to introduce homotopy-theoretic techniques to other fields. For instance, there are simplicial techniques. Because simplicial techniques are more combinatorial, and so easier to handle. There are also spectral sequences, which is an algebraic way to reassemble filtration quotients into the original object.

There are many things we can do. We will first cover the homotopy theory of spaces, generalized cohomology, stable homotopy theory, and spectral sequences. But then there are three directions we can go.

- "manifold direction": cobordism theory, cobordism category, chromatic homotopy theory.
- "linear direction": topological K-theory, chromatic homotopy theory, vector fields on spheres, algebraic K-theory.
- "categorical direction": homotopical algebra, algebraic K-theory, higher categories.

1.2 Category theory

The philosophy of category theory is that objects are determined by maps in and out of them.

Definition 1.4. A **category** $\mathcal{C}$ is a class $\mathrm{ob}(\mathcal{C})$ and a class $\mathrm{mor}(\mathcal{C})$ with functions $s, t : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ called "source" and "target" and $i : \mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ called "identity" and composition

$$\circ : \{(f, g) : t(f) = s(g)\} \to \mathrm{mor}(\mathcal{C})$$

satisfying some reasonable axioms:

- $(h \circ g) \circ f = h \circ (g \circ f)$
- $\mathrm{id}_{t(f)} \circ f = f = f \circ \mathrm{id}_{s(f)}$ where $\mathrm{id}_X = i(X)$

Example 1.5. Set denotes the category of sets. Objects are sets and morphisms are functions between sets.

We will always assume that our categories or locally small, i.e., $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{mor}(\mathcal{C}) : s(f) = X, t(f) = Y\}$ is a set.

Definition 1.6. A **functor** is a "map" between categories: functions $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ and $\mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$ that are compatible with $s, t, i, \circ$.

Example 1.7. Given a set K, we have a functor $- \times K$ that sends X to $X \times K$ and sends f to $f \times \mathrm{id}_K$.

Example 1.8. Given a set K, we have a functor $\mathrm{Fun}(K, -) : \mathsf{Set} \to \mathsf{Set}$ that sends f to the function $g \circ f \circ g$.

These two functors have the following relation: there is a bijection

$$\mathrm{Hom}_{\mathsf{Set}}(X \times K, Y) \quad \longleftrightarrow \quad \mathrm{Hom}_{\mathsf{Set}}(X, \mathrm{Fun}(K, Y)).$$

This bijection is also natural in X and Y.

Definition 1.9. An **adjoint pair** is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

The previous example is the case $\mathcal{C} = \mathcal{D} = \mathsf{Set}$, $F = - \times K$, and $G = \mathrm{Fun}(K, -)$. We want to do something similar for topological spaces. Consider the category Top of topological spaces, with objects given by topological spaces and morphisms given by continuous maps. Given a topological space K, we have two functors

$$- \times K : \mathsf{Top} \to \mathsf{Top}, \quad \mathrm{Map}(K, -) : \mathsf{Top} \to \mathsf{Top}.$$

Definition 1.10. The **mapping space** $\mathrm{Map}(K, Y)$ is the set of continuous maps $K \to Y$ with the **compact-open topology**. This topology is generated by

$$W(L, U) = \{(f : K \to Y) : f(L) \subseteq U\}$$

where $L \subseteq K$ is compact and $U \subseteq Y$ is open.

We would want there to be a bijection

$$\mathrm{Hom}_{\mathsf{Top}}(X \times K, Y) \cong \mathrm{Hom}_{\mathsf{Top}}(X, \mathrm{Map}(K, Y)). \tag{$*$}$$

But this is not always true, e.g., when $K = \mathbb{R} \setminus \{1/n : n \in \mathbb{N}\}$.

Lemma 1.11. *If K is compact Hausdorff, then $(*)$ is true.*

1.3 A convenient category

Because the above property fails in general, we are going to restrict to a convenient category.

Definition 1.12. A subcategory $\mathcal{C} \subseteq \mathsf{Top}$ is **convenient** if

(i) $\mathcal{C}$ is full (which means that if $X, Y \in \mathrm{ob}(\mathcal{C})$ then all continuous $f : X \to Y$ are in $\mathrm{mor}(\mathcal{C})$)

(ii) $\mathcal{C}$ is complete and cocomplete (which means that we don't leave $\mathcal{C}$ by taking disjoint unions, products, quotients, passing to subspaces specified by equations)

(iii) $\mathcal{C}$ is closed (which means that $- \times K$ and $\text{Map}(K, -)$ is adjoint for all $K \in \text{ob}(\mathcal{C})$)

(iv) $\mathcal{C}$ should contain locally compact Hausdorff spaces, CW-complexes, metric spaces.

Theorem 1.13 (McCord)**.** *There is a convenient category of spaces.*

The idea is that compact Hausdorff spaces are well-behaved. So we use spaces whose topologies are detected by compact Hausdorff spaces. These are called compactly generated spaces.

Definition 1.14. A space X is **compactly generated** if $C \subseteq X$ is closed if and only if $u^{-1}(C)$ is closed for all $u : L \to X$ continuous with L compact Hausdorff.

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Most people were interested in the cobordism direction, so we will do this later. Today we will start actually doing homotopy theory.

2.1 Homotopy

Recall that $f, g : X \to Y$ are **homotopic** if there is a **homotopy** between them. This is a map $H : X \times I \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{Y\times\{1\}} = g$.

Lemma 2.1. *Homotopy is an equivalence relation on the set of continuous maps $X \to Y$.*

Proof. $f \sim f$ by using the constant homotopy. $f \sim g$ implies $g \sim f$ by reversing the homotopy. $f \sim g$ and $g \sim h$ implies $f \sim h$ by concatenating homotopies: if H is the homotopy $f \sim g$ and K is the homotopy $g \sim h$ then

$$(x,t) \mapsto \begin{cases} H(x,2t) & t \le \frac{1}{2}, \\ K(x,2t-1) & t \ge \frac{1}{2}. \end{cases}$$

□

Since Top is closed, we can reinterpret this in terms of mapping spaces. The maps f, g are points in $\mathrm{Map}(X, Y)$, and a homotopy H between them is a path $I \to \mathrm{Map}(X, Y)$ between them.

Definition 2.2. The **homotopy category** $\mathrm{Ho}(\mathsf{Top})$ has objects topological spaces X and morphisms $X \to Y$ given by the set $[X, Y] = \pi_0 \mathrm{Map}(X, Y)$ of homotopy classes of continuous maps.

There is a functor $\gamma : \mathsf{Top} \to \mathrm{Ho}(\mathsf{Top})$ that identifies homotopic maps. This is in fact the universal such functor; any functor $\mathsf{Top} \to \mathcal{C}$ that identifies homotopic maps factors uniquely over $\mathrm{Ho}(\mathsf{Top})$.

But homotopy groups are based, so the natural input is a pointed space.

Definition 2.3. The category Top_* of **pointed spaces** is the category with objects (X, x_0) of a space X and $x_0 \in X$. Morphisms $(X, x_0) \to (Y, y_0)$ are continuous maps sending x_0 to y_0.

Example 2.4. The sphere

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1\}$$

with basepoint $(1, 0, \dots, 0)$ is a pointed space, and is isomorphic to $I^n/\partial I^n$ with basepoint the equivalence class of ∂I^n.

Many constructions in Top can be adapted to Top_*. The categorical product is going to be

$$(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0)).$$

The categorical coproduct is

$$(X, x_0) \vee (Y, y_0) = (X \amalg Y/(x_0 \sim y_0), [x_0]),$$

which is also called the **wedge sum**. The mapping space becomes

$$\mathrm{Map}_*((X, x_0), (Y, y_0)) \subseteq \mathrm{Map}(X, Y)$$

is a subspace consisting of those f with $f(x_0) = y_0$. We use the subspace topology, and the basepoint is the constant map to y_0. Something different from unpointed spaces is that

$$\mathrm{Hom}_{\mathsf{Top}_*}((X, x_0) \times (K, k_0), (Y, y_0)) \not\cong \mathrm{Hom}_{\mathsf{Top}_*}((X, x_0), \mathrm{Map}_*((K, k_0), (Y, y_0))).$$

The condition on the right hand side is much stronger, because the adjoint $\tilde{f} : X \times K \to Y$ of any f on the right hand side should send $X\{k_0\} \cup \{x_0\} \times K$ to y_0. To fix this, we define the **smash product**

$$(X, x_0) \wedge (Y, y_0) = (X \times Y/X \times \{y_0\} \cup \{x_0\} \times Y, [(x_0, y_0)]).$$

Then it is true that

$$\mathrm{Hom}_{\mathsf{Top}_*}((X, x_0) \wedge (K, k_0), (Y, y_0)) \cong \mathrm{Hom}_{\mathsf{Top}_*}((X, x_0), \mathrm{Map}_*((K, k_0), (Y, y_0))).$$

Lemma 2.5. *$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$. $X \wedge Y \cong Y \wedge X$. $S^0 \wedge X \cong X \cong X \wedge S^0$.*

The first one is a bit subtle, and will be true because we're working in a convenient category. This lemma shows that $(\mathsf{Top}_*, \wedge, S^0)$ is a symmetric monoidal category.

A pointed homotopy between pointed spaces is not a map $(X, x_0) \times (I, ?) \to (Y, y_0)$ but a map $I \to \mathrm{Map}_*(X, Y)$. This is the same thing as a pointed map $I_+ \to \mathrm{Map}_*(X, Y)$. So by adjunction, this is the same as a pointed map

$$X \wedge I_+ \to Y.$$

Lemma 2.6. *Pointed homotopy is an equivalence relation on the set of pointed continuous maps $X \to Y$.*

Definition 2.7. The category $\mathrm{Ho}(\mathsf{Top}_*)$ is the category with objects pointed topological spaces (X, x_0) and morphisms $(X, x_0) \to (Y, y_0)$ are pointed homotopy classes $[(X, x_0), (Y, y_0)]_+ = \pi_0(\mathrm{Map}_*(X, Y))$.

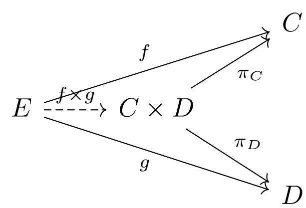
This has a similar universal property.

Definition 2.8. If (X, x_0) is a pointed space and $n \geq 0$, then the n**th homotopy group** $\pi_n(X, x_0)$ is given by $[S^n, X]_+$.

The set $[S^n, X]_+$ always contains the constant map to the basepoint, so using this we can consider it as pointed sets.

2.2 Group and cogroup objects

Definition 2.9. Given two objects C, D of $\mathcal{C}$, their **product** is an object $C \times D$ with morphisms $\pi_C : C \times D \to C$ and $\pi_D : C \times D \to D$ with the following property: for every $f : E \to C$ and $g : E \to D$, there exists a unique $f \times g : E \to C \times D$ such that the following diagram commutes:



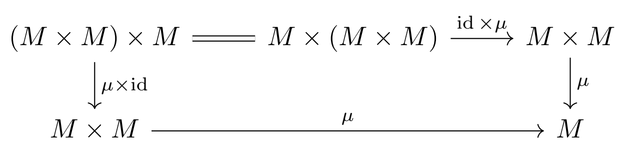
Lemma 2.10. *If a product of C and D exists, it is unique up to unique isomorphism.*

Proof. If $(C \times D, \pi_C, \pi_D)$ and $(\widetilde{C \times D}, \tilde{\pi}_C, \tilde{\pi}_D)$ are products, we get unique maps $\alpha : C \tilde{\times} D \to C \times D$ and $\beta : C \times D \to \widetilde{C \times D}$. Now $\beta \circ \alpha = \mathrm{id}$ by uniqueness and likewise $\alpha \circ \beta = \mathrm{id}$. ☐

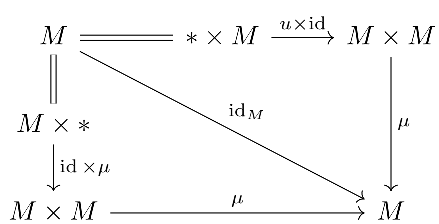
Let's assume that in $\mathcal{C}$, all finite products exist. Then $(C \times D) \times E \cong C \times (D \times E)$ by spelling out maps into them. Similarly, we get isomorphisms $C \times D \cong D \times C$ and $* \times C \cong C \cong C \times *$ where $*$ is the terminal object. (Finite products includes the empty product.)

To see when in general $\mathrm{Hom}_{\mathcal{C}}(C, D)$ as a (abelian) group structure, we study structures on objects in $\mathcal{C}$.

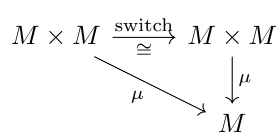
Definition 2.11. A **monoid structure** on $M \in \mathrm{ob}(\mathcal{C})$ is a multiplication map $\mu : M \times M$ and a unit $u : * \to M$ such that



and



commutes. It is said to be **commutative** if



commutes as well.

Example 2.12. In Set, and object M with a monoid structure is a (unital associative) monoid.

Definition 2.13. A **group structure** on an object G of $\mathcal{C}$ is a monoid structure (μ, u) together with an inverse map $i : G \to G$ such that

$$\begin{array}{ccccc} G\times G & \xleftarrow{\Delta} & G & \xrightarrow{\Delta} & G\times G \\ \Big\downarrow{\scriptstyle i\times \mathrm{id}} & & \downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}\times i} \\ & & * & & \\ & & \downarrow{\scriptstyle u} & & \\ G\times G & \xrightarrow{\mu} & G & \xleftarrow{\mu} & G\times G \end{array}$$

commutes.

Dually, we can define a (co-commutative) co-monoid structure on an object N of $\mathcal{C}$. This is a (commutative) monoid structure on N in $\mathcal{C}^{\mathrm{op}}$. (For this to make sense, we need $\mathcal{C}$ to have finite coproducts.) Similarly, we can define co-group objects.

Example 2.14. In Set, the only comonoid object is $\emptyset$, because not many objects map to $* = \emptyset$.

Lemma 2.15. *If G is a (commutative) group object in $\mathcal{C}$, then for all $C \in \mathrm{ob}(\mathcal{C})$ the set $\mathrm{Hom}_{\mathcal{C}}(C, G)$ has a (abelian) group sstructure. If H is a (co-commutative) co-group object in $\mathcal{C}$, then for all $C \in \mathrm{ob}(\mathcal{C})$ the set $\mathrm{Hom}_{\mathcal{C}}(H, C)$ has a (abelian) group structure.*

Proof. Let's see where multiplication on $\mathrm{Hom}_{\mathcal{C}}(C, G)$ comes from. We can compose μ to

$$\mathrm{Hom}_{\mathcal{C}}(C, G)^2 \cong \mathrm{Hom}_{\mathcal{C}}(C, G \times G) \xrightarrow{\mu\circ -} \mathrm{Hom}_{\mathcal{C}}(C, G).$$

Then applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, G) : \mathcal{C} \to \mathsf{Set}$ will preserve group objects. □

In fact, by Yoneda, a group structure on G is equivalent to a lift of $\mathrm{Hom}_{\mathcal{C}}(-, G) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ to Grp.

2.3 Algebraic structures on homotopy groups

First note that $\times$ and $\vee$ give functors on the homotopy category $\mathrm{Ho}(\mathsf{Top}_*)$, and give finite products and coproducts. So to get (abelian) group structures on $\pi_n(X, x_0)$, it suffices to give (co-commutative) co-group structures on S^n in $\mathrm{Ho}(\mathsf{Top}_*)$.

Example 2.16. S^0 cannot have a co-group structure. But S^1 has a co-group structure. We need to give a co-unit $S^1 \to *$, which is going to be the constant map. We need to give a co-multiplication map $S^1 \to S^1 \vee S^1$ by going around each of the wedge S^1s once. We need a inverse $S^1 \to S^1$, which is just going to be reflecting the circle. You can check co-associativity and the inverse being the inverse.

Example 2.17. S^n for $n \geq 2$ has a co-commutative co-group structure. The counit $S^n \to *$ is the constant map, comltiplication $S^n \to S^n \vee S^n$ is going to be collapsing a great circle to a point, and an inverse $S^2 \to S^2$ given by reflection.

But there is a fancier argument.

Lemma 2.18 (Eckmann–Hilton)**.** *If X is a set with two multiplications given by $\cdot$ and $\otimes$ which have the same unit and the following interchange property:*

$$(x \cdot y) \otimes (z \cdot w) = (x \otimes z) \cdot (y \otimes w).$$

Then $\cdot = \otimes$ and it is abelian.

Proof. We have

$$x \cdot y = (x \otimes e) \cdot (e \otimes y) = (x \cdot e) \otimes (e \cdot y) = x \otimes y.$$

Then we have

$$x \cdot y = (e \otimes x) \cdot (y \otimes e) = (e \cdot y) \otimes (x \cdot e) = y \otimes x = y \cdot x.$$

□

Lemma 2.19. *For $n \geq 2$, S^n has a co-commutative cogroup structure.*

Proof. We have $S^n \cong S^1_L \wedge S^1_R \wedge S^{n-2}$. So

$$[S^n, X]_+ = [S^1_L, \mathrm{Map}_*(S^1_R \wedge S^{n-2}, X)]_+ = [S^1_R, \mathrm{Map}_*(S^1_L \wedge S^{n-2}, X)]$$

has two lifts to groups. It can by shown that the two group structures $\cdot$ and $\otimes$ satisfies conditions of the previous lemma. Then these are all abelian groups, and Yoneda gives us a co-commutative cogroup structure on S^n. □

2.4 Fundamental groupoid

How does $\pi_n(X, x_0)$ depend on x_0?

Definition 2.20. The **fundamental groupoid** $\Pi(X)$ is the category with objects points x_0 of X and morphisms $x_0 \to x_1$ given by homotopy classes of γ from x_0 to x_1.

There is a functor

$$\pi_n : \Pi(X) \to \begin{cases} \mathsf{Set}_* & n = 0 \\ \mathsf{Grp} & n = 1 \\ \mathsf{Ab} & n \geq 2 \end{cases}$$

given by $x_0 \mapsto \pi_n(X, x_0)$. A path $[\gamma]$ gets mapped to a homomorphism $\pi_n(X, x_0) \to \pi_n(X, x_1)$ in the following way: given any $[f] \in \pi_n(X, x_0)$, construct an element of $\pi_n(X, x_1)$ by considering f as a map $I^n \to X$, putting a smaller version of f inside I^n, and filling the surrounding part with γ.

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Last time, we saw which structure on a based space X gives $[-, X]_+$ and $[X, -]_+$ a (abelian) group structure. The goal is to give examples when $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces exact sequences on $[-,-]_+$.

Definition 3.1. A sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of maps of pointed sets is **exact** if $\mathrm{im}(\alpha) = \beta^{-1}(c_0)$.

Definition 3.2. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is **exact** if

$$[-, X]_+ \xrightarrow{f_*} [-, Y]_+ \xrightarrow{g_*} [-, Z]_+$$

is exact. It is **co-exact** if

$$[Z, -]_+ \xrightarrow{g_*} [Y, -]_+ \xrightarrow{f_*} [X, -]_+$$

is exact.

Our goal is to give examples.

3.1 Mapping cone construction

We would like to show that every $f : X \to Y$ can be extended to a co-exact sequence.

Definition 3.3. Let $f : X \to Y$ be a based map. We first define the **based mapping cylinder**

$$Mf = (M \wedge I_+ \vee Y)/(x, 1) \sim f(x).$$

This is the pushout

$$\begin{array}{ccc} X & \xhookrightarrow{-\times\{1\}} & X \wedge I_+ \\ \downarrow f & & \downarrow \\ Y & \longrightarrow & Mf. \end{array}$$

There is also the unbased version $\tilde{M}g = (X \times I) \cup_g Y$ for $g : X \to Y$.

Definition 3.4. Using this, we define the **based mapping cone** $Cf = Mf/(X \times \{0\})$. This is the pushout

$$\begin{array}{ccc} X \times \{0\} & \hookrightarrow & Mf \\ \downarrow & & \downarrow \\ * & \longrightarrow & Cf. \end{array}$$

Let us write $CX = C(\mathrm{id}_X)$. Then we have $Cf = CX \cup_f Y$, and in particular, we have $i_f : Y \hookrightarrow Cf$.

Lemma 3.5. *The sequence* $X \xrightarrow{f} Y \xrightarrow{i_f} Cf$ *is co-exact.*

Proof. For each based Z, we need to show that

$$[Cf, Z]_+ \xrightarrow{i_f^*} [Y, Z]_+ \xrightarrow{f^*} [X, Z]_+$$

is exact. It is clear that $\operatorname{im}(i_f^*) \subseteq (f^*)^{-1}(*)$ because $X \to Y \to Cf$ is null-homotopic. This is because this is the same map as $X \hookrightarrow CX \simeq * \to Cf$. For the other direction $\operatorname{im}(i_f^*) \supseteq (f^*)^{-1}(*)$, we start with $g : Y \to Z$ with $f^*(g) \simeq *$. Let $H : X \times I \to Z$ be the based homotopy from $*$ to $g \circ f$. Take $g : Y \to Z$ and $\tilde{H} : CX \to Z$ and glue them to a map $\tilde{H} \circ fg : Cf \to Z$. It is now clear that $(i_f^*)(\tilde{H} \cup_f g) = \tilde{H} \cup_f g|_Y = g$. □

The cone Cf only depends up to based homotopy on the based homotopy class of f. Also, the entire diagram $X \to Y \to Cf$ is natural in f. That is, we can think of this construction as a functor $\mathsf{Top}_*^{[1]} \to \mathsf{Top}_*^{[2]}$ where $[1]$ is the category $0 \to 1$ and $[2]$ is the category $0 \to 1 \to 2$.

We can iterate this based mapping cone construction to shift one over:

$$X \xrightarrow{f} Y \xrightarrow{i_f} Cf \xrightarrow{i_{i_f}} C(i_f) = CY \cup_{if} (CX \cup_f Y) \to \cdots .$$

So we want to simply these spaces up to homotopy. If we introduce $\pi_f : Cf \to \Sigma X$ given by $CX \cup_f Y \to (CX \cup_f Y)/Y$, then we can consider the sequence

$$\begin{array}{l} X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad i_f \quad} Cf \xrightarrow{\quad \pi_f \quad} \\ \\ \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i_f} \Sigma Cf \xrightarrow{-\Sigma \pi_f} \\ \\ \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 i_f} \Sigma^2 Cf \xrightarrow{\Sigma^2 \pi_f} \cdots . \end{array}$$

Here, the $-$ sign indicates that you flip the orientation in the Σ, that is, $-\Sigma f$ maps (t, x) to $(1-t, f(x))$.

Proposition 3.6. *This sequence is isomorphic in* $\operatorname{Ho}(\mathsf{Top}_*)^{\mathbb{N}_0}$ *to iterating the based mapping cone construction.*

Consider the map

$$\pi_f : C(i_f) = CY \cup (CX \cup_f Y) \to (CY \cup_{i_f} (CX \cup_f Y))/CY = \Sigma X.$$

If we write $j = i_f$, we have commuting triangles

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{j=i_f} & Cf & \xrightarrow{i_f} & C_f & \longrightarrow & C_{i_j} \\ & & & & & \searrow^{\pi_f} & \downarrow{\scriptstyle p_f} & \searrow^{\pi_j} & \downarrow{\scriptstyle p_j} \\ & & & & & & \Sigma f & \overset{-\Sigma f}{\dashrightarrow} & \Sigma Y \end{array}$$

Lemma 3.7. *p_f is a based homotopy equivalence.*

Proof. Just draw a picture and see it. □

Lemma 3.8. *If we put $-\Sigma f$ in the bottom, the triangle commutes up to homotopy.*

Proof. Again draw a picture. □

So we can keep iterating the procedure, and we get a long co-exact sequence. Mapping into Z gives us the sequence

$$[X,Z]_+ \leftarrow [Y,Z]_+ \leftarrow [Cf,Z]_+ \leftarrow [\Sigma X,Z]_+ \leftarrow [\Sigma Y,Z]_+ \leftarrow [\Sigma Cf,Z]_+ \leftarrow \cdots.$$

By what we did last time, we have that it is a exact sequence of groups from $[\Sigma X,Z]_+$ and on, and it is an exact sequence of abelian groups from $[\Sigma^2 X,Z]$ and on. Actually there is a $[\Sigma X,Z]_+$-action on $[Cf,Z]_+$ coming from the co-action $Cf \to Cf \vee \Sigma X$. Then the map $[\Sigma X,Z]_+ \to [Cf,Z]_+$ contains some information.

Corollary 3.9. *If f is a based homotopy equivalence, then $Cf \simeq *$.*

Proof. We know from the exact sequence that $[Cf,Z]_+$ has a single element for all Z. Taking $Z = Cf$, we get that id_{CX} and $*$ are based homotopic. This means that Cf is contractible. □

3.2 Mapping path cone construction

There is a very similar story for exact sequences. This type of duality is called Eckmann–Hilton duality. I don't know a formal statement, and it is more of a philosophy.

Definition 3.10. Let $f : X \to Y$ be a based map. The **based mapping path cylinder** is defined as

$$Pf = \{(x,\gamma) : \gamma(1) = f(x)\} \subseteq X \wedge Y^{I_+},$$

which is the pullback

$$\begin{array}{ccc} Pf & \longrightarrow & Y^{I_+} \\ \big\downarrow & & \big\downarrow \mathrm{ev}_1 \\ X & \xrightarrow{f} & Y. \end{array}$$

There is also the unbased version $\tilde{P}f = \{(x,\gamma) : \gamma(1) = f(x)\} \subseteq X \times Y^I$.

Definition 3.11. The **mapping path cone** is defined as

$$Q_f = \{(x,\gamma) : \gamma(1) = f(x), \gamma(0) = y_0\} \subseteq X \vee Y^I.$$

Lemma 3.12. *The sequence $Q_f \xrightarrow{q_f} X \xrightarrow{f} Y$ is exact where $q_f(x,\gamma) = x$.*

Proof. Fix Z. The composite $Q_f \to X \to Y$ is null-homotopic and so $\operatorname{im}(q_f)_* \subseteq (f_*)^{-1}(*)$. For the other direction, If we are given $g : Z \to X$ satisfying $f \circ g \simeq *$, then we have $H : Z \times I \to Y$ a homotopy from $*$ to $f \circ g$. Then you can define $\tilde{H} : Z \to Q_f$ by $z \mapsto ((g(z), s \mapsto H(z, s))$ that satisfies $q_f \circ \tilde{H} = g$. ☐

Now you can do the same thing

$$\begin{array}{ccccccccc} Q(q_h) & \longrightarrow & Q_h & \xrightarrow{q_h} & Q_f & \xrightarrow{h=q_f} & X & \longrightarrow & Y \\ \simeq\uparrow & \nearrow & \simeq\uparrow & \nearrow^{i_f} & & & & & \\ \Omega X & \xrightarrow{-\Omega f} & \Omega Y & & & & & & \end{array}$$

and get a similar diagram

$$\cdots \to \Omega^2 X \to \Omega Qf \to \Omega X \to \Omega Y \to Qf \to X \to Y$$

isomorphic in $\mathrm{Ho}(\mathsf{Top}_*)^{\mathbb{N}^{\mathrm{op}}}$ to the iterated based mapping path cone construction. Mapping in Z, we get a long exact sequence of pointed sets/groups/abelian groups.

3.3 Relative homotopy groups and n-connected maps

Let us look at the case $Z = S^0$. Then we get

$$\cdots \to \pi_2(X) \to \pi_1(Qf) \to \pi_1(Y) \to \pi_1(X) \to \pi_0(Qf) \to \pi_0(X) \to \pi_0(Y).$$

We'll interpret $\pi_n(Qf)$ more concretely.

Definition 3.13. For $A \subseteq X$ containing the basepoint, define

$$\pi_n(X, A, x_0) = \frac{\{I^n \to X \text{ such that } \partial I^n \to A \text{ and } \partial I^n \setminus \{1\} \times I^{n-1} \to \{x_0\}\}}{\text{homotopy of such maps}}.$$

Lemma 3.14. $\pi_n(Qf, q_0) \cong \pi_{n+1}(Mf, X \times \{0\}, x_0)$.

Proof. Let us take an element of $\pi_n(Qf, q_0)$ is represented by $g : I^n \to X$ and $G : I^{n+1} \to Y$ such that

(i) $G(s, 0) = y_0$,

(ii) $G(s, 1) = f(g(s))$,

(iii) $g(s) = x_0$ if $s \in \partial I^n$, $G(s, t) = y_0$ if $s \in \partial I^n$.

Then you can glue them to get a map $I^{n+1} \to Mf$ in $\pi_{n+1}(Mf, X \times \{0\}, x_0)$. ☐

Definition 3.15. A continuous map $f : X \to Y$ is n**-connected** if for all x_0, $\pi_i(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism for $i < n$ and surjective for $i = n$. This is equivalent to that for all x_0, $\pi_i(Y, X, x_0)$ vanishes for $i \leq n$. (Here, we define $\pi_0(Y, X, x_0) = \pi_0(Y, y_0)/\pi_0(X, x_0)$.)

Definition 3.16. A map $f : X \to Y$ is a **weak homotopy equivalence** if it is n-connected for all n.

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Let time, we had these two long-exact sequences associated to the based map $f : X \to Y$,

$$\cdots \to [\Sigma Cf, Z]_+ \to [\Sigma Y, Z]_+ \to [\Sigma X, Z]_+ \to [Cf, Z]_+ \to \cdots$$

and

$$\cdots \to [Z, \Omega Qf]_+ \to [Z, \Omega X]_+ \to [Z, \Omega Y]_+ \to [Z, Qf]_+ \to \cdots .$$

To really make these useful, we really need to simplify Cf and Qf (under conditions on f).

4.1 Hurewicz cofibration

Let's start with Cf. First, by replacing f by its mapping cylinder $X \hookrightarrow Mf = X\sigma I_+ \cup_f Y$, we can assume that f is an inclusion. Now let us use the notation $i : A \hookrightarrow X$. In this case, we can take

$$Mi = (A \wedge I_+ \cup X \times \{1\}) \subseteq X \wedge I_+$$

and $Ci = Mi/A \times \{0\}$. So there is always a based map $Ci \hookrightarrow X/A$ given by

$$p : Ci \to Ci/CA \cong X/A.$$

Lemma 4.1. *If M_i is a retract of $X \times I_+$, then p is a based homotopy equivalence.*

Proof. Let us find the homotopy inverse. Here is what we can do.

$$\begin{array}{ccccc} X & \xrightarrow{-\times \mathrm{id}} & X \wedge I_+ & \xrightarrow{r} & Mi \\ \downarrow & & & & \downarrow \\ X/A & & \overset{q}{\dashrightarrow} & & Ci \end{array}$$

The map X/Ci sends A to a point, so we can find a unique $q : X/A \to Ci$. Now we need to find a homotopy from $p \circ q$ to $\mathrm{id}_{X/A}$. Intuitively, this is sliding X/A from $\{0\}$ to $\{1\}$. This is given by

$$(X/A) \wedge I_+ \to (X \wedge I_+)/(A \wedge I_+) \xrightarrow{\bar{r}} M_i/A \wedge I_+ \to Ci/CA = X/A.$$

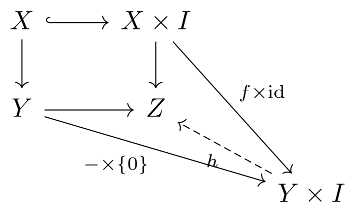
The other homotopy from $q \circ p$ to id_{Ci} is given in the following way. You define

$$Ci \wedge I_+ \to Ci; \quad \begin{cases} r & \text{on } X \wedge I_+ \\ (a, s, t) & \text{on } (a, \min(s, t)). \end{cases}$$

This shows that p is a homotopy equivalence. □

If Mi is the unbased mapping cylinder $A \times I \cup X \times \{1\} \subseteq X \times I$, it being a retract of $X \times I$ implies that Mi is a retract of $X \wedge I_+$, and this implies that $Ci \to X/A$ is a homotopy equivalence.

Definition 4.2. A continuous map $f : X \to Y$ has the **homotopy extension property** with respect to Z if in each commutative diagram, there exists an $h : Y \times I \to Z$ making the diagram commute.

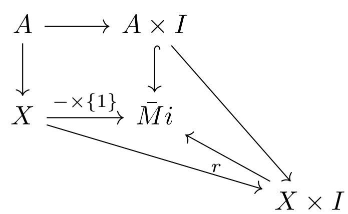


Definition 4.3. A **Hurewicz cofibration** is a map f having the homotopy extension property with respect to all Z.

In the category CGWH, every Hurewicz cofibration is the inclusion of a closed subspace. The inclusion part is easy, but the closed subspace part is annoying.

Lemma 4.4. *Let $i : A \hookrightarrow X$ be an inclusion. Then i is a Hurewicz cofibration if and only if Mi is a retract of $X \times I$.*

Proof. For convenience, we will use $\bar{M}i = A \times I \cup X \times \{0\} \subseteq X \times I$. We take the universal object. We get a diagram



Then r is the retraction. In the other direction, $\bar{M}i$ is the pushout of the diagram, so we have a map $\bar{M}i \to Z$ always. So we can set $X \times I \xrightarrow{r} \bar{M}i \to Z$ to be the extension. $\square$

So if $i : A \to X$ is a Hurewicz cofibration, then $Ci \simeq X/A$.

Definition 4.5. A **NDR-pair** or **neighborhood retract** (X, A) is a pair $A \subseteq X$ with $u : X \to I$ such that $A = u^{-1}(0)$ and a homotopy $H : X \times I \to X$ fixing A pointwise from id_X to a map sending $u^{-1}([0, 1))$ into A.

Then any NDR-pair is a cofibration. For example, $X \times \{0\} \hookrightarrow Mf$ is an NDR-pair, and so a cofibration. In particular, any map $f : X \to Y$ factors as

$$X \xrightarrow{\text{cof}} Mf \xrightarrow{\simeq} Y.$$

Another example of an NDR-pair is (D^n, S^{n-1}).

Here are also some formal properties. Hurewicz cofibrations are closed under

- (transfinite) composition
- pushouts
- retracts
- products with a fixed space C.

For instance, to show the last item, you can adjoint and move C out to the test space.

4.2 Hurewicz fibrations

Now the goal is to identify Qf with something simpler.

Definition 4.6. We say that a map $f : X \to Y$ has the **homotopy lifting property** with respect to Z if in each commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle -\times\{0\}}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle f} \\ Z \times I & \longrightarrow & Y \end{array}$$

there exists a map $\ell : Z \times I \to X$ making the diagram commute.

Definition 4.7. A **Hurewicz fibration** is a map $f : X \to Y$ having the homotopy lifting property with respect to all Z.

Again, we want a characterization by looking a universal diagram. The data of the diagram is equivalent to a map $Z \to \bar{P}f$ where

$$\bar{P}f = \{(x, \gamma) : f(x) = \gamma(0)\} \subseteq X \times Y^I.$$

So if we take $Z = \bar{P}f$, we can write out

$$\begin{array}{ccc} \bar{P}f & \xrightarrow{q} & X \\ \downarrow & & \downarrow \\ \bar{P}f \times I & \longrightarrow & Y \end{array}$$

If this were a Hurewicz fibration, we would get a map $\tilde{\lambda} : \bar{P}f \times I \to X$ and get $\lambda : \bar{P}f \to X^I$. This map taking a point in X and a path γ in Y, and producing a path in X that lifts γ. This is called the "path lifting function".

Lemma 4.8. *f is a Hurewicz fibration if and only if it has a path lifting function.*

Proof. We showed this in one direction. For the converse, let us take a diagram, and we need to produce a diagram $Z \times I \to X$. But the diagram is the same as the data of $Z \to \bar{P}f$, and then we can compose with λ to get $Z \to X^I$. Then we get $Z \times I \to X$. $\square$

Let's now take $\bar{Q}f = \{(x,\gamma) : f(x) = \gamma(0), \gamma(1) = y_0\} \subseteq X \times Y^I$. Consider the fiber $F = f^{-1}(y_0) \hookrightarrow \bar{Q}f$ given by $x \mapsto (x, c_{y_0})$.

Lemma 4.9. *If f has a path lifting function, then j is a homotopy equivalence.*

Proof. Let us restrict to $\lambda|_{\bar{Q}f}$. This takes (x,γ) to a path $\tilde{\gamma}$ in X, starting at x and covering γ. Since $\gamma(1) = y_0$, the path $\tilde{\gamma}$ lies in $f^{-1}(y_0) = F$. Then we can take

$$\mathrm{ev}_1 \circ \lambda|_{\bar{Q}_f} : \bar{Q}_f \to F.$$

We can check that this is the homotopy inverse. □

This is unsatisfactory, because this is not a based homotopy equivalence. So what we can do is

- either use based Hurewicz fibrations
- or use the lemma that if X, Y are well-based spaces and $f : X \to Y$ is a cofibrations, ($\{x_0\} \hookrightarrow X$ and $\{y_0\} \hookrightarrow Y$ are Hurewicz cofibrations) then a based path lifting function exists.

The conclusion is if f is a Hurewicz fibration ($+\epsilon$) then $F \hookrightarrow \bar{Q}f$ is a based homotopy equivalence. This is very powerful, because fibers are easy to describe. Hurewicz fibrations are closed under

- (transfinite) composition,
- pullbacks,
- retracts,
- mapping in a fixed space.

There is also the following fact.

Theorem 4.10. *If $U = \{U_i\}$ is a open cover of a paracompact space B, then $p : E \to B$ is a Hurewicz fibration if and only if for all i, $p|_{p^{-1}(U_i)} : p^{-1}(U_i) \to U_i$ is a Hurewicz fibration.*

4.3 Cofibrations and fibrations interacting

Definition 4.11. A Hurewicz cofibration of fibration is said to be **trivial** if it is also a homotopy equivalence.

We say that $f : C \to D$ has the **left lifting property** with respect to some class $X \subseteq \mathrm{mor}(\mathcal{C})$, if each commuting diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \in X} \\ D & \longrightarrow & D' \end{array}$$

has a lift. Similarly, we can define the **right lifting property**.

Theorem 4.12. *(i) f is trivial fibration if and only if f has the right lifting property with respect to all cofibrations.*

(ii) f is a fibration if and only if f has the right lifting property with respect to all trivial cofibrations.

(iii) f is a trivial cofibration if and only if f has the left lifting property with respect to all fibrations.

(iv) f is a cofibration if and only if f has the left lifting property with respect to all trivial fibrations.

Proof. Let's do (iii), $\Leftarrow$, assuming that every $f : X \to Y$ can be factored as a trivial cofibration followed by a fibration. Let us factor f as $X \hookrightarrow Z \twoheadrightarrow Y$ where $X \hookrightarrow Z$ is a trivial cofibration. We lift

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow{\text{id}} & Y. \end{array}$$

Then

$$\begin{array}{ccccc} X & \xrightarrow{\text{id}} & X & \xrightarrow{\text{id}} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

and so f is a retract of a trivial cofibration. This shows that f is a trivial cofibration. □

4.4 Serre fibrations

We we proved that $F \hookrightarrow \bar{Q}f$ is a homotopy equivalence, this is often too strong. In most cases, we only need that it is a weak homotopy equivalence. So can we find a version of the Hurewicz fibration that is more easily verified?

Definition 4.13. A map $f : X \to Y$ is a **Serre fibration** if it has the homotopy lifting property with respect to $D^n \hookrightarrow D^n \times I$ for all n.

Lemma 4.14. *If p is a Serre fibration that is a surjection on path components, then $F \hookrightarrow \bar{Q}f$ is a weak homotopy equivalence.*

Proof. It suffices to prove that $\pi_n(\bar{Q}f, F)$ vanish. π_0 vanishes by the surjectivity condition, so we don't worry about this. For each commutative diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & F \\ \downarrow & \nearrow & \downarrow \\ D^n & \xrightarrow{g} & \bar{Q}f \end{array}$$

we need to find a lift such that the top triangle commutes and the bottom diagram triangle commutes up to homotopy relative to S^{n-1}. We can do this by first using the lifting property on

$$\begin{array}{ccc} D^n & \xrightarrow{X\text{-component of } g} & X \\ \downarrow & \nearrow & \downarrow f \\ D^n \times I & \xrightarrow{Y\text{-component of } g} & Y \end{array}$$

and then we can use this map on $\partial D^n \times I \cup D^n \times \{1\} \cong D^n$. $\square$

5 February 6, 2018

Today we are going to study a particular class spaces, called CW-complexes. These are

- close to the definition of homotopy groups,
- comes with a filtration you can induct over.

5.1 CW-complexes

Definition 5.1. Attaching an n**-cell** to a space is taking a pushout

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X \cup_f D^n. \end{array}$$

Lemma 5.2. *The homotopy type of $X \cup_f D^n$ only depends on the homotopy class of f.*

It is more convenient to do this more generally. We should also allow $n = 0$, in which case $S^{-1} = \emptyset$, and we should also allow attaching multiple n-cells at the same time. In that case, we have

$$\begin{array}{ccc} \coprod_{i \in I} S_i^{n-1} & \xrightarrow{\coprod f_i} & X \\ \downarrow & & \downarrow \\ \coprod_{i \in I} D_i^{n-1} & \longrightarrow & X'. \end{array}$$

Definition 5.3. A **CW-complex** of dimension -1 is $\emptyset$. A **CW-complex** of dimension n is a space X obtained from a CW-complex of dimension $n-1$ by attaching a collection of n-cells. A **CW-complex** is a space obtained as a colimit of sequences

$$\emptyset = X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

where each X_i is a CW-complex of dimension i and $X_i \to X_{i+1}$ is the inclusion.

For a CW-complex X, we get a filtration of X by subspaces given by the image of X_i. This is called the **skeletal filtration** with X_i the i-th skeleton. For each i-cell, there is a map $e_j^i : D_j^i \to X$, which is called the **characteristic map**. In fact,

$$\coprod_{i \geq 0} \coprod_{j \in I_j} D_j^i \to X$$

is a quotient map. (This is a reshuffling of colimits.) We think of skeletal filtration and characteristic maps as part of the data of the CW-complex.

In the word CW-complex, C stands for "closure-finite".

Lemma 5.4. *For a compact space K and a continuous $f : K \to X$, the image intersects the interior of only finitely many cells.*

In particular, attaching maps also hits finitely many cells. The second letter W stands for "weak topology". That is, $C \subseteq X$ is closed if and only if $(e^i_j)^{-1}(C)$ is closed for all cells. For us, this is a consequence of the definition in terms of colimits.

Lemma 5.5. *Each CW-complex is paracompact Hausdorff.*

A generalization is a **relative CW-complex**. Here we replace $X_{-1} = \emptyset$ by $X_{-1} = A$. Then we look at

$$A \to (X, A)_0 \to (X, A)_1 \to \cdots$$

and take the colimit.

Lemma 5.6. *$A \hookrightarrow X$ is a Hurewicz cofibration for a relative CW-complex.*

Proof. We know that $\coprod S^{i-1} \to \coprod D^i$ is a cofibration because it is a neighborhood retract, and then each $(X, A)_{i-1} \hookrightarrow (X, A)_i$ is a cofibration. Then transfinite compositions of cofibrations are closed. □

Example 5.7. Let $A \subseteq X$ be both CW-complex. Then A is said to be a **subcomplex** if each characteristic map of A is also a characteristic map of X. Then $i : A \to X$ is a relative CW-complex. Also i is a filtration-preserving map, i.e., $i(A_n) \subseteq X_n$.

Definition 5.8. A map $f : X \to Y$ of CW-complexes is **cellular** if $f(X_n) \subseteq Y_n$.

Example 5.9. The n-sphere S^n is $D^0 \cup_{\text{constant}} D^n$. Another way you can obtain this is by attaching 2 n-cells on S^{n-1}, and doing this inductively. This CW-structure has two i-cells for all $i \leq n$. The map $-1 : S^n \to S^n$ sends cells onto cells. So if we quotient by this map, we get a cellular structure on $\mathbb{R}P^n$. This has a single i-cell for $i \leq n$.

Example 5.10. If M is a smooth closed manifold, then any smooth map $f : M \to \mathbb{R}$ which has non-degenerate critical points gives a CW-structure on M (up to homotopy). This is Morse theory, and is a standard source of CW-structures.

We can construct new CW-complexes out of old ones.

Lemma 5.11. *If $A \subseteq X$ is a subcomplex and $f : A \to Y$ is cellular, then $X \cup_f Y$ is a CW-complex. In particular, if $Y = *$ then X/A is a CW-complex.*

Lemma 5.12. *A product $X \times Y$ of CW-complexes is a CW-complex.*

Proof. This uses that we work in CGWH, so that product of quotient maps are quotient maps. In particular,

$$\left(\coprod_{p \geq 0} \coprod_{j \in I^X_p} D^p_j\right)\left(\coprod_{q \geq 0} \coprod_{h \in I^Y_q} D^q_k\right) \to X \times Y$$

is a quotient map, and from this you can reconstruct the CW-structure. □

Let's look at an application. Excision states that the map

$$H_*(X, A) \to \tilde{H}_*(X/A)$$

is an isomorphism if (X, A) is excisive, i.e., there is an open containing A such that U deformation retracts onto A. (If $A \hookrightarrow X$ is a Hurewicz cofibration, this is satisfied.) Let us look at $X_{n-1} \hookrightarrow X_n$, which is a inclusion of subcomplexes, which is in particular a Hurewicz cofibration. Then we get

$$\cdots \to H_i(X_{n-1}) \to H_i(X_n) \to \tilde{H}_i(X_n/X_{n-1}) \to H_{i-1}(X_{n-1} \to \cdots .$$

But X_n/X_{n-1} is just a wedge $\bigvee_{j\in I_n} S^n_j$. In particular,

$$\tilde{H}_i(X_n/X_{n-1}) = \begin{cases} \mathbb{Z}\langle I_n\rangle & i = n \\ 0 & i \neq n. \end{cases}$$

So this is helpful in computing homology. We can look at the map

$$\begin{array}{ccc} \partial : \mathbb{Z}\langle I_n\rangle \cong \tilde{H}_n(X_n/X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}) & & \\ \| & & \\ H_{n-1}(X_{n-1}) \longrightarrow \tilde{H}_{n-1}(X_{n-1}/X_{n-2}) \cong \mathbb{Z}\langle I_{n-1}\rangle. & & \end{array}$$

This is a chain complex, and you can show that if you take homology of this chain complex

$$\cdots \to \mathbb{Z}\langle I_n\rangle \xrightarrow{\partial} \mathbb{Z}\langle I_{n-1}\rangle \xrightarrow{\partial} \mathbb{Z}\langle I_{n-2}\rangle \to \cdots .$$

You should have learned in you first algebraic topology class that

$$H^{\mathrm{CW}}_*(X) \cong H_*(X).$$

5.2 Whitehead's theorem

Let's look at a second application of the skeletal filtration.

Theorem 5.13 (Whitehead)**.** *Let X be a CW-complex, and $f : Y \to Z$ be a map. Then*

(i) if f is n-connected, $[X, Y] \to [X, Z]$ is a bijection if $\dim X < n$, and is a surjection if $\dim X = n$.

(ii) if f is a weak homotopy equivalence, then $[X, Y] \to [X, Z]$ is a bijection.

Deducing (ii) from (i) is quite subtle. But let me state some corollaries.

Corollary 5.14. *If $f : X \to Y$ is a weak homotopy equivalence between CW-complexes, it is a homotopy equivalence.*

Proof. Consider $f_* : [Y, X] \to [Y, Y]$, which is a bijection. This implies that there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$. Then $f \circ g \circ f \simeq f$, but since $f_* : [X, X] \to [X, Y]$ is bijective, we get $g \circ f \simeq \mathrm{id}_X$. $\square$

The main tool is the following lemma.

Lemma 5.15. *If (X, A) is a relative CW-complex of dimension $\leq n$ and $f : Y \to Z$ is n-connected, then given $g : A \to Y$ and $h : X \to Z$ such that $h|_A = f \circ g$, we can find an $\tilde{h} : X \to Y$ such that $\tilde{h}|_A = g$ and $f \circ \tilde{h} \simeq h$ relative to A.*

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \big\downarrow & \overset{\tilde{h}}{\nearrow} & \big\downarrow \\ X & \longrightarrow & Z \end{array}$$

We need to find a $\tilde{h}$ such that the top triangle commutes and the bottom commutes relative to A.

Proof. We first reduce to the case $X = A \cup D^i$ for $i \leq n$. Consider the partially ordered set consisting of $(X', \tilde{h}, K)$ with $A \subseteq X' \subseteq X$ is a subcomplex and $\tilde{h} : X' \to Y$ is such that $\tilde{h}|_A = g$ and $f \circ \tilde{h} \simeq h|_{X'}$ relative to A through the homotopy K. We say that $(X'_0, \tilde{h}_0, K_0) \leq (X'_1, \tilde{h}_1, K_1)$ if $X'_0 \subseteq X'_1$ and $\tilde{h}_1|_{X'_0} = \tilde{h}_0$ and $K_1|_{K'_0 \times I} = K_0$. Then every chain has an upper bound, so there is a maximal element by Zorn's lemma.

Let $(X', \tilde{h}, K)$ be a maximal element. It suffices to show that if a cell of X is not in X' we can extend $\tilde{h}, K$ over it. Now we can assume that (X, A) is just $(A \cup D^i, A)$ where $i \leq n$. We need to find this lift

$$\begin{array}{ccccc} S^{i-1} & \longrightarrow & A & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow & \nearrow & \big\downarrow \\ D^i & \longrightarrow & A \cup D^i & \longrightarrow & Z, \end{array}$$

but by the definition of a pushout, it's enough to prove this for $(X, A) = (D^i, S^{i-1})$.

This is now one of the equivalent definitions of n-connectedness of f, but let's do this again. Consider any $[g] \in \pi_{i-1}(Y, y_0)$, which should map to $0 \in \pi_{n-1}(Z, z_0)$. Because f_* is injective since $i - 1 < n$, we can find a null-homotopy of g, which is a candidate for $\tilde{h}$. But to find a homotopy on the bottom triangle, the class of

$$H : h \cup_{S^{i-1}} f \circ \tilde{h} : S^i \to Z$$

should be zero in $\pi_i(Z, z_0)$. It might not be, but because $\pi_i(Y, y_0) \to \pi_i(Z, z_0)$ is surjective, there is some $[\tilde{H}]$ that maps to $[H]$. We can modify $\tilde{h}$ by "wedging on" $-\tilde{H}$ to kill off this obstruction. $\square$

Proof of Whitehead's theorem. (ii) We want to show that $f_* : [X, Y] \to [X, Z]$ is surjective or injective. Let's first do surjectivity. In this case, we can just find a lift

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \big\downarrow & \overset{\tilde{h}}{\nearrow} & \big\downarrow f \\ X & \xrightarrow{h} & Z. \end{array}$$

For injectivity, assume that $\tilde{h}_0, \tilde{h}_1 : X \to Y$ become homotopic after composing with f. Then

$$\begin{array}{ccc} X \times \{0, 1\} & \xrightarrow{\tilde{h}_0 \cup \tilde{h}_1} & Y \\ \big\downarrow & \nearrow & \big\downarrow \\ X \times I & \xrightarrow{H} & Z \end{array}$$

gives the homotopy. □

The next thing we are going to prove is the following.

Theorem 5.16. *If* (X, A) *and* (Y, B) *are relative CW-complexes, then*

$$f : (X, A) \to (Y, B)$$

is a cellular map relative to A.

6 February 8, 2018

Last time, we talked about the definition of CW-complexes and proved Whitehead's theorem. Today we will prove two approximation results:

- every map between two CW-complexes is homotopy equivalent to a cellular map,
- every space is weakly homotopic to a CW-complex.

The basic tool is something that is called compression theorem, which about connectivity of $X_n \hookrightarrow X$. The geometric input here is approximating continuous by PL functions.

6.1 Cellular approximation

Definition 6.1. An **ordered simplicial complex** consists of a set of vertices V and for each $p \geq 1$ a collection V_p of ordered $p+1$-tuples of elements of V (called the p-simplices) that is closed under taking subsets.

Given an ordered simplicial complex, you can take a 0-cell for each V, a p-cell for each $\sigma \in V_p$, and glue them. You can barycentrically subdivide such triangulated spaces. Formally, you are now taking all simplices as points, and k-cells are going to be length k chain of simplices.

Proposition 6.2 (Simplicial approximation)**.** *Let $(A \cup D^n, A)$ be a relative CW-complex with a single n-cell. Suppose we have a map of pairs $f : (|K|, |L|) \to (A \cup D^n, A)$, where K is finite and L is a subcomplex. Then there exists a $r \geq 0$ and a map*

$$f' : (|Sd^r K|, |Sd^r L|) \to (A \cup D^n, A)$$

such that

- $f|_{f^{-1}(A)} = f'|_{f^{-1}(A)}$,
- $f \simeq f'$,
- *if σ is a simplex of $Sd^r K$ which meets $D^n_{\frac{1}{2}} \subseteq D^{\circ n} \subseteq A \cup D^n$ then $\sigma \subseteq D^\circ$ and $f'|_\sigma$ is affine linear.*

Proof. You can first subdivide so that each σ meeting $D_{1/2}$ is in D^n. Then linearly interpolate. ☐

Theorem 6.3. *If (X, A) is a relative CW-complex, then $(X, A)_n \hookrightarrow X$ is n-connected.*

Proof. For $i \leq n$ we need to find a dotted lift making the top triangle commute and the bottom triangle commute up to homotopy relative to S^{i-1}.

$$\begin{array}{ccc} S^{i-1} & \longrightarrow & (X, A)_n \\ \downarrow & \nearrow & \downarrow \\ D^i & \xrightarrow{\ g\ } & X \end{array}$$

First we observe that without loos of generality X has finitely many cells, and not it suffices to inductively push $g$of the j-cells for $j > n$. Given a j-cell, we apply simplicial approximation to get a homotopic g' (rel S^{i-1}) such that D^i has a triangulation by σ such that if σ meet $D^j_{1/2}$ then it maps into $(D^j)^\circ$ and $g'|_\sigma$ is affine linear. Now the image of g inside $D^j_{1/2}$ is a finite union of planes of dimension $< n$, and then there exists an x_0 not in the image. The we can push radially outwards off D^j. □

Corollary 6.4. *If (X, A) and (Y, B) are relative CW-complexes, then $f : (X, A) \to (Y, B)$ is homotopic to a cellular map.*

Proof. We inductively make $f|_{(X,A)_n}$ cellular relative to $(X, A)_{n-1}$. We start with $f|_A = A = (X, A)_{-1} \to B = (Y, B)_{-1}$. Apply Whitehead's lemma to $(X, A)_{n-1} \hookrightarrow (X, A)_n$ and the n-connected $(Y, B)_n \hookrightarrow Y$. Then we get a homotopy to a map such that $(X, A)_n$ is mapped to $(Y, B)_n$. We can do this for all n, and then glue them together. □

6.2 CW-approximation

Theorem 6.5. *(i) Given a space X, there exists a CW-approximation, a CW-complex $\mathcal{C}$ and a weak homotopy equivalence $\mathcal{X} \to X$.*

(ii) If X is n-connected, we can find $\mathcal{X}$ with a single 0 and no i-cells for $0 < i \leq n$.

(iii) Given CW-approximations and a map $f : X \to Y$, there exists a $\phi : \mathcal{X} \to \mathcal{Y}$, unique up to homotopy, making the diagram commute.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\xi} & X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{\nu} & Y \end{array}$$

Proof. (i) Without loss of generality, assume that X is path-connected. We will inductively construct the i-skeleton $\mathcal{X}_i$ with a map $\xi_i : \mathcal{X}_i \to X$ that is i-connected.

Take $\mathcal{X}_0 = *$ and map it arbitrarily into X. Let's assume we found the i-skeleton $\xi_i : \mathcal{X}_i \to X$. Consider $\pi_{i+1}(X, \mathcal{X}_i)$, which is the first relative homotopy group that could be non-vanishing. We need to kill this by attaching $(i+1)$-cells. Luckily, each nontrivial element $\pi_{i+1}(X, \mathcal{X}_i)$ is represented by a

$$\begin{array}{ccc} S^i & \xrightarrow{\gamma} & \mathcal{X}_i \\ \downarrow & & \downarrow{\scriptstyle g_i} \\ D^{i+1} & \xrightarrow{G} & X. \end{array}$$

Then we can take the pushout

$$\begin{array}{ccc} \coprod_{[\gamma,G]} S^i & \xrightarrow{\amalg\gamma} & \mathcal{X}_i \\ & & \downarrow \\ \coprod_{[\gamma,G]} D^{i+1} & \longrightarrow & \mathcal{X}_{i+1} \end{array}$$

and extend $\xi_{i+1} : \mathcal{X}_{i+1} \to X$. We need to show that this is $(i+1)$-connected, but it is i-connected because $\mathcal{X}_i \to \mathcal{X}_{i+1}$ and $\mathcal{X}_i \to X$ is i-connected. To show that $\pi_{i+1}(X, \mathcal{X}_{i+1})$, we note that in any

$$\begin{array}{ccc} S^i & \longrightarrow & \mathcal{X}_{i+1} \\ \downarrow & & \downarrow \\ D^{i+1} & \longrightarrow & X, \end{array}$$

we can factor $S^i \to \mathcal{X}_{i+1}$ through $\mathcal{X}_i$ up to homotopy, and then it is trivial by construction.

(ii) follows from construction. (iii) follows from Whitehead's theorem applied to $[\mathcal{X}, \mathcal{Y}] \cong [\mathcal{X}, Y]$. $\square$

6.3 Homotopy excision

Suppose $B \subseteq X$ is open and $A \subseteq X$ is such that $\overline{A} \subseteq B$. For homology,

$$H_*(X \setminus A, B \setminus A) \xrightarrow{\cong} H_*(X, B).$$

But it is not always true that $\pi_*(X \setminus A, B \setminus A) \to \pi_*(X, B)$ is an isomorphism.

Example 6.6. Consider $(X, B) = (S^2, D^2)$ and let $A = * \in (D^2)^\circ$. But $\pi_3(S^2, D^2) \cong \mathbb{Z}$ and $\pi_3(D^2, S^1) = 0$.

Excision is equivalent to that if $A, B \subseteq X$ are open and $A \cup B = X$ (this is called an **excisive triad**) then $H_*(A, A \cap B) \cong H_*(X, B)$.

Definition 6.7. A map $f : (X, A) \to (X, B)$ of pairs of (path-connected) spaces is n-connected if $\pi_i(X, A, x_0) \to \pi_i(X, Y, f(x_0))$ is bijective for $i < n$ and surjective for $i = n$, for all x_0.

Theorem 6.8 (Homotopy excision)**.** *If $A, B \subseteq X$ is an excisive triad of path-connected spaces, with $A \cap B$ path-connected, $(A, A \cap B)$ is m-connected ($m \geq 1$), and (X, B) is n-connected ($n \geq 0$), then*

$$(A, A \cap B) \to (X, B)$$

is $(m+n)$-connected.

Proof. You first rephrase connectivity in terms of vanishing of triad homotopy groups. Then you reduce to the case when A and B are attaching one cells from $A \cap B$. You can then do some geometric argument about things being in general position and pushing things out. □

For X a space, consider the unreduced suspention $CX \cup_X CX$. (If X is based and well-pointed, this is homotopy equivalent to $\Sigma X = S^1 \wedge X$.) If we consider two pairs $(SX, C^{\epsilon}_{-}X)$ and $(C^{\epsilon}_{+}X, X)$, we see that (SX, CX_+) is n-connected and (CX_-, X) is n-connected. This shows that

$$\pi_{i-1}(X) \cong \pi_i(CX, X) \to \pi_i(SX, CX) \cong \pi_i(\Sigma X)$$

is an isomorphism for $i \leq 2n - 1$ and surjective for $i = 2n$.

Theorem 6.9 (Freudenthal). *If X is an $(n-1)$-connected based well-point map, then*

$$\pi_{i-1}(X) \to \pi_i(\Sigma X)$$

is an isomorphism $i \leq 2n - 1$ and surjective for $i = 2n$.

So $\pi_{*+h}(\Sigma^h X)$ stabilizes, and we define

$$\pi^s_*(X) = \varinjlim_h \pi_{*+h}(\Sigma^h X).$$

Example 6.10. The map $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is a surjection for $n = 1$ and an isomorphism for $n \geq 2$. You can prove that $\pi_2(S^2) \cong \mathbb{Z}$ by using Hurewicz maps, and so $\pi_n(S^n) \cong \mathbb{Z}$.

7 February 13, 2018

Simplicial sets are a way to make algebraic topology combinatorial.

7.1 The singular simplicial set

Definition 7.1. Δ is the category with objects non-empty finite ordered sets and morphisms order-preserving maps.

So a skeleton consists of $[p] = \{0 < \cdots, p\}$ and generating morphisms

$$\delta_i : [p-1] \to [p] \text{ given by skipping } i$$

for $0 \leq i < p$ and

$$\sigma_j : [p] \to [p-1] \text{ given by collapsing } j, j+1$$

for $0 \leq j \leq p-1$.

Definition 7.2. A **simplicial set** is a functor

$$X_\bullet : \Delta^{\text{op}} \to \mathsf{Set}.$$

A simplicial map $f_\bullet : X_\bullet \to Y_\bullet$ is a natural transformation, and the category of simplicial sets is written as sSet.

More concretely, a simplicial set consist of sets X_p for $p \geq 0$, called the "p-simplices of $X_\bullet$", face maps $d_i : X_p \to X_{p-1}$, and degeneracy maps $s_j : X_{p-1} \to X_p$.

Example 7.3. Let $\mathcal{C}$ be a category. Its **nerve** is the simplicial set with p-simplices

$$N_p\mathcal{C} = \{X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{p-1}} X_p\}$$

with face maps either dropping X_0, f_0, composing f_{i-1}, f_i, or dropping X_p, f_{p-1}. Degeneracy maps will be inserting an identity morphism at the jth spot. The axioms for the relation between face and degeneracy maps encode associativity and identity.

Define the following functor:

$$\delta^\bullet : \Delta \to \mathsf{Top}; \quad [p] \mapsto \{(t_0, \ldots, t_p) \in [0,1]^{p+1} : \textstyle\sum_i t_i = 1\}.$$

Here, the maps $\delta_i : \Delta^{p-1} \to \Delta^p$ are given by face inclusions and $\sigma_j : \Delta^p \to \Delta^{p-1}$ given by projections.

Definition 7.4. The **geometric realization** $|X_\bullet| \in \mathsf{Top}$ is given by

$$\left(\coprod_{p \geq 0} \Delta^p \times X_p\right) / ((\delta_i \vec{t}, x) \sim (\vec{t}, d_i x),\ (\sigma_j \vec{t}, x) \sim (\vec{t}, s_j x)).$$

If we didn't quotient out by the degeneracy maps, it is a thick geometric realization and obviously a CW-complex. But the geometric realization is also a CW-complex. This is because every $x \in X_p$ is an iterated degeneracy of a unique non-degenerate simplex. Then $|X_\bullet|$ is a CW-complex with a single p-cell for each non-degenerate p-simplex.

Example 7.5. Consider $\Delta^n_\bullet = \mathrm{Hom}_\Delta([p],[n])$. Non-degenerate simplices here are $[p] \hookrightarrow [n]$, and so the geometric realization is really just the n-simplex Δ^n.

Lemma 7.6. *Geometric realization preserves finite products.*

Proof. We write $X_\bullet$ as a colimit of $\Delta^n_\bullet$:

$$\operatorname*{colim}_{\Delta_b ullet^n \to X_\bullet} \Delta^n_\bullet \xrightarrow{\cong} X_\bullet.$$

We will soon see that $|-|$ preserves colimits, and so

$$\begin{aligned} |X_\bullet \times Y_\bullet| &= \operatorname*{colim}_{\substack{\Delta^n_\bullet \to X_\bullet, \\ \Delta^m_\bullet \to X_\bullet}} |\Delta^n_\bullet \times \Delta^m_\bullet| \\ &= \left(\operatorname*{colim}_{\Delta^n_\bullet \to X_\bullet} |\Delta^n_\bullet| \right) \times \left(\operatorname*{colim}_{\Delta^m_\bullet \to Y_\bullet} |\Delta^m_\bullet| \right) = |X_\bullet| \times |Y_\bullet| \end{aligned}$$

because we are working in a nice category of spaces. □

Definition 7.7. Let X be a topological space. The **singular simplicial set** $\mathcal{S}(X)$ has p-simplices given by the set of continuous maps $\Delta^p \to X$ and

$$\begin{aligned} d_i &: \mathcal{S}(X)_p \to \mathcal{S}(X)_{p-1} \text{ induced by } \Delta^{p-1} \xrightarrow{s_i} \Delta^p \to X, \\ s_j &: \mathcal{S}(X)_{p-1} \to \mathcal{S}(X)_p \text{ induced by } \Delta^p \xrightarrow{\sigma_j} \Delta^{p-1} \to X. \end{aligned}$$

Lemma 7.8. *$\mathcal{S}$ is the right adjoint to $|-|$.*

So $|-|$ commutes with colimits.

Proof. We use the map $\epsilon_X : |\mathcal{S}(X)| \to X$ defined by evaluation, and $\eta_{X_\bullet} : X_\bullet \to \mathcal{S}(|X_\bullet|)$ defined by inclusion. Then any $f : |X_\bullet| \to Y$ give you a map $\eta(f) : X_\bullet \to \mathcal{S}(Y)$ given by $\mathcal{S}(f) \circ \eta_X$. We can recover f uniquely by $|X_\bullet| \xrightarrow{|\eta(f)|} |\mathcal{S}(Y)| \xrightarrow{\epsilon_Y} X$. So

$$\mathrm{Hom}_{\mathsf{Top}}(|X_\bullet|, Y) \to \mathrm{Hom}_{\mathsf{sSet}}(X_\bullet, \mathcal{S}(Y))$$

is bijective. □

Why would taking the singular simplicial set be a good idea?

Proposition 7.9. *The counit map $\epsilon_X : |\mathcal{S}(X)| \to X$ is always a weak homotopy equivalence.*

Proof. Our goal is to show that

$$\begin{array}{ccc} |\partial\Delta^{i+1}_\bullet| & \longrightarrow & |\mathcal{S}(X)| \\ \big\downarrow & & \big\downarrow \\ |\Delta^{i+1}_\bullet| & \longrightarrow & X \end{array}$$

has a lift. By simplicial approximation, there exists an $r \geq 0$ such that the diagram is homotopic through commutative diagrams to

$$\begin{array}{ccc} |Sd^r(\partial\Delta^{i+1}_\bullet)| & \xrightarrow{|f_\bullet|} & |\mathcal{S}(X)| \\ \big\downarrow & & \big\downarrow \\ |Sd^r(\Delta^{i+1})| & \longrightarrow & X. \end{array}$$

Now the bottom data is the same as a map $Sd^r r(\Delta^{i+1} \to \mathcal{S}(X)$, and it is going to be a compatible lift. (The map $|f_\bullet|$ being a simplicial map is important because this is what makes the upper triangle commute.) □

Corollary 7.10. *$|\mathcal{S}(f)|$ is a homotopy equivalence if $f : X \to Y$ is a weak homotopy equivalence.*

Proof. We have $|\mathcal{S}(X)| \to X \to Y$ a weak homotopy equivalence and $|\mathcal{S}(Y)| \to Y$ a weak homotopy equivalence, and so $|\mathcal{S}(X)| \to \mathcal{S}(Y)$ is a weak homotopy equivalence. Then by Whitehead's theorem, it is a homotopy equivalence. □

Proposition 7.11. *If $A, B \subseteq X$ is an excisive triad, then*

$$|\mathcal{S}(A) \cup_{\mathcal{S}(A\cap B)} \mathcal{S}(B)| \to |\mathcal{S}(X)|$$

is a homotopy equivalence.

7.2 Singular homology

Definition 7.12. If $X_\bullet$ is a simplicial set, there is a **chain complex**

$$C_*(X_\bullet) \text{ with } C_p(X_\bullet) = \mathbb{Z}\langle X_p\rangle$$

and differentials

$$d = \sum_i (-1)6id_i : \mathbb{Z}\langle X_p\rangle \to \mathbb{Z}\langle X_{p-1}\rangle.$$

There is also

$$N_p(X_\bullet) = C_p(X_\bullet)/D_p(X_\bullet)$$

where $D_p(X_\bullet)$ is the subgroup on degenerate simplices. This is really the composite

$$\begin{array}{ccccc} \mathsf{sSet} & \xrightarrow{\mathbb{Z}\langle - \rangle} & \mathsf{sAb} & \xrightarrow{C} & \mathsf{Ch}_*^{\geq 0} \\ & & & \searrow^{N} & \\ & & & & \mathsf{Ch}_*^{\geq 0}. \end{array}$$

Theorem 7.13 (Dold–Kan)**.** *The natural transformation $C_*(X_\bullet) \to N_*(X_\bullet)$ is a quasi-isomorphism (and N_* has a right-adjoint Γ which is part of the Quillen equivalence).*

Definition 7.14. The **singular homology** is

$$H_*(X) = H_*(C_*(\mathcal{S}(X))).$$

For example,

$$H_*(\mathrm{pt}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This is also a natural in X, i.e., a functor

$$H_*(-) : \mathsf{Top} \to \mathsf{GrAb}.$$

Here are some properties:

(i) There is a long exact sequence of pairs. For

$$H_*(X, A) = H_*(C_*(\mathcal{S}(X))/C_*(\mathcal{S}(A)))$$

we have

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \to \cdots .$$

(ii) $H_*(-)$ is invariant under homotopy. If $H : I \times X \to Y$, we get a simplicial map

$$\Delta^1_\bullet \times \mathcal{S}(X) \to \mathcal{S}(I \times X) \to \mathcal{S}(Y).$$

Now you can prove that simplicial homotopy induces chain homotopy on C_*.

(iii) It satisfies exicision. Using $|\mathcal{S}(A) \cup_{\mathcal{S}(A\cap B)} \mathcal{S}(\mathcal{B})| \to |\mathcal{S}(X)|$, you can show that if $A, B \subseteq X$ is excisive then

$$H_*(A, A \cap B) \to H_*(X, B)$$

is an isomorphism.

Corollary 7.15 (Mayer–Vietoris)**.** *If $A, B \subseteq X$ is an excisive triad, then*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \cdots .$$

Proof. You can chase diagrams in

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_n(A \cap B) & \longrightarrow & H_n(A) & \longrightarrow & H_n(A, A \cap B) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, B) & \longrightarrow & \cdots . \end{array}$$

□

Then you get suspension isomorphisms on the reduced theory. There are some additional properties singular homology has.

(i) H_* take weak homotopy equivalences to isomorphisms. This is because first $H_*(X) \cong H_*^{\mathrm{CW}}(|\mathcal{S}(X)|) \cong H_*(\mathcal{S}(X))$, and then if $X \to Y$ is a weak homotopy equivalence, then $H_*(X) \cong H_*(|\mathcal{S}(X)|) \cong H_*(|\mathcal{S}(Y)| \cong H_*(Y)$.

(ii) $\tilde{H}_n$ satisfies the wedge axiom:

$$\bigoplus_{i\in I} H_*(X_i) \xrightarrow{\cong} H_*\left(\bigvee_{i\in I} X_i\right).$$

You want these axioms because you want to control the theory only on $*$.

7.3 Hurewicz theorem

If you have an element $\pi_n(X, x_0)$, this is represented by $\Delta^n/\partial\Delta^n \cong S^n \xrightarrow{g} X$ and induces an element of $\Delta^n \to \Delta^n/\partial\Delta^n \xrightarrow{g} X$.

Theorem 7.16 (Hurewicz)**.** *If X is $(n-1)$-connected and $n \geq 2$, then*

$$\pi_n(X, x_0) \xrightarrow{\cong} H_n(X), \quad \pi_{n+1}(X, x_0) \twoheadrightarrow H_{n+1}(X).$$

You can read the proof in the notes.

Definition 7.17. A **unreduced generalized homology theory** is a sequence of functors $E_n : \mathsf{Ho}(\mathsf{Top})^2 \to \mathsf{Ab}$ with natural transformations $\delta_n : E_n(X, A) \to E_{n-1}(A)$ such that

(i) there is a long exact sequence of pairs

$$\cdots \to E_n(A) \to E_n(X) \to E_n(X, A) \xrightarrow{\delta_n} E_{n-1}(A) \to \cdots,$$

(ii) If $A, B \subseteq X$ is an excisive triad, then $E_n(A, A \cap B) \to E_n(X, B)$ is an isomorphism.

Definition 7.18. A **reduced generalized homology theory** is a sequence of functors $\tilde{E}_n : \mathrm{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}$ and natural isomorphisms $\tilde{E}_n(X) \to \tilde{E}_{n+1}(\Sigma X)$ such that

(i) exact sequence of based pairs

$$\tilde{E}_n(A) \to \tilde{E}_n(X) \to \tilde{E}_n(X \cup CA).$$

8 February 15, 2018

From generalized cohomology theories, we will use Brown representability to define spectra. Then $\mathrm{Ho}(\mathsf{Sp})$ will be the stable homotopy category.

8.1 Generalized (co)homology theory

Definition 8.1. An **unreduced generalized cohomology theory** is a sequence $E^n : \mathrm{Ho}(\mathsf{Top}^2)^{\mathrm{op}} \to \mathsf{Ab}$ with natural transformations $\partial^n : E^{n-1}(A) \to E^n(X, A)$ such that

(i) for a pair (X, A), there is a long exact sequence

$$\cdots \leftarrow E^{n+1}(X, A) \xleftarrow{\partial^{n+1}} E^n(A) \leftarrow E^n(X) \leftarrow E^n(X, A) \leftarrow \dots$$

(ii) for an excisive triad $A, B \subseteq X$, $E^n(A, A \cap B(\leftarrow E^n(X, B)$ is an isomorphism.

Definition 8.2. A **reduced generalized cohomology theory** is a sequence of functors $\tilde{E}^n : \mathrm{Ho}(\mathsf{Top}) \to \mathsf{Ab}$ with natural isomorphisms $\sigma^n : \tilde{E}^n \circ \Sigma \to E^{n+1}$ such that for a based pair (X, A) we get an exact sequence

$$\tilde{E}^n(A) \leftarrow \tilde{E}^n(X) \leftarrow \tilde{E}^n(X \cup CA).$$

We impose additional axioms:

- WHE axiom—$\tilde{E}^n$ send weak homotopy equivalence to isomorphisms
- wedge axiom—$\tilde{E}(\bigvee_{i\in I} X_i) \cong \prod_{i\in I} \tilde{E}^n(X_i)$.

Under the weak homotopy equivalence axiom, an unreduced generalized cohomology theory is equivalent to a reduced generalized cohomology theory.

A generalized cohomology theory $\tilde{E}^*$ is determined by their values on the full subcategory $\mathrm{Ho}(\mathsf{CW}_*)$ of $\mathrm{Ho}(\mathsf{Top}_*)$, spaces homotopy equivalent to a CW-complex.

Example 8.3. Stable homotopy

$$\pi_n^s(X) = \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X)$$

form a generalized homology theory. To see this, first note that it satisfies the WHE axiom and so are functors $\pi_n^s(-) : \mathrm{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}$. The suspension axiom is tautologically true. We only need to check the exactness property now. For sufficiently large k, we only need to check that

$$\pi_{n+k}(\Sigma^k A) \to \pi_{n+k}(\Sigma^k X) \to \pi_{n+k}(X \cup CA)$$

is exact by Freudenthal suspension theorem. Then this is homotopy excision. The Hurewicz map

$$\pi_n^s \to \tilde{H}_n$$

induces an isomorphism $\pi_0^s(S^0) \to \tilde{H}_0(S^0)$ but non in higher degrees.

Other examples include:

- (un)oriented bordism theory
- K-theory
- Brown–Comenetz dual: note that for an injective abelian group $I = \mathbb{Q}/\mathbb{Z}$, $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact. This shows that

  $$I_{\mathbb{Q}/\mathbb{Z}}(\pi^s)^n(X) = \mathrm{Hom}(\pi_n^s(X), \mathbb{Q}/\mathbb{Z})$$

  is an unreduced generalized cohomology theory.

8.2 Brown representability and spectra

Let's start by restricting to $\mathrm{Ho}(\mathsf{CW}_*)$. We will later extend the theory to $\mathrm{Ho}(\mathsf{Top}_*)$ by CW approximation. If Y is a based CW-complex, then

$$A \hookrightarrow X \to X \cup CA$$

gives an exact sequence

$$[A, Y]_+ \leftarrow [X, Y] \leftarrow [X \cup CA, Y]_+.$$

So this looks something like $\tilde{E}^n(-)$. It also satisfies the wedge axiom as $\bigvee_{i \in I} X_i$ is a coproduct in $\mathsf{Ho}(\mathsf{CW}_*)$.

Theorem 8.4 (Brown)**.** *Given $F : \mathrm{Ho}(\mathsf{Top}_*^c)^{\mathrm{op}} \to \mathsf{Set}_*$ (Top^c is the connected spaces) such that*

(1) for each based CW-pair (X, A),

$$F(A) \leftarrow F(X) \leftarrow F(X \cup CA)$$

is exact,

(2) $F(\bigvee_{i \in I} X_i) \to \prod_{i \in I} F(X_i)$ is an isomorphism,

then there exists a Y such that $F \cong [-, Y]_+$. This isomorphism will be of the following form: for some $u \in F(Y)$,

$$T_u : [-, Y]_+ \to F(-); \quad f \in [X, Y]_+ \mapsto f^*(u).$$

This pair (Y, u) is going to be unique up to isomorphism and pulling back u. So for a generalized cohomology theory $\tilde{E}^*$, we get spaces Y_n such that

$$\tilde{E}^n(-) \cong [-, Y_n]_+.$$

Now the suspension isomorphisms give

$$[-, \Omega Y_n]_+ \cong [\Sigma -, Y_n] \xrightarrow{\cong} [-, Y_{n-1}].$$

So Yoneda gives $Y_{n-1} \to \Omega Y_n$, and so maps $\Sigma Y_{n-1} \to Y_n$ whose adjoint is a homotopy equivalence. (Here, we're secretly using the fact that Ω of a CW-complex is homotopy equivalent to a CW-complex.) This data of $\{Y_n\}_{n \in \mathbb{Z}}$ with maps $\Sigma Y_{n-1} \to Y_n$ is roughly what a spectrum is.

Definition 8.5. A **prespectrum** Y is a collection of based spaces Y_n with maps $\Sigma Y_{n-1} \to Y_n$.

- A **CW prespectrum** is a prespectrum Y such that all Y_n are CW and $\Sigma Y_{n-1} \to Y_n$ is an inclusion of subcomplex.
- A **spectrum** Y is a prespectrum such that $Y_{n-1} \to \Omega Y_n$ are homotopy equivalences.

One can always replace a prespectrum Y be a CW prespectrum by levelwise homotopy equivalent spaces with compatible maps $\Sigma Y_{n-1} \to Y_n$. Also, one can always replace any CW pre-spectrum Y be a spectrum by replacing Y_n with

$$\operatorname*{colim}_{k\to\infty} \Omega^k Y_{n+k}.$$

The upshot is that given a spectrum Y, we get a reduced generalized cohomology theory satisfying the wedge axiom $[-, Y_n]_+$, and given a cohomology theory $\tilde{E}$, we get Y_E by Brown representability.

There are spaces $K(A,n)$ with

$$\pi_i(K(A,n)) = \begin{cases} A & i = n \\ 0 & i \neq n \end{cases}$$

and they are unique up to homotopy. So $\Omega K(A,n) \simeq K(A, n-1)$ and then we have a spectrum with nth space $K(A,n)$ for $n \geq 0$ and $*$ otherwise. Let us call this the **Eilenberg–Mac Lane spectrum** HA.

We have

$$HA^n(S^0) = [S^0, K(A,n)]_+ = \pi_0 K(A,n) = \begin{cases} A & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Because HA^* and $\tilde{H}(-, A)$ have the same value on S^0, and have the wedge axiom, we know that they are the same on all spaces, if we can find a natural transformation between them inducing isomorphism on S^0.

By the Hurewicz theorem, we have

$$H_n(K(A,n); \mathbb{Z}) = A,$$

and so we have a id_A in

$$H^n(K(A,n); A) = \operatorname{Hom}(H_n(K(A,n)), A).$$

So we have a natural transformation

$$[-, K(A,n)]_+ \to H^*(-; A).$$

This assembles to a natural transformation $HA^* \to H^*(-, A)$.

8.3 Stable homotopy category

The first thing we would like is a functor $\Sigma^\infty : \mathrm{Ho}(\mathsf{Top}_*) \to \mathrm{Ho}(\mathsf{Sp})$ with the property that
$$[\Sigma^\infty X, E] \cong \tilde{E}^0(X),$$
whatever the left hand side is. This is going to be the spectrum associated to the prespectrum
$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ * & \text{otherwise} \end{cases}$$
There should also be the 0th space $\Omega^\infty : \mathrm{Ho}(\mathsf{Sp}) \to \mathrm{Ho}(\mathsf{Top}_*)$ which is something like
$$\Omega^\infty \Sigma^\infty X \simeq \operatorname*{colim}_{k\to\infty} \Omega^k \Sigma^k X$$
that gives an adjunction $\Sigma^\infty \dashv \Omega^\infty$.

Another thing we want is invertibility of suspension. We have that $\Sigma^\infty \Sigma X$ is homotopy equivalent to the shifted $\Sigma^\infty X$ by 1. We want a functor $\Sigma : \mathrm{Ho}(\mathsf{Sp}) \to \mathrm{Ho}(\mathsf{Sp})$ that is invertible and
$$\begin{array}{ccc} \mathrm{Ho}(\mathsf{Sp}) & \xrightarrow{\Sigma} & \mathrm{Ho}(\mathsf{Sp}) \\ {\scriptstyle \Sigma^\infty}\uparrow & & {\scriptstyle \Sigma^\infty}\uparrow \\ \mathrm{Ho}(\mathsf{Top}_*) & \xrightarrow{\Sigma} & \mathrm{Ho}(\mathsf{Top}_*). \end{array}$$
The inverse is going to be right adjoint, and will satisfy
$$\begin{array}{ccc} \mathrm{Ho}(\mathsf{Sp}) & \xrightarrow{\Omega} & \mathrm{Ho}(\mathsf{Sp}) \\ \downarrow{\scriptstyle \Omega^\infty} & & \downarrow{\scriptstyle \Omega^\infty} \\ \mathrm{Ho}(\mathsf{Top}_*) & \xrightarrow{\Omega} & \mathrm{Ho}(\mathsf{Top}_*). \end{array}$$
Then we can write
$$\tilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E].$$

We also want this to be an additive category, that is, $\mathrm{Ho}(\mathsf{Sp})$ is enriched in abelian groups. If we take $E \vee F$ and $E \times F$ levelwise categorical products and coproducts, then $E \times F \to E \vee F$ is an equivalence. So $\mathrm{Ho}(\mathsf{Sp})$ is additive.

Finally, we want the triangulated category that sort of allows us to do homological algebra. If you have this, you will be able to form cones and stuff.

9 February 20, 2018

Last time we said we wanted a stable homotopy category.

9.1 Desiderata for stable homotopy category

- There are adjoint functors $\mathrm{Ho}(\mathsf{Top}_*) \xrightarrow{\Sigma^\infty} \mathrm{Ho}(\mathsf{Sp})$ and $\mathrm{Ho}(\mathsf{Sp}) \xrightarrow{\Omega^\infty} \mathrm{Ho}(\mathsf{Top}_*)$.
- The functor $\Sigma : \mathrm{Ho}(\mathsf{Sp}) \to \mathrm{Ho}(\mathsf{Sp})$ should have an inverse Ω, with $\Sigma^\infty \Sigma = \Sigma\Sigma^\infty$ and $\Omega^\infty\Omega = \Omega\Omega^\infty$.
- $\mathrm{Ho}(\mathsf{Sp})$ is additive, i.e., enriched in abelian groups, and finite products and coproducts exists and coincide.
- $\mathrm{Ho}(\mathsf{Sp})$ is triangulated, i.e., there is a collection of distinguished triangles with lots of axioms. A consequence is that there are cofibers and fibers of maps. So we we have an extra long coexact sequence

$$\begin{aligned}\cdots \to \Omega F \to \Omega C(f) \to E \xrightarrow{f} F \to C(f) \\ \to \Sigma E \to \Sigma F \to \Sigma C(f) \to \Sigma^2 E \to \cdots .\end{aligned}$$

  Also, cofibers and fibers coincide up to shifting, so that the fiber of $F \to C(f)$ is $E \xrightarrow{f} F$. So this long coexact sequence is also exact, i.e., mapping into it gives a long exact sequence of abelian groups.
- There exists a smash product with pointed spaces, which is taking level-wise smash product with pointed spaces:

$$\wedge : \mathrm{Ho}(\mathsf{Sp}) \times \mathrm{Ho}(\mathsf{Top}_*) \to \mathrm{Ho}(\mathsf{Sp}); \quad (E, X) \mapsto E \wedge X.$$

  For example, $\Sigma^\infty X = \mathbb{S} \wedge X$ where $\mathbb{S} = \Sigma^\infty S^0$. We want to extend this to spectra

$$\wedge : \mathrm{Ho}(\mathsf{Sp}) \times \mathrm{Ho}(\mathsf{Sp}) \to \mathrm{Ho}(\mathsf{Sp}); \quad (E, F) \mapsto E \wedge F$$

  such that $E \wedge \Sigma^\infty X \cong E \wedge X$.
- $\mathrm{Ho}(\mathsf{Sp}, \wedge, \mathbb{S})$ is a closed symmetric monoidal structure, and $- \wedge E$ has a right adjoint $\mathrm{Fun}(E, -)$.

Recall we can define a generalized reduced cohomology theory $\tilde{E}^* : \mathrm{Ho}(\mathsf{Top}_*)^{\mathrm{op}} \to \mathsf{Ab}$ given by

$$\tilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E].$$

But then, $\tilde{E}^n$ extends to spectra by

$$\tilde{E}^n(F) = [F, \Sigma^n E].$$

Also, we should have

$$\Sigma^\infty(X \wedge Y) = \mathbb{S} \wedge (X \wedge Y) \cong (\mathbb{S} \wedge X) \wedge Y \cong \Sigma^\infty X \wedge \Sigma^\infty Y.$$

So Σ^∞ should send Σ of pointed space to smash products of spectra.

Let us rewrite $\tilde{E}^n(X)$ in terms of this data. Let us first define

$$\pi_n(E) = [\Sigma^n \mathbb{S}, E]$$

for $n \in \mathbb{Z}$. Then we have

$$\begin{aligned}\pi_{-n}(\mathrm{Fun}(\Sigma^\infty X, E)) &\cong [\Sigma^{-n}\mathbb{S}, \mathrm{Fun}(\Sigma^\infty X, E)] \cong [\Sigma^{-n}\mathbb{S} \wedge \Sigma^\infty X, E] \\ &\cong [\Sigma^{-n}\Sigma^\infty X, E] \cong [\Sigma^\infty X, \Sigma^n E].\end{aligned}$$

Definition 9.1. A **homotopy commutative ring spectrum** is a commutative monoid object in $(\mathrm{Ho}(\mathsf{Sp}), \wedge, \mathbb{S})$. That is, there should be $\mu : R \wedge R \to R$ and $1 : \mathbb{S} \to R$ satisfying some axioms.

Example 9.2. $\mathbb{S}$ is a commutative ring spectrum, and Hk is a commutative ring spectrum for a commutative ring k. This follows from brown representability with ordinarily defined cup products.

If we have elements $\tilde{R}^m(X) = [\Sigma^\infty, \Sigma^m R]$ and $\tilde{R}^n(Y) = [\Sigma^\infty, \Sigma^n R]$, we can map

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y \to \Sigma^n R \wedge \Sigma^m R \cong \Sigma^{m+n}(R \wedge R) \to \Sigma^{m+n} R.$$

This gives an external cup product $\tilde{R}^m(X) \otimes \tilde{R}^n(Y) \to \tilde{R}^{m+n}(X \wedge Y)$. To get the internal cup product, we can just

$$\tilde{R}^m(X) \otimes \tilde{R}^n(X) \to \tilde{R}^{m+n}(X \wedge X) \xrightarrow{\Delta^*} \tilde{R}^{m+n}(X).$$

Proposition 9.3. *If E is a spectrum, then*

$$\tilde{E}_n(X) = \pi_n(E \wedge X)$$

is a generalized reduced homology theory.

For instance, we have the suspension isomorphism

$$\begin{aligned}\tilde{E}_n(\Sigma X) &= \pi_n(E \wedge \Sigma X) = \pi_n(E \wedge \Sigma^\infty \Sigma X) = [\Sigma^n \mathbb{S}, \Sigma(E \wedge \Sigma^\infty X)] \\ &= [\Sigma^{n-1}\mathbb{S}, E \wedge \Sigma^\infty X] = \pi_{n-1}(E \wedge \Sigma^\infty X) = \tilde{E}_{n-1}(X).\end{aligned}$$

Also, this extends to spectra $\tilde{E}_n(F) = \pi_n(E \wedge F)$.

Let's see what happens to a homology theory associated to R a homotopy commutative ring spectrum. In this case, we are going to have a pairing

$$\tilde{R}_m(X) \otimes \tilde{R}^n(X) \to \tilde{R}_{m-n}(S^0).$$

(Here, note that $\tilde{R}_{m-n}(S^0) = \pi_{m-n}(R) = [\Sigma^{m-n}\mathbb{S}, R] = [\mathbb{S}, \Sigma^{n-m} R] = \tilde{R}^{n-m}(S^0)$.) This is given by

$$[\Sigma^m \mathbb{S}, R \wedge \Sigma^\infty R] \otimes [\Sigma^\infty, \Sigma^n R] \to [\Sigma^m \mathbb{S}, R \wedge \Sigma^n R] \to [\Sigma^{m-n}\mathbb{S}, R \wedge R] \to [\Sigma^{m-n}\mathbb{S}, R].$$

9.2 Classical construction

This was already considered outdated in the 1970s, but it gives you an idea of how to construct these things. Recall that a pre-spectrum E is a sequence E_n for $n \in \mathbb{Z}$ of based spaces with maps $\Sigma E_n \to E_{n+1}$. A CW pre-spectrum is a pre-spectrum E such that each E_n is a CW-complex and $\Sigma E_n \to E_{n+1}$ inclusion of sub-complexes. This is the basis of the classical construction. Certainly, maps

$$\begin{array}{ccc} \Sigma E_n & \longrightarrow & E_{n+1} \\ \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_n & \longrightarrow & F_{n+1} \end{array}$$

should qualify as morphisms of CW-pre-spectra, bu we want more.

We interpret CW-pre-spectra as CW-objects with stable cells. Each r-cell e in E_n gives an $(r+1)$-cell Σe in E_{n+1}, a $(r+2)$-cell $\Sigma^2 e$ in E_{n+2}, and so on. So we can think of this as a stable $(r-n)$-cell which only eventually exists. So we want to add partially defined morphisms.

Definition 9.4. A sub-CW-pre-spectrum $E' \subseteq E$ is **cofinal** if every stable cell of E is in E' eventually, i.e., for every r-cell e of E_n, there exists a $k \geq 0$ such that $\Sigma^k e$ lies in E'_{n+k}.

Definition 9.5. A **map** $f : E \to F$ of CW-pre-spectra is an equivalence class (E', f') where $E' \subseteq E$ is cofinal and $f' : E' \to F$ a "naive" morphism. Here, the equivalence relation is agreeing on a smaller (cofinal) sub-CW-pre-spectum.

Definition 9.6. A **homotopy** is a map $E \wedge I_+ \to F$. The category $\mathrm{Ho}(\mathsf{Sp})^{cl}$ has objects CW-pre-spectra and morphisms homotopy classes of maps.

The homotopy group is can be defined as

$$\pi_n(E) = \operatorname*{colim}_{k\to\infty} \pi_{n+k}(E_k).$$

Then any map $f : E \to F$ induced a map on homotopy groups. Then f is a homotopy equivalence if and only if it is a weak homotopy equivalence. Let's look at few of the desiderata.

- shift gives the suspension, and this is clearly invertible.
- $\mathrm{Ho}(\mathsf{Sp})$ is enriched in abelian groups, because

  $$[E, F] = [\Sigma^2 E, \Sigma^2 F] = [S^2 \wedge E, S^2 \wedge F]$$

  and S^2 is a cocommutative cogroup object.

Lemma 9.7. *For $f : E \to F$, we define $C(f)$ eventually levelwise. Then $[H, E] \xrightarrow{f_*} [H, F] \xrightarrow{g_*} [H, C(f)]$ is exact.*

Proof. Clearly $g \circ f \simeq 0$. Now suppose that $\varphi \in \ker(g_*)$. Then we have

$$\begin{array}{ccccccc} H & \longrightarrow & CH \simeq * & \longrightarrow & \Sigma H & \xrightarrow{-\mathrm{id}} & \Sigma H \\ \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \Sigma\varphi} \\ F & \xrightarrow{g} & C(f) & \longrightarrow & \Sigma E & \xrightarrow{-\Sigma f} & \Sigma F. \end{array}$$

So $\Sigma f \circ \psi = \Sigma\varphi$ and we can desuspend $f \circ \Omega\psi = \varphi$. So $\varphi \in \mathrm{im}(f_*)$. □

9.3 Model category of orthogonal spectra

This classical construction is really annoying. So we are going to give another definition.

Definition 9.8. A **model category** C is a complete, cocomplete category with three classes of morphism $\mathcal{W}, \mathcal{C}, \mathcal{F}$ together with functorial factorizations of morphisms into either $\mathcal{C} \cap \mathcal{W}$ followed by $\mathcal{F}$ or $\mathcal{C}$ followed by $\mathcal{F} \cap \mathcal{W}$ satisfying

- the 2-out-of-3 axiom,
- retracts,
- lifting properties.

The point is that each of these has a homotopy category $\mathrm{Ho}(\mathsf{C}) = \mathsf{C}_{\mathrm{cf}}/\sim$ where C_{cf} is the full subcategory of cofibrant fibrant objects. This has the universal property that if $F : \mathcal{C} \to \mathsf{D}$ is a functor that sends $\mathcal{W}$ to isomorphisms, then there exists a unique lift to $\mathrm{Ho}(\mathsf{C}) \to \mathsf{D}$.

Definition 9.9. An **orthogonal spectrum** is a sequence E_n of based spaces with $\mathrm{O}(n)$-action with structure map $E_n \wedge S^1 \to E_{n+1}$ such that

$$E_n \wedge S^k \to E_{n+k}$$

is $\mathrm{O}(n) \times \mathrm{O}(k)$-equivariant.

The point is to make the structure rigid, so that there isn't too many choices. A morphism $f : E \to F$ is a sequence of $\mathrm{O}(n)$-equivariant maps $f_n : E \to F$ with

$$\begin{array}{ccc} E_n \wedge S^1 & \longrightarrow & E_{n+1} \\ \downarrow{\scriptstyle f_n \wedge S^1} & & \downarrow{\scriptstyle f_{n+1}} \\ F_n \wedge S^1 & \longrightarrow & F_{n+1}. \end{array}$$

Let Sp^O be the category of orthogonal spectra. The weak equivalences are π_*-isomorphism with $\pi_n(E) = \mathrm{colim}_{k\to\infty} \pi_{n+k}(E_k)$. The cofibrant objects are $\mathrm{O}(n)$-versions of cell complexes with transfinite compositions. Fibrant objects are Ω-spectra.

Definition 9.10. The **smash product** is defined as $(E \wedge F)_n$ being the coequalizer of

$$\bigvee_{p+1+q=n} \mathrm{O}(n) \wedge_{\mathrm{O}(p)\times \mathrm{id} \otimes \mathrm{O}(q)} (E_q \wedge S \wedge F_q) \to \bigvee_{p+q=n} \mathrm{O}(n) \times_{\mathrm{O}(p)\times\mathrm{O}(q)} (E_p \wedge E_q).$$

10 February 22, 2018

The goal today is to introduce how to compute $\tilde{E}_*(X)$ of a (based) CW-complex. In doing so, we will develop spectral sequences.

10.1 Generalized cellular homology

Fix a spectrum E, so that $\tilde{E}_*(-)$ is a generalized homology theory, satisfying the wedge axiom and the WHE axiom. Also fix a based CW-complex X. Then we can build X by

$$\begin{array}{ccccccccc} X_{-1} = * & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \bigvee_{I_0} S^0 & & \bigvee_{I_1} S^1 & & \bigvee_{I_2} S^2 & & \end{array}$$

and $X = \operatorname{colim}_{q\to\infty} X_q$. Each of $X_{k-1} \to X_k \to \bigvee_{I_k} S^k$ is a cofiber sequence. So for each of these, we get a long exact sequence

$$\cdots \to \tilde{E}_q(X_{k-1}) \to \tilde{E}_q(X_k) \to \bigoplus_{I_k} \tilde{E}_{q-k}(S^0) \to \tilde{E}_{q-1}(X_{k-1}) \to \cdots .$$

In principle, we can then inductively compute $\tilde{E}_*(X_k)$ and get $\tilde{E}_*(X) = \operatorname{colim}_{k\to\infty} \tilde{E}_*(X_k)$. (This is using the wedge axiom.) Spectral sequences will package this infinite computation to a single algebraic object. This will be a much more efficient way to think about this.

10.2 The Atiyah–Hirzebruch spectral sequence

We'll begin by collecting each of these long exact sequences into a single bigraded object. So we define

$$D_{p,q} = \tilde{E}_{p+q}(X_p), \quad E_{p,q} = \tilde{E}_{p+q}(\textstyle\bigvee_{I_p} S^p) \cong \bigoplus_{I_p} E_q(\mathrm{pt}).$$

If we sum all of these together, we get

$$\begin{array}{ccc} D_{*,\bullet} = \tilde{E}_{*+\bullet}(X_*) & \xrightarrow[i]{[1,-1]} & D_{*,\bullet} \\ {\scriptstyle k,\,[-1,0]} \nwarrow & & \swarrow {\scriptstyle j,\,[0,0]} \\ & E_{*,\bullet} = \bigoplus_{I_*} E_\bullet(\mathrm{pt}) & \end{array}$$

where each of these maps i, j, k shift the bidegree.

Definition 10.1. An **exact couple** is (D, E, i, j, k) where D, E are abelian groups and fit into

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ {\scriptstyle k} \nwarrow & & \swarrow {\scriptstyle j} \\ & E & \end{array}$$

which is exact at each vertex. (In many cases, D, E will be bigraded with i, j, k shifting the grading in some way, but sometimes they will be trigraded or graded by non-integers or have some other complicated form.)

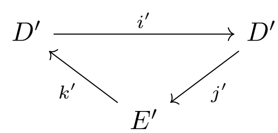
From this, we can extract a chain complex on E with differential $\partial = j \circ k$. This is a differential because $j \circ k \circ j \circ k = j \circ 0 \circ k = 0$. Using this, we can get a **derived couple** with

$$E' = \frac{\ker(\partial)}{\operatorname{im}(\partial)}, \quad D' = \operatorname{im}(i),$$

and

$$i'(d) = i(d), \quad j'(i(d)) = [j(d)], \quad k'([e]) = k(e).$$

Lemma 10.2. *The derived couple*



is an exact couple.

Proof. First we check that the maps are well-defined. For j', $j(d)$ in the kernel of ∂ because $\partial \circ j = j \circ k \circ j = 0$. Also, we need to check that $i(d) = i(d')$ implies $[j(d)] = [j(d')]$. This is because $i(d - d') = 0$ implies $d - d' = k(e)$ and then $j(d - d') = \partial(e)$. You can also check that k' is well-defined.

Next we need to check that it is exact at each vertex. On the top right corner, we can first check that $\operatorname{im}(i') \subseteq \ker(j')$ because $j' \circ i'(d) = [j \circ i(d)] = 0$. For the other direction, suppose that $j'(i(d)) = [j(d)] = 0$. Then $j(d) = j(k(e))$ for some e and so $d - k(e) \in \ker(j) = \operatorname{im}(i)$. So $d - k(e) = i(\tilde{d})$ and so $i(d) = i(k(e) + i(\tilde{d})) = i(i(\tilde{d})) = i'(i(\tilde{d}))$. For the other vertices, you can check this. ☐

If i, j, k have bigradings, $\operatorname{bideg}(i') = \operatorname{bideg}(i)$ and $\operatorname{bideg}(k') = \operatorname{bideg}(k)$ and $\operatorname{bideg}(j') = \operatorname{bideg}(j) - \operatorname{bideg}(i)$. In our special case, E' will be the homology of

$$\bigoplus_{I_0} E_q(*) \xleftarrow{\partial} \bigoplus_{I_1} E_q(*) \xleftarrow{\partial} \bigoplus_{I_2} E_q(*) \xleftarrow{\partial} \cdots.$$

In the case of ordinary chain complex, this is going to be $\mathbb{Z}[I_0] \leftarrow \mathbb{Z}[I_1] \leftarrow \cdots$, tensored with $E_*(\mathrm{pt})$. So we get

$$(E')_{p,q} = \tilde{H}_p(X, E_q(\mathrm{pt})).$$

We can iterate the derived couple construction many times and get the r-th derived couple $(D^r, E^r, i_r, j_r, k_r)$. Usually, the original exact couple is used with the index $r = 1$.

Definition 10.3. A **spectral sequence** is a collection $(E^r, d^r, \varphi^r)_{r\geq 1}$ of abelian groups E^r with differentials $d^r : E^r \to E^r$ with isomorphisms $\ker(d^r)/\operatorname{im}(d^r) \cong E^{r+1}$.

Taking the original exact couple to be $r = 1$, we get the **Atiyah–Hirzebruch spectral sequence**. The E^1-page is

$$E^1_{p,q} \cong \bigoplus_{I_p} E_q(\mathrm{pt}),$$

and the E^2-page is

$$E^2_{p,q} \cong \tilde{H}_p(X, E_q(\mathrm{pt})).$$

Then the d^r-differential has bidegree $(-r, r+1)$.

It is clear that E^r is a subquotient of E, because we are only taking homology. So $E^r \cong Z^r/B^r$ for some $B_r \subseteq Z^r \subseteq E$.

Lemma 10.4. *$D^r = \operatorname{im}(i^{r-1})$ and $Z^r = h^{-1}(\operatorname{im}(i^{r-1}))$ and $B^r = j(\ker(i^{r-1}))$. So*

$$E^r = \frac{Z^r}{B^r} = \frac{h^{-1}(\operatorname{im}(i^{r-1}))}{j(\ker(i^{r-1}))}.$$

This looks really useless, but let us look at the case of the Atiyah–Hirzebruch spectral sequence. Here, D^r is

$$D^r = \operatorname{im}(\tilde{E}_{p+q}(X_{q-r+1}) \to \tilde{E}_{p+q}(X_q)).$$

Then Z^r are those elements in $\tilde{E}_{p+q}(X_p/X_{p-1})$ whose image in $\tilde{E}_{p+q-1}(X_{p-1})$ lies in the image of $\tilde{E}_{p+q-1}(X_{p-r})$. In particular, if $r > p$ then

$$Z^r = \ker(\tilde{E}_{p+q}(X_p/X_{p-1}) \to \tilde{E}_{p+q-1}(X_{p-1})) = \operatorname{im}(\tilde{E}_{p+q}(X_p) \to \tilde{E}_{p+q}(X_p/X_{p-1})).$$

We also see that B_r are those elements in $\tilde{E}_{p+q}(X_p/X_{p-1})$ coming from an element in $\tilde{E}_{p+q}(X_p)$ dying in $\tilde{E}_{p+q}(X_{p+q-1})$. So these are sort of trying to compute $\tilde{E}_{p+q}(X_p)$ as $r \to \infty$.

Let me make this precise. If $E^r_{p,q}$ is first quadrant, then for $r > \max(p+1, q+2)$ there is no nonzero differential into or out of $E^r_{p,q}$. That means that from that point on, $E^r_{p,q} = E^{r+1}_{p,q} = \cdots$, and we can write this stable value as $E^\infty_{p,q}$. But the Atiyah–Hirzebruch spectral sequence is not first quadrant in general, so we have to do something else.

In general, we have

$$0 = B_1 \subseteq B_2 \subseteq B_3 \subset \cdots \subset Z_3 \subseteq Z_2 \subseteq Z_1 = E.$$

So it is reasonable to define

$$Z^\infty = \bigcap_n Z_n, \quad B^\infty = \bigcup_r B_r, \quad E^\infty = \frac{Z^\infty}{B^\infty} = \frac{k^{-1}(\bigcap_r \operatorname{im}(i^{r-1}))}{j(\bigcup_r \ker(i^{r-1}))}.$$

In Atiyah–Hirzebruch, we have $\bigcap_r \operatorname{im}(i^{r-1}) = 0$ because the filtration is a point at -1.

So consider the following convergence condition:

$$\ker(i) \cap \bigcap_r \operatorname{im}(i^{r-1}) = 0.$$

Since $\operatorname{im}(k) \subseteq \ker(i)$, this implies

$$k^{-1}\left(\bigcap_r \operatorname{im}(i^{r-1})\right) = k^{-1}(0) = \ker(k) = \operatorname{im}(j).$$

But I want a different expression for E^∞. Let

$$D^\infty = \operatorname{colim}(\cdots \xrightarrow{i} D \xrightarrow{i} D \xrightarrow{i} \cdots),$$

and let F be the image of $D \to D^\infty$. Then its kernel is given by $\bigcup_i \ker(i^{r-1})$. Also, $\bigcup_r i^r(F) = D^\infty$. This is saying that the filtration

$$\cdots \subseteq i^2(F) \subseteq i(F) \subseteq F \subseteq i^{-1}(F) \subseteq i^{-2}(F) \subseteq \cdots$$

is exhaustive.

Lemma 10.5. $E^\infty = F/i(F)$.

Proof. We already know

$$E^\infty \cong \frac{\operatorname{im}(j)}{j(\bigcup_r \ker(i^{r-1}))}$$

and so $j : D \to E$ gives $D \twoheadrightarrow \operatorname{im}(f)$ with kernel $i(D)$. This shows that

$$\frac{D}{i(D) + \bigcup_r \ker(i^{r-1})} \cong E^\infty.$$

Now $E \to D^\infty$ gives $D \twoheadrightarrow F$ with kernel $\bigcup_r \ker(i^{r-1})$ and then this gives

$$\frac{D}{i(D) + \bigcup_r \ker(i^{r-1})} \cong \frac{F}{i(F)}.$$

This proves the lemma. □

Let's go back to the Atiyah–Hirzebruch spectral sequence. Since $\tilde{E}_{p+q}(X_r) = 0$ for $r < 0$, we have $\bigcap_r i^r(D) = 0$ so the convergence condition is satisfied. Now we have

$$D^\infty = \bigoplus_{\mathbb{Z}} \operatorname*{colim}_r \tilde{E}_{p+q}(X_r) = \bigoplus_{\mathbb{Z}} p \tilde{+} q(X).$$

So F is the image of $\tilde{E}_{p+q}(X_p)$ in $\tilde{E}_{p+q}(X)$, and iF is the image of $\tilde{E}_{p+q}(X_{p-1})$ in $\tilde{E}_{p+q}(X)$. Therefore E^∞ is the associated of the filtration of $\tilde{E}_{p+q}(X)$ by images of $\tilde{E}_{p+q}(X_r)$.

Theorem 10.6 (Atiyah–Hirzebruch spectral sequence)**.** *If X is a based CW-complex, then there is a spectral sequence*

$$E^2_{p,q} = \tilde{H}_p(X, E_q(\mathrm{pt}))$$

converging to $\tilde{E}_{p+q}(X)$.

This is not optimal because it says CW-complex. You can actually remove this condition by using a CW-approximation. From the E^2-page, this is going to be independent of the CW-approximation. Cellular map induces a map of exact couples and that gives an isomorphism on E^2 onwards.

Example 10.7. Let us take $X = \mathbb{C}P^\infty$ and $E = KU$. Here, I am going to use that $KU_*(\mathrm{pt})$ is concentrated in even degrees. If you try and write down the spectral sequence, we will get From this picture you immediately see that

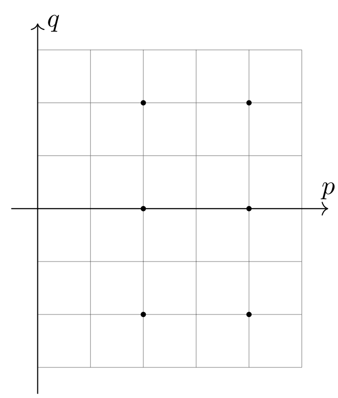


Figure 1: Computing $\widetilde{KU}(\mathbb{C}P^\infty)$

$\widetilde{KU}(\mathbb{C}P^\infty)$ is concentrated in even degree.

11 February 27, 2018

Today we are going to lift the construction of the Atiyah–Hirzebruch spectral sequence for B a CW-complex to $p : X \to B$ a Hurewicz fibration over a CW-complex.

11.1 Atiyah–Hirzebruch–Serre spectral sequence

We will start by comparing fibers using path lifting. If $p : X \to B$ is a Hurewicz fibration, for $b \in B$ we will look at fibers $p^{-1}(b)$.

Lemma 11.1. *If b_0, b_1 are fibers in the same path-component of B, then $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.*

Proof. We will construction a functor $\Pi(B) \to \mathrm{Ho}(\mathsf{Top})$. The functor will sent $b_0 \in B$ to $p^{-1}(b_0)$, and send homotopy class of paths to path lifting. If we can show that this is a functor, it is going to send isomorphisms to isomorphisms. So let us construct this functor. Pick $\gamma : I \to B$ a representative. Then we lift

$$\begin{array}{ccc} p^{-1}(b_0) & \longrightarrow & X \\ {\scriptstyle -\times\{0\}}\downarrow & \overset{G_\gamma}{\nearrow} & \downarrow \\ p^{-1}(b_0) \times I & \xrightarrow{\gamma\circ\pi_2} & B. \end{array}$$

Then for G_γ, we restrict to $p^{-1}(b_0) \times \{1\}$. We need to show that this is well-defined. If we lift a homotopy of paths γ and γ', this gives a homotopy of between g_γ and $g_{\gamma'}$. Functoriality then can be checked by explicitly constructing appropriate lifts. ☐

Similarly,

Lemma 11.2. *If B deformation retracts to b_0, then there is a homotopy equivalence $X \to p^{-1}(b_0) \times B$ over B.*

Proof. We can lift

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[H\circ(p\times\mathrm{id})]{} & B. \end{array}$$

Then we can take $h : \tilde{H} \times X \times \{1\}$ and $(h, p) : X \to p^{-1}(b_0) \times X$. ☐

For $p : X \to B$ a Hurewicz fibration, we can take a CW-approximation $\tilde{B} \to B$ and then pull back

$$\begin{array}{ccc} \tilde{X} = X \times \tilde{B}B & \longrightarrow & X \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \tilde{B} & \longrightarrow & B. \end{array}$$

Then $\tilde{p}$ has the same same fiber, and the exact sequence

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_i(\tilde{p}^{-1}(\tilde{b}_0)) & \longrightarrow & \pi_i(\tilde{X}) & \longrightarrow & \pi_i(\tilde{B}) & \longrightarrow & \cdots \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ \cdots & \longrightarrow & \pi_i(p^{-1}(b_0)) & \longrightarrow & \pi_i(X) & \longrightarrow & \pi_i(B) & \longrightarrow & \cdots \end{array}$$

shows that $\tilde{X} \to X$ is a weak homotopy equivalence. So if we care about $E_*(\tilde{X}) \cong E_*(X)$, we may assume that the base is a CW-complex.

Now we can look at the skeletal filtration on B and pull it back to X.

$$\begin{array}{ccccccccc} X_{-1} = \emptyset & \longrightarrow & X_0 = p^{-1}(B_0) & \longrightarrow & X_1 = p^{-1}(B_1) & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & \downarrow p_0 & & \downarrow p_1 & & \downarrow p_2 & & \\ B_{-1} = \emptyset & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \end{array}$$

By definition of a CW-complex, we have

$$\begin{array}{ccc} \coprod_{I_k} S^{k-1} & \xrightarrow{\coprod \partial e_i} & B_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{I_k} D^k & \xrightarrow{\coprod e_i} & B_k \end{array}$$

a pushout diagram, and then we get

$$\begin{array}{ccc} \coprod_{I_k} \partial e_i^* p & \longrightarrow & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{I_k} e_i^* p & \longrightarrow & X_k. \end{array}$$

Here, e^*p is a Hurewicz fibration over D^k, so $e^*p \simeq p^{-1}(b_i) \times D^k$. So we can write $\partial e_i^* p \simeq p^{-1}(b_i) \times S^{k-1}$. You can show that in Top pullback of a Hurewicz cofibration along a Hurewicz fibration is a Hurewicz cofibration by lifting the NDR-pair structure. (This is a theorem of Kieboom.) So we get a long exact sequence

$$\cdots \to E_*(X_{k-1}) \to E_*(X) \to \bigoplus_{I_k} E_*(p^{-1}(b_i) \times D^k, p^{-1}(b_i) \times S^{k-1}) \to \cdots,$$

where

$$\bigoplus_{I_k} E(p^{-1}(b_i) \times D^k, p^{-1}(b_i), S^{k-1}) \cong \bigoplus_{I_k} \tilde{E}_k(p^{-1}(b_i)_+ \wedge S^k) \cong \bigoplus_{E_{*-k}} (p^{-1}(b_i)).$$

Now we can build an exact couple

$$\begin{array}{ccc} D^1_{p,q} = \bigoplus E_{p+q}(X_p) & \xrightarrow{[1,-1]} & D^1_{p,q} \\ {\scriptstyle [1,0]}\nwarrow & & \swarrow{\scriptstyle [0,0]} \\ & E^1_{p,q} = \bigoplus_{I_p} E_q(p^{-1}(b_i)). & \end{array}$$

Then we get a spectral sequence. But what is E^2? By the lemma, we know that $p^{-1}(b_i) \simeq p^{-1}(b_0)$ by your favorite b_0 in the same path component, but not canonically so. This is because if you pick a different path not homotopic to the first one, might get a different isomorphism. That is, $\pi_1(B, b_0)$ might not act trivially on $E_*(p^{-1}(b_0))$.

So to get away with this issue, we assume that B is path-connected and $E_*(p^{-1}(b_0)$ is simple, i.e., $\pi_1(B, b_0)$ acts trivially on $E_*(p^{-1}(b_0))$. Then we can compute

$$E^2_{p,q} = H_p(B; E_q(F)).$$

The same E^∞ convergence criterion holds since $X_{-1} = \emptyset$. So $E^\infty_{p,q}$ for $p+q = n$ are associated graded of the filtration $\mathrm{colim}_{k\to\infty} E_n(X_k) \cong E_n(X)$, by images of $E_n(X_k)$.

Theorem 11.3 (Atiyah–Hirzebruch–Serre spectral sequence)**.** *If $p : X \to B$ is a Hurewicz fibration, B is path-connected, and $E_*(F)$ are simple where $F = p^{-1}(b_0)$ is the fiber, then there is a spectral sequence*

$$E^2_{p,q} = H_p(B, E_q(F)) \quad \Longrightarrow \quad E_{p+q}(X).$$

The case $E_*(-) = H_*(-; A)$ is called the Serre spectral sequence. Let me make list some properties:

- This also works for Serre fibrations if E_* satisfies the weak homotopy equivalence axiom.
- If we use a functorial CW-approximation, e.g., $|\mathrm{Sing}(-)|$, then you see that the AHS spectral sequence is functorial. That is, if we have

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \longrightarrow & B' \end{array}$$

with $g(b_0) = b_0'$, we get

$$\begin{array}{ccc} H_p(B; E_q(F)) & \Longrightarrow & E_{p+q}(X) \\ \downarrow & & \downarrow \\ H_p(B'; E_q(F')) & \Longrightarrow & E_{p+q}(X'). \end{array}$$

Using

$$\begin{array}{ccccc} X & \longleftarrow & X^I & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B & \longleftarrow & B^I & \longrightarrow & B \end{array}$$

with γ mapping to b_0 on the left and b_1 on the right, we get a non-canonical isomorphism between the Atiyah–Hirzebruch–Serre spectral sequence with fibers over different basepoints.

- If it is not simple, you get

$$E^2_{p,q} = H_p(B; \mathcal{E}_q(F)) \implies E_{p+q}(X)$$

with $\mathcal{E}_q(F)$ being a local system.

11.2 First examples

For $p = \mathrm{id} : B \to B$, we get

$$E^2_{p,q} = H_p(B; E_q(*)) \implies E_{p+q}(B)$$

and this is an unbased version of the Atiyah–Hirzebruch spectral sequence. We also have

$$\Omega S^n \to PS^n \simeq * \to S^n.$$

So we get a spectral sequence

$$H_p(S^n, H_q(\Omega S^n)) \Longrightarrow H_{p+q}(PS^n) = \begin{cases} \mathbb{Z} & p+q=0 \\ 0 & \text{otherwise.} \end{cases}$$

But for $n \geq 2$, we can get $H_p(S^n) \otimes H_q(\Omega S^n)$. So our spectral sequence looks like Figure 2 on the E^2-page.

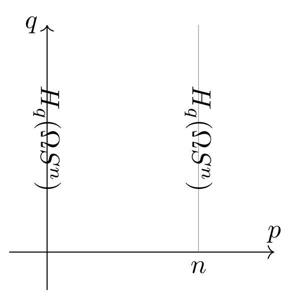


Figure 2: The E^2 page of the Serre spectral sequence for $\Omega S^n \to PS^n \to S^n$

Because the only possible nonzero differentials are d_n, and they should knock everything out, we can inductively show that only multiples of $n-1$ can have nonzero homology, and others should be zero.

Theorem 11.4. *If $n \geq 2$, then*

$$H_q(\Omega S^n) \cong \begin{cases} \mathbb{Z} & i \geq 0, (n-1) \mid i \\ 0 & \textit{otherwise.} \end{cases}$$

If X is such that $\bigoplus_i H_i(X; \mathbb{F})$ is finite-dimensional, we can define its **Euler characteristic** as

$$\chi_{\mathbb{F}}(X) = \sum_i (-1)^i \dim H_i(X; \mathbb{F}).$$

Also, the Künneth theorem says that $H_*(B \times F; \mathbb{F}) \cong H_*(B;\mathbb{F}) \otimes_{\mathbb{F}} H_*(F;\mathbb{F})$, so we should get

$$\chi_{\mathbb{F}}(B \times F) = \chi_{\mathbb{F}}(B)\chi_{\mathbb{F}}(F).$$

Then if a fibration is something like a twisted product, we should expect the same for fibrations.

Theorem 11.5. *Suppose B is 1-connected with finite-dimensional $\mathbb{F}$-homology. Let $p : X \to B$ be a fibration such that $F = p^{-1}(b)$ has finite-dimensional $\mathbb{F}$-homology. Then*

(i) X has finite-dimensional $\mathbb{F}$-homology,

(ii) $\chi_{\mathbb{F}}(X) = \chi_{\mathbb{F}}(B)\chi_{\mathbb{F}}(F)$.

Proof. We have

$$E^2_{p,q} = H_p(B;\mathbb{F}) \otimes_{\mathbb{F}} H_q(F;\mathbb{F}) \Longrightarrow H_{p+q}(X;\mathbb{F}).$$

So the E^2 has bounded support, and is finite-dimensional everywhere. So the same is true for E^∞. This proves the first part. For the second part, we note that

$$\chi_{\mathbb{F}}(E^r_{*,*}) = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{F}} E^r_{p,q}$$

stays invariant under passing to the next page by taking homology. So $\chi_{\mathbb{F}}(X) = \chi_{\mathbb{F}}(E^\infty_{*,*}) = \chi_{\mathbb{F}}(E^2_{*,*}) = \chi_{\mathbb{F}}(B)\chi_{\mathbb{F}}(F)$. □

Theorem 11.6. *Let $\mathbb{K}$ be a PID. If B is 1-connected, $p : X \to B$ a Hurewicz fibration with fiber $F = p^{-1}(b_0)$, then all of B, F, X have degree-wise finitely generated $H_*(-;\mathbb{K})$ if and only if two of these do.*

Corollary 11.7. *IF A is a finitely generated abelian group, then $K(A,n)$ has degree-wise finitely generated homology.*

Proof. For $K(A,1) = K(\mathbb{Z},1)^r \times \prod_i K(\mathbb{Z}/p_i^{r_i}, 1)$ you do this by hand. Then for $n > 1$ you do induction. We have

$$K(A,n) \to * \to K(A,n+1)$$

and then use the theorem. □

You can imagine feeding this into Postnikov towers and getting information.

12 March 1, 2018

We will continue doing spectral sequences. We spent a first couple of lectures doing general homotopy theory of spaces, and that led to the description of some more refined invariants of generalized (co)homology theories coming from spectra. To compute them we have developed the Atiyah–Hirzebruch spectral sequence.

12.1 Atiyah–Hirzebruch–Serre spectral sequence for cohomology

The goal is to have a spectral sequence

$$E_2^{p,q} = H^p(B; E^q(F)) \Longrightarrow E^{p+q}(X)$$

with an algebra structure if E is a homotopy-commutative ring spectrum.

To construct this we start as before. Let $p : X \to B$ is a Serre fibration, and without loss of generality let B be a CW-complex. Then if we filter B by skeleta, and pull back to X, we get a filtration

$$X_{-1} = \emptyset \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots .$$

These are cofibrations and obtained by pushing out along $S^{k-1} \times p^{-1}(b_i) \hookrightarrow D^k \times p^{-1}(b_i)$. If we apply $E^*(-)$ instead of $E_*(-)$, we get a long exact sequence

$$\cdots \leftarrow E^*(X_{k-1}) \leftarrow E^*(X_k) \leftarrow \bigoplus_{I_k} E^{*-k}(p^{-1}(b_i)) \leftarrow \cdots .$$

Here, we are going to assume that the system is **simple**, so that we have a canonical isomorphism

$$\bigoplus_{I_k} E^{*-k}(p^{-1}(b_i)) \cong \bigoplus_{I_k} E^{*-k}(p^{-1}(b_0)).$$

So we get an exact couple

$$D^1_{p,q} = E^{p+q}(X_p) \xrightarrow[i]{[-1,1]} D^1_{p,q}$$

$$D^1_{p,q} \xrightarrow{[0,0]} E^1_{p,q} = \bigoplus_{I_q} E^q(p^{-1}(b_0)).$$

Then the E^2-page can be idenfied with

$$E^2_{p,q} = H^q(B; E^q(F)).$$

But now the convergence condition fails. If you recall, the convergence condition was that $\lim_i D^1 = 0$ shows that you converges to $\operatorname{colim}_i D^i$. Now, we have $\operatorname{colim}_i D^i = 0$, so you might expect this to converge to $\lim_i D^i = D^{-\infty}$. But there are two problems:

- $\lim_i D^1$ is degreewise $\lim_{r\to\infty} E^p(X_r)$, but limit is not exact. So we only get the **Milnor exact sequence**

$$0 \to \lim_{r\to\infty}^1 E^p(X_r) \to E^p(X) \to \lim_{r\to\infty} E^p(X_r) \to 0.$$

- The sequence might not converge to $D^{-\infty}$ but only conditionally convergent. To avoid these issues, we impose additional conditions.

Theorem 12.1. *If we are given a Serre fibration $p : X \to B$ with B path-connected, with fiber $F = p^{-1}(b_0)$ such that $E^*(F)$ is a simple local coefficient system, then there is a spectral sequence*

$$E^2_{p,q} = H^p(B; E^q(F)) \Longrightarrow E^{p+q}(X)$$

if one of the following two conditions is satisfied:

- *for fixed p, q, there are only finitely many differentials which are nonzero and have $E^r_{p,q}$ as source or target.*
- *$E^*(X_p)$ satisfy the Mittag–Leffler condition.*

The first one is helpful if $E^*(F)$ is bounded below, or if B is finite-dimensional, or if each $H^p(B; E^q(F))$ is finite-dimensional vector space.

If E is a homotopy commutative ring spectrum, we want a spectral sequence of algebras. I want each page $E^r_{*,*}$ to be a **graded-commutative algebra**, graded by the total degree, with d_r being a derivation so that $H(E^r_{*,*}, d^r$ is again a graded-commutative algebra. I also want $H(E^t_{*,*}) \cong E^{r+1}_{*,*}$ to be an isomorphism of graded-commutative algebras.

A filtered graded-commutative algebra A^* is a filtered graded abelian group such that $F^qA \cdot F^qA \subseteq F^{p+q}A$. Then

$$\mathrm{gr}(A) = \bigoplus_{p,q} F^pA^{p+q}/F^{p+1}A^{p+q}$$

has a graded-commutative algebra structure. Then we are going to say that a spectral sequence of algebras converges to A^* if $E^\infty_{*,*} \cong \mathrm{gr}(A)$. We shall not construct such algebra structures, because we should have used Cartan–Eilenberg systems instead.

Proposition 12.2. *If E is a homotopy-commutative ring spectrum, then the cohomological Atiyah–Hirzebruch–Serre spectral sequence is one of algebras converging to $E^*(X)$ with cup product (if all conditions are satisfied) and E^2-page*

$$E^2_{p,q} = H^p(B; E^q(F)).$$

This is a graded-commutative algebra with cup product in cohomology with coefficients in $E^*(F)$.

12.2 More examples

Let us first compute cohomology of Eilenberg–Mac Lane spaces. We know that $H_*(K(A,n);B) = 0$ if $0 < * < n$, and also

$$A = \pi_n(K(A,n)) \cong H_n(K(A,n))$$

by the Hurewicz theorem. Then by the universal coefficients theorem, we get

$$H^*(K(A,n),B) = \begin{cases} B & * = 0 \\ 0 & 0 < * < n \\ \operatorname{Hom}(A,B) & * = n. \end{cases}$$

Also, if A and B are finitely generated then $H^p(K(A,n),B)$ is finitely generated.

Let us assume that A is finitely-generated. This is not such a big restriction, because $K(A,n)$ is he homotopy colimit of $K(A',n)$ for $A' \subseteq A$ finitely generated. Using $K(A \times A', n) \cong K(A,n)$, we see that it suffices to compute cohomology of $K(\mathbb{Z},n)$ and $K(\mathbb{Z}/p^r\mathbb{Z},n)$.

Proposition 12.3. $H^*(K(\mathbb{Z}/p^r\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0 \\ 0 & \textit{otherwise.} \end{cases}$

Proof. Write $A = \mathbb{Z}/p^r\mathbb{Z}$. It suffices to show that $H_*(K(A,n);\mathbb{Q})$ that vanishes in positive degree. For $n = 1$, we use that the universal cover of $K(A,1)$ is contractible. Then we have

$$A \to \widetilde{K(A,1)} \xrightarrow{p} K(A,1).$$

Then there is a transfer map

$$\tau : H_*(K(A,1);\mathbb{Q}) \to H_*(\widetilde{K(A,1)};\mathbb{Q}),$$

induced by the chain map on singular chains $\sigma \mapsto \sum \tilde{\sigma}$ by adding all the lifts. (This is works because A is finite and so we are adding only p^r of them.) Then $p_*\tau$ is multiplication by p^r on $H_*(K(A,1);\mathbb{Q})$. But p^r is invertible, and this implies that τ is injective. Then $H_*(K(A,1);\mathbb{Q}) \hookrightarrow H_*(\mathrm{pt};\mathbb{Q})$.

Now we do the case $n \geq 2$ by induction. There is a fiber sequence

$$K(A,n-1) \to * \to K(A,n).$$

So we get a Serre spectral sequence

$$E^2_{p,q} = H_p(K(A,n);H_*(K(A,n-1);\mathbb{Q})) \Longrightarrow H_{p+q}(*;\mathbb{Q}).$$

The E^2-page can be simplified because there are no Tor terms over $\mathbb{Q}$. Then the induction hypothesis shows that $H_p(K(A,n);\mathbb{Q}) = H_p(*;\mathbb{Q})$. The same argument actually shows that

$$H_p(K(A,n);\mathbb{Z}[\tfrac{1}{p}]) = H_p(*;\mathbb{Z}[\tfrac{1}{p}])$$

or that $H_*(K(A,n))$ is p-torsion. □

Let $\Lambda_{\mathbb{Q}}(-)$ be the free graded-commutative algebra, so that

$$\Lambda_{\mathbb{Q}}(x_{2n}) \cong \mathbb{Q}[x_{2n}], \quad \Lambda_{\mathbb{Q}}(x_{2n-1}) \cong E_{\mathbb{Q}}[x_{2n-1}]$$

where $E_{\mathbb{Q}}$ is the exterior algebra.

Proposition 12.4. *For $n \geq 1$, $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_n)$.*

Proof. For $n = 1$, we have $K(\mathbb{Z}, 1) \simeq S^1$. For $n \geq 2$, we use

$$K(\mathbb{Z}, n-1) \to * \to K(\mathbb{Z}, n)$$

and use the cohomological Serre spectral sequence. Let us do the n odd case. Then we get $\Lambda_{\mathbb{Q}}(x_{2m})$ at $p = 0$, and so $d_{2m+1}x_{2m} = x_{2m+1}$ for something lying in $H^{2m+1}(K(\mathbb{Z}, 2m+1); \mathbb{Q})$. We can compute

$$d^{2m+1}(x_{2m}^k) = kd^{2m+1}(x_{2m})x_{2m}^{k-1} = kx_{2m+1}x_{2m}^{k-1}$$

and so the differentials kill everything. This shows that

$$H^p(K(\mathbb{Z}, 2m+1); \mathbb{Q}) = \begin{cases} \mathbb{Q} & p = 0, 2m+1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, there is only one algebra structure. □

Let $H \hookrightarrow G$ be a closed subgroup of Lie groups, and let $p : G \to G/H$ be the quotient map.

Lemma 12.5. *If $\mathcal{U} = \{U_i\}$ is an open cover of B, then $p : X \to B$ is a Hurewicz fibration if and only if each $p|_{p^{-1}(U_i)} : p^{-1}(U_i) \to U_i$ is a Hurewicz fibration.*

Lemma 12.6. *$p : G \to G/H$ has local sections.*

Proof. Lie theory will tell us that H is a sub-Lie group, and then $p : G \to G/H$ is smooth with derivative at $\{e\}$ given by $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Now we can use the implicit function theorem to find a local section section near H, and then translate this local section around. □

Now we have a Hurewicz fibration

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}.$$

Theorem 12.7. *$H^*(\mathrm{SU}(n)) \cong \Lambda_{\mathbb{Z}}(x_3, x_5, \dots, x_{2n-1})$.*

Proof. The case $n = 1$ is trivial. For $n \geq 2$, we induce using the Serre spectral sequence for $\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$. Take $n = 5$ for example. There are going to be generators x_3, x_5, x_7 on $H^q(\mathrm{SU}(n-1))$ and x_9 on $H^p(S^{2n-1})$. But note that d^r of all of these generators are 0, and so the E^∞-page is going to be the same. That is, the associated graded for $H^*(\mathrm{SU}(n))$ is equal to

$$\Lambda_{\mathbb{Z}}[x_3, \dots, x_{2n-1}].$$

This implies that $H^*(\mathrm{SU}(n))$ is actually equal to that after doing some lifting. □

13 March 6, 2018

Today we want to do some geometry by generalizing $G \to G/H$, quotienting by a Lie group, to a principal bundle. Then we will build a single space containing information about these.

13.1 Principal G-bundles

Fix a topological group G.

Definition 13.1. A **principal G-bundle** $\xi = (E, B, p)$ is a map $p : E \to B$ with a fiber-preserving right G-action on E which is locally trivial. That is, there exists an open cover $\mathcal{U} = \{U_i\}$ of B and G-equivariant homeomorphisms $\varphi_i : G \times U_i \to p^{-1}(U_i)$ over U_i.

Example 13.2. The map $\pi_2 : G \times B \to B$ is a **trivial** principal G-bundle. A restriction of $\xi = (E, B, p)$ to any $U \subseteq B$ given by $\xi|_U = (p^{-1}(U), U, p|_{p^{-1}(U)})$ is a principal G-bundle.

Definition 13.3. A map of principal G-bundles $\xi = (E, B, p) \to \xi' = (E', B', p')$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

with f being G-equivariant.

Example 13.4. If $\xi' = (E', B', p')$ and $g : B \to B'$ are given, its **pullback** $g^*\xi$ is a principal G-bundle

$$\begin{array}{ccc} B \times_{B'} E' & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B'. \end{array}$$

Then $\xi = (E, B, p) \to \xi' = (E', B', p')$ over g is the same as a map $\xi \to g^*\xi'$ over id_B.

The local characterization of Hurewicz fibrations tells us that

Proposition 13.5. *If $\xi = (E, B, p)$ is a principal G-bundle, then $p : E \to B$ is a Hurewicz fibration.*

So $E \to B \times I$ is homotopy equivalent to $E|_{B\times\{0\}} \times I \to B \times I$ over $B \times I$.

Definition 13.6. $\xi = (E, B, p)$ is **numerable** if there exists a trivializing open cover $\mathcal{U}$ such that there exists a partition of unity subordinate to $\mathcal{U}$. This is always true for B paracompact. Also, pullback of numerable bundles is numerable.

Theorem 13.7. *Let ξ be a numerable principal G bundle over $B \times I$. Then $\xi \cong \xi|_{B\times\{0\}} \times I$ as principal G-bundles.*

Corollary 13.8. *If $f_0, f_1 : B \to B'$ and $\xi' = (E', B', p')$ is numerable, then $f_0^* \cong f_1^*$.*

Proof. Let $H : B \times I \to B'$ be a homotopy. Then $H^*\xi$ is numerable over $B \times I$, and hence isomorphic to $(H^*\xi)|_{B\times\{0\}} \times I = f_0^* \times I$. Then restricting to $B \times \{1\}$ gives $f_0^*\xi \cong f_1^*\xi$. □

This shows that the set $\mathrm{Prin}_G^{\mathrm{num}}(B)$ of the set of isomorphism classes of numerable principal G-bundles only depends on the homotopy type of B and moreover fixing $\xi = (E, B, p)$ numerable gives a natural transformation

$$[-, B] \to \mathrm{Prin}_G^{\mathrm{num}}(-); \quad f \mapsto f^*\xi$$

of functors $\mathrm{Ho}(\mathsf{Top})^{\mathrm{op}} \to \mathsf{Set}$.

Definition 13.9. A **classifying space** for principal G-bundles is a $\gamma = (EG, BG, p)$ such that pullback of γ gives a natural isomorphism

$$[-, BG] \to \mathrm{Prin}_G^{\mathrm{num}}(-).$$

Then $p : EG \to BG$ is called the **universal bundle**.

If it exists, it is going to be unique up to homotopy, because they are defined as representing a functor. We will different definitions: the first definition is due to Milnor, and we will give two alternative constructions. Note that we can apply Brown representability to a slightly different functor

$$\mathrm{Prin}_{G,*}^{\mathrm{num}}(-) : \mathrm{Ho}(\mathsf{CW}_*^{\geq 0})^{\mathrm{op}} \to \mathsf{Set}.$$

The $\mathrm{Prin}_{G,*}^{\mathrm{num}}$ is the isomorphism classes of principal G-bundles whose fiber over the basepoint is identified with G. We need

(1) Meyer–Vietoris: given an excisive triad, we can just glue the principal bundles on both sides.
(2) wedge axiom: here, we use the fact that the bundle is trivialized over the basepoint.

Then we get a classifying space that works for path-connected based CW-complexes. You can extend to all CW-complexes by some geometric argument given in Kirby–Siebenmann.

13.2 Transition functions and the bar construction

If $\xi = (E, B, p)$ we can record it by taking $\mathcal{U} = \{U_i\}$ an trivializing open cover and $\varphi_i : G \times U_i \to p^{-1}(U_i)$. Then we get

$$G \times (U_i \cap U_j) \xrightarrow{\varphi_i} p^{-1}(U_i \cap U_j) \xrightarrow{\varphi_j^{-1}} G \times (U_i \times U_j).$$

This is G-equivariant, so there is a map

$$g_{ij} : U_i \cap U_j \to G$$

such that $\varphi_j^{-1}\varphi_i(g,b) = (g_{ij}(b)g, b)$. This has some obvious properties:

(i) g_{ii} is contant at $e \in G$.

(ii) $g_{ij} = g_{ji}^{-1}$.

(iii) $g_{ij}g_{jk} = g_{ik}$.

These g_{ij} are called **transition functions**, and (i)–(iii) are called the **cocycle conditions**.

From the transition functions we can recover ξ up to isomorphisms from

$$\left(\coprod_i G \times U_i\right)/((i,g,b) \sim (j, g_{ij}(b)g, b)).$$

Note that this is really is an equivalence relation by the cocycle condition. This still has the right G-action and the quotient is going to be B. It is locally trivial using U_i, so it is a principal G-bundle.

This data $(U_i, \{g_{ij}\})$ is not unique. First, you can always refine $\mathcal{U}$. Then you can change φ_i by taking $h_i : U_i \to G$ and replacing φ_i by

$$(g,b) \mapsto \varphi_i(h_i(b)g, b).$$

Then we replace g_{ij} by $h_j g_{ij} h_i^{-1}$. This is some kind of non-abelian Čech cohomology.

So we have a map

$$\left\{\begin{matrix}\mathcal{U} + \text{transition} \\ \text{functions}\end{matrix}\right\} \Big/ \text{refine} + \text{replace} \quad \longrightarrow \quad \left\{\begin{matrix}\text{principal} \\ G\text{-bundles over } B\end{matrix}\right\} \Big/ \text{iso.}$$

This is going to be an isomorphism, but we are not going to be use this.

Definition 13.10. A **simplicial space** is a functor $\Delta^{\mathrm{op}} \to \mathsf{Top}$, and a map between simplicial spaces is a natural transformation.

Definition 13.11. Let X be a right G-space, and Y a left G-space. Then we define $B_\bullet(X,G,Y)$ as the simplicial space with p-simplices given by

$$B_p(X,G,Y) = X \times G^p \times Y.$$

the face maps are

$$d_i(x, g_1, \dots, g_p, y) = \begin{cases} (xg_1, g_2, \dots, g_p, y) & i = 0, \\ (x, g_1, \dots, g_i g_{i+1}, \dots, g_p, y) & 0 < i < p, \\ (x, g_1, \dots, g_{p-1}, g_p y) & i = p. \end{cases}$$

The degeneracy map s_i is going to insert $e \in G$ at the jth slot. The **two-sided bar construction** is its geometric realization $B(X,G,Y) = |B_\bullet(X,G,Y)|$.

This can be considered as the space of lists of points $\{0, t_1, \dots, t_n, 1\} \subseteq [0,1]$, with a point x attached to 0 and y attached to 1, and $g_i \in G$ attached to each t_i. Then when two points collide, they are going to become the product, and you can insert e anywhere.

Now we are going to consider

$$EG = B(*, G, G) \to B(*, G, *) = BG.$$

Since we are working in CGWH, we have $B(*, G, G) \times G \cong B(*, G, G \times G)$ and then map to $B(*, G, G)$ by multiplication. So EG has a right G-action, and BG is the quotient map.

Definition 13.12. We say that G is **well-pointed** if $\{e\} \hookrightarrow G$ is a Hurewicz cofibration. (This is not very obvious for very large groups.)

All Lie groups are well-pointed. To see how this helps, let us go back to the geometric realization $|X_\bullet|$. We can filter this by the image of $\coprod_{0 \le p \le k} \Delta^p \times X_p$. This is the skeletal filtration.

$$\emptyset = \mathrm{sk}_{-1}(|X_\bullet|) \to \mathrm{sk}_0(|X_\bullet|) \to \mathrm{sk}_1(|X_\bullet|) \to \cdots.$$

This is exhaustive, and there are pushout diagrams

$$\begin{array}{ccc} \partial\Delta^p \times L_p(|X_\bullet|) & \longrightarrow & \mathrm{sk}_{p-1}(|X_\bullet|) \\ \downarrow & & \downarrow \\ \Delta^p \times X_p & \longrightarrow & \mathrm{sk}_p(|X_\bullet|) \end{array}$$

where $L_p(|X_\bullet|)$ is the pth latching object $\bigcup_j s_j(X_{p-1})$.

Definition 13.13. $X_\bullet$ is **proper** of $L_p(|X_\bullet|) \hookrightarrow X_p$ is a Hurewicz cofibration. This is implied by each $s_j(X_{p-1}) \hookrightarrow X_p$ being a Hurewicz cofibration. Such $X_\bullet$ are called "good".

Example 13.14. If G is well-pointed then $B_\bullet(X, G, Y)$ is good. This is because

$$X \times G^{i-1} \times \{e\} \times G^{p-i} \times Y \hookrightarrow X \times G^p \times Y$$

is a cofibration.

The conclusion is that the inclusion

$$\mathrm{sk}_{p-1}(B_\bullet(X, G, Y)) \hookrightarrow \mathrm{sk}_p(B_\bullet(X, G, Y))$$

has a NDR-pair structure. This can be used to show that $EG \to BG$ has local sections. So it is going to be locally trivial and numerable. The upshot is the if G is well-pointed, then $EG \to BG$ is a numerable principal G-bundle.

Now we should map into BG. The easiest way to do this is to realize a map $X_\bullet \to B_\bullet(*, G, *)$.

Definition 13.15. If $\mathcal{U} = \{U_i\}$ is an open cover of B, we define $N_\bullet\mathcal{U}$ to be the simplicial space with p-simplices given by

$$\coprod_{(i_0,\ldots,i_p)} U_{i_0} \cap \cdots \cap U_{i_p}$$

with duplicates allowed.

There is a simplicial map $N_\bullet\mathcal{U} \to \mathrm{const}_\bullet(B)$ where p-simplices in $\mathrm{const}_\bullet(B)$ are identities. Then we get $|N_\bullet\mathcal{U}| \to |\mathrm{const}_\bullet(B)| \cong B$.

Lemma 13.16. *If $\mathcal{U}$ is numerable, then $|N_\bullet\mathcal{U}| \to B$ is a homotopy equivalence.*

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Last time we had the bar construction $|B_\bullet(X,G,Y)|$. Last time we stated that if $\{e\} \to G$ is a cofibration, then $EG = B(*,G,G) \to BG = B(*,G,*)$ is a numerable principal G-bundle γ. We wanted to show that this is universal:

$$[-,BG] \xrightarrow{\cong} \mathrm{Prin}_G^{\mathrm{num}}(-); \quad f \mapsto f^*\gamma.$$

14.1 Bar construction of the classifying space

First we show that this is surjective. Given $\xi = (E,B,p)$, we want to find $g : B \to BG$ such that $g^*\gamma \cong \xi$. To map into BG, we want to write B as a geometric realization of a simplicial space, and map this into $B_\bullet(*,G,*)$.

Given $\mathcal{U} = \{U_i\}$ an open cover of B, we defined

$$N_\bullet\mathcal{U} : [p] \mapsto \amalg_{(i_0,\dots,i_p)} U_{i_0} \cap \cdots \cap U_{i_p}.$$

Lemma 14.1. *If $\mathcal{U}$ is numerable, then $|N_\bullet\mathcal{U}| \to B$ is a homotopy equivalence.*

Proof. Pick a partition of unity $\{\eta_i : U_i \to [0,1]\}$ and an ordering on the indexing set I of $\mathcal{U}$. There is a map

$$B \to |N_\bullet\mathcal{U}|,$$

given by, for each $b \in B$ look at the finitely many $j_0(b),\dots,j_p(b)$ with $\eta_j(b)$ nonzero and then looking at this coordinate inside the simplex.

$$s : b \mapsto ((\eta_{j_0}(b),\dots,\eta_{j_p}(b)), ((j_0(b),\dots,j_p(b)), b))$$

Then $q \circ s = \mathrm{id}$ and $s \circ q \simeq \mathrm{id}_{|N_\bullet\mathcal{U}|}$ by linear interpolation. □

If ξ trivializes over $\mathcal{U}$, there is a simplicial map $\Phi_\bullet : N_\bullet\mathcal{U} \to B_\bullet(*,G,*)$ that is defined on $U_{i_0} \cap \cdots \cap U_{i_p}$ by

$$b \mapsto (g_{i_0 i_1}(b),\dots,g_{i_{p-1} i_p}) \in G^p.$$

The cocycle conditions tell us that these are compatible with face and degeneracy maps.

Proposition 14.2. *$|\Phi_\bullet|^*\gamma \cong \varphi^*\xi$, and so $s^*|\Phi_\bullet|^*\gamma \cong \xi$.*

Proof. Note that both $|\Phi_\bullet|^*\gamma$ and $q^*\xi$ are realizations of simplicial principal G-bundles

$$[p] \mapsto \coprod_{(i_0,\dots,i_p)} U_{i_0} \cap \cdots \cap U_{i_p} \times G \cong \coprod_{(i_0,\dots,i_p)} \xi|_{U_{i_0}\cap\cdots\cap U_{i_p}},$$

where the two are isomorphic by the trivialization. □

We have shown that $[-,BG] \to \mathrm{Prin}_G^{\mathrm{num}}(-)$ is surjective.

Proposition 14.3. *EG is contractible.*

Proof. The point is that $B_\bullet(*, G, G)$ has an extra degeneracy map. For instance, $(*, g_1, g_2) \in B_1(*, G, G)$ has $s_0 = (*, e, g_1, g_2)$ and $s_1 = (*, g_1, e, g_2)$ and also $(*, g_1, g_2, e)$. Then we can contract everything to the $*$ side. $\square$

Corollary 14.4. *ΩBG is weakly equivalent to G.*

For injectivity, let $f : B \to BG$ classify $\xi = (E, B, p)$. This is the same as a G-equivariant map $E \to EG$.

Proposition 14.5. *This is the same as a section of $E \times_G EG \to B$.*

Proof. For the trivial bundle $E = B \times G$, we note that

$$\mathrm{Map}_G(E, EG) = \mathrm{Map}(B, EG) = \Gamma_B(B \times EG) = \Gamma_B(E \times_G EG).$$

Now note that $\mathrm{Map}_G(p^{-1}(-), EG)$ and $\Gamma_B(p^{-1}(-) \times_G EG)$ are sheaves of sets on B. $\square$

So given two f_0, f_1 classifying maps, this is the same as two sections s_{f_0}, s_{f_1} of $E \times_G EG \to B$. But note that this is a trivial Hurewicz fibration. So we have the lifting for

$$\begin{array}{ccc} \{0,1\} \times B & \longrightarrow & E \times_G EG \\ \downarrow & \nearrow & \downarrow \\ I \times B & \longrightarrow & B. \end{array}$$

This gives a homotopy between f_1 and f_2.

Theorem 14.6. *$\gamma = (EG, BG)$ is a universal numerable principal bundle.*

14.2 Bar spectral sequence

We can then use the skeletal filtration to interpret $H_*(BG)$. Then we have

$$\mathrm{sk}_{p-1}(BG) \hookrightarrow \mathrm{sk}_p(BG) \to \Delta^p/\partial\Delta^p \wedge G^{\wedge p}.$$

Then we get a spectral sequence with E^1-page

$$E^1_{p,q} = \tilde{H}_{p+q}(\Delta^p/\partial\Delta^p \wedge G^{\wedge p}) \Rightarrow \tilde{H}_{p+q}(BG).$$

If G is discrete, only $q = 0$ will be nonzero, and the groups will be $\mathbb{Z}[G \setminus \{e\}]^{\otimes p}$. The chain complex is going to be

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G] \otimes \mathbb{Z}[G \setminus \{e\}]^{\otimes p})$$

which is the free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module. Then the E^2-page is

$$\mathrm{Tor}_p^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = H_p^{\mathrm{group}}(G).$$

The conclusion is that $H_*(BG)$ is isomorphic to the group homology of G.

In general, (if $X_\bullet$ is proper) we have

$$E^1_{p,q} = H_q(X_p) \Rightarrow H_{p+q}(|X_\bullet|)$$

with d^1-differential given by $\sum(-1)^i(d_i)_*$. For $X_\bullet = B_\bullet(X, G, Y)$ with $\{e\} \hookrightarrow G$ a cofibration and coefficients given in a field $\mathbb{F}$, we have the **bar spectral sequence**

$$E^2_{p,q} = \operatorname{Tor}_p^{H_*(G;\mathbb{F})}(H_*(X;\mathbb{F}), H_*(Y;\mathbb{F})) \Rightarrow H_{p+q}(B(X, G, Y); \mathbb{F}).$$

14.3 Vector bundles

Whenever you have a left G-space X and a principal G-bundle $\xi = (E, B, p)$, you can form an **associated bundle** $\tilde{p} : E \times_G X \to B$. This is locally trivial with fiber X. You can think of this has having transition functions in G, in the sense that there are maps $g_{ij} : U_i \cap U_j \to G$ such that

$$X \times (U_i \cap U_j) \xrightarrow{\tilde{\varphi}_i} \tilde{p}^{-1}(U_i \cap U_j) \xrightarrow{\tilde{\varphi}_j^{-1}} X \times (U_i \cap U_j)$$

is given by $(x, b) \mapsto (g_{ij}(b) \cdot x, b)$.

A **real n-dimensional vector bundle** is the case $G = \mathrm{GL}_n(\mathbb{R})$ and $X = \mathbb{R}^n$. Then we get a locally trivial $\mathbb{R}^n$-bundle with transition functions in $\mathrm{GL}_n(\mathbb{R})$. If G acts effectively on X, then you can recover ξ from the associated bundle. So an n-dimensional vector bundle is the same as a principal $\mathrm{GL}_n(\mathbb{R})$-bundle. This can be explicitly described as the frame bundle $\mathrm{Fr}^{\mathrm{GL}}(\xi)$.

Therefore numerable vector bundles are classified by $B\,\mathrm{GL}_n(\mathbb{R})$. Note that $\mathrm{O}(n) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ is a homotopy equivalence by Gram–Schmidt. This shows that

$$B\,\mathrm{O}(n) \simeq_w B\,\mathrm{GL}_n(\mathbb{R}).$$

There is an explicit model for $B\,\mathrm{O}(n)$. Consider $V_n(\mathbb{R}^N)$ the space of injective linear maps $\mathbb{R}^n \hookrightarrow \mathbb{R}^N$, which is called the **Stiefel manifold**. Then we have a fiber sequence

$$S^{N-n} \to V_n(\mathbb{R}^N) \to V_{n-1}(\mathbb{R}^N).$$

By induction, we see that $V_n(\mathbb{R}^N)$ is $N-n$-connected, and so $V_n(\mathbb{R}^\infty)$ is weakly contractible. Then we can look at the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^\infty) = V_n(\mathbb{R}^\infty)/\mathrm{GL}_n(\mathbb{R})$. Then we get

$$\begin{array}{ccc}
\mathrm{GL}_n(\mathbb{R}) & \xrightarrow{\cong} & \mathrm{GL}_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
V_n(\mathbb{R}^\infty) & \xrightarrow{\simeq_w} & E\,\mathrm{GL}_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_n(\mathbb{R}^\infty) & \longrightarrow & B\,\mathrm{GL}_n(\mathbb{R}).
\end{array}$$

This makes it easier to make things to compute. For instance, we can use differential topology, and also try to use cellular decompositions.

15 March 20, 2018

For the next two weeks, we will accelerate towards proving $\Omega^0_*(*) \cong \pi_* MO$. To prove this, one has to know about the homology on MO. From now on, all vector bundles will be numerable.

15.1 Characteristic classes

Let us fix a commutative ring k. We want a natural invariance of n-dimensional vector bundles over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, valued in $H^*(-;k)$.

Definition 15.1. An $H^k(-;k)$-valued **characteristic class** c of n-dimensional $\mathbb{F}$-vector bundles is an assignment of $c(\zeta) \in H^k(B;k)$ to each n-dimensional $\mathbb{F}$-vector bundle $\zeta = (E, B, p)$ satisfying $c(f^*\zeta) = f^*c(\zeta)$.

Then there is an bijection

$$\left\{\begin{matrix} H^k(-,k)\text{-valued characteristic classes} \\ \text{of } n\text{-dimensional } \mathbb{F}\text{-vector bundles}\end{matrix}\right\} \quad \longleftrightarrow \quad H^k(BGL_n(\mathbb{F}); k).$$

This is because the characteristic class of the universal bundle determines everything. If $\mathbb{F} = \mathbb{R}$, then $BGL_n(\mathbb{R}) \simeq BO_n(\mathbb{R})$.

15.2 Gysin sequence and Thom isomorphism

Let us start with the easiest case $\mathbb{C}P^\infty \simeq BU(n)$. Let us fix a spectrum E.

Definition 15.2. An **E-orientable spherical fibration of dimension** n is a Serre fibration $p : X \to B$ where B is path-connected and fiber $F \simeq S^n$ such that the $\pi_1(B)$-action on $E^*(F)$ is trivial.

If $E = Hk$ is the Elienberg–Mac Lane spectrum, we have the cohomological Serre spectral sequence. This is going to look like two rows of $H^p(B;k)$ converging to $H^{p+q}(X;k)$. The only nontrivial trivial differential is d^{n+1}, and it is going to be given by

$$d^{n+1}(x) = e \smile x,$$

where $e = d^{n+1}(1)$ is called the **Euler class**. If we stitch together the short exact sequence, we get a long exact sequence called the **Gysin sequence**:

$$\cdots \to H^k(B;k) \xrightarrow{p^*} H^k(X;k) \to H^{k-n}(B;k) \xrightarrow{e\smile -} H^{k+1}(B;k) \to B^{k+1}(X;k) \to \cdots.$$

Example 15.3. Consider $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, given by quotienting out by the U(1)-action on unit vectors in $\mathbb{C}^{n+1}$. This is Hk-orientable because $\mathbb{C}P^n$ is simply connected.

Proposition 15.4. *$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[c_1]/(c_1^{n+1})$ for some $c_1 \in H^2(\mathbb{C}P^n)$. Also, for $0 < m < n$, the natural map $\mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ pulls back c_1 to c_1.*

Proof. We have a Gysin sequence

$$\to H^k(\mathbb{C}P^n) \to H^k(S^{2n+1}) \to H^{k-1}(\mathbb{C}P^n) \xrightarrow{-\smile c_1} H^{k+1}(\mathbb{C}P^n) \to \cdots.$$

So for $0 < k < 2n$, we get that $H^{k-1}(\mathbb{C}P^n) \to H^{k+1}(\mathbb{C}P^n)$ given by $- \smile c_1$ is an isomorphism. It follows that $H^{2j}(\mathbb{C}P^n) \cong \mathbb{Z}\{c_1^j\}$ and $H^{2j+1}(\mathbb{C}P^n) \cong 0$ for $2j+1 \leq 2n+1$. The rest vanishes because $\mathbb{C}P^n$ is $2n$-dimensional. Naturality of c_1 follows from the naturality of the Serre spectral sequence. The Gysin sequence is natural, and we can apply this to

$$\begin{array}{ccccc} S_1 & \longrightarrow & S^{2m+1} & \longrightarrow & \mathbb{C}P^m \\ \downarrow\cong & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^{2n+1} & \longrightarrow & \mathbb{C}P^n. \end{array}$$

□

Since $H^*(\mathbb{C}P^n) \twoheadrightarrow H^*(\mathbb{C}P^m)$ is surjective if $m < n$, the $\lim^1$ term vanishes. This shows that

$$H^*(\mathbb{C}P^\infty) = H^*(\operatorname{colim}_n \mathbb{C}P^n) \cong \lim_n H^*(\mathbb{C}P^n) \cong \mathbb{Z}[[c_1]].$$

A slightly different application is when there is a section $s : B \to X$ of p. Then we can denote its image by X_∞, and we get a relative fibration $(S^n, *) \to (X, X_\infty) \to B$. Now we can run a relative cohomological Serre spectral sequence.

Theorem 15.5 (Thom isomorphism)**.** *There is an isomorphism $H^k(B; k) \to H^{k+n}(X, X_\infty; k)$ given by $p^*(-) \smile u$, where $u \in H^n(X, X_\infty; k)$ is the **Thom class**.*

For example, if $\zeta = (E, B, p)$ is an n-dimensional vector bundle, then we can look at the fiberwise one-point compactification $(\mathrm{Fr}^{\mathrm{GL}}(\zeta) \times_{\mathrm{GL}_n(\mathbb{F})} (\mathbb{F}^n)^+)$. This is going to be a spherical fibration of dimension n (or $2n$) with canoncial section s_∞ given by that added point at ∞. In this case, $X_\infty \hookrightarrow X$ is a Hurewicz cofibration, so $H^n(X, X_\infty; k) \cong \tilde{H}(X/X_\infty; k)$. This space $\mathrm{Th}(\zeta) = X/X_\infty$ is called the **Thom space** of ζ.

Here is another way you can construct it. If we pick a metric on ζ and $D(\zeta)$ the unit disk bundle and $S(\zeta)$ the unit sphere bundle, then

$$D(\zeta)/S(\zeta) \cong \mathrm{Th}(\zeta).$$

We can recover the Gysin sequence for $S(\zeta) \to B$ by

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^*(D(\zeta); k) & \longrightarrow & H^*(S(\zeta); k) & \longrightarrow & H^{*+1}(D(\zeta), S(\zeta); k) & \longrightarrow & \cdots \\ & & \cong\uparrow & & \cong\uparrow & & \cong\uparrow \text{Thom} & & \\ \cdots & \longrightarrow & H^*(B; k) & \longrightarrow & H^*(S(\zeta); k) & \longrightarrow & H^{*-n+1}(B; k) & \longrightarrow & \cdots. \end{array}$$

15.3 Chern and Stiefel–Whitney classes

We will use this idea in a more general setting.

Theorem 15.6 (Leray–Hirsch)**.** *Let E be a homotopy commutative ring spectrum, and for convenience assume that E is bounded below. Suppose we have a Serre fibration $p : X \to B$, where B is path-connected and the fiber is F, with the inclusion $i : F \hookrightarrow X$ such that there exist $x_1, \dots, x_r \in E^*(X)$ such that $i^*(x), \dots i^*(x_r)$ are the basis of $E^*(F)$ as a $E^*(\mathrm{pt})$-module. Then*

$$E^*(B) \otimes_{E^*(\mathrm{pt})} E^*(\mathrm{pt})\{z_1, \dots, z_n\} \to E^*(X); \quad y \otimes z_i \mapsto p^*(y) \smile x_i$$

is an $E^*(B)$-module isomorphism.

E being bounded below is not a great condition, so you sometimes want to replace this by other conditions that guarantee convergence.

Proof. Define $F^*(-) = E^*(-) \otimes_{E^*(\mathrm{pt})} E^*(\mathrm{pt})\{z_1, \dots, z_r\}$. This is a new generalized cohomology theory because we still have long exact sequences and the wedge axioms is satisfied since it is finitely presented. Then we get two exact couples $D^1_{p,q} = F^{p+q}(B_p)$, $E^1_{p,q} = \prod_{I_p} F^q(b_i)$ and $\tilde{D}^1_{p,q} = E^{p+q}(X_p)$, $\tilde{E}^1_{p,q} = \prod_{I_p} E^q(p^{-1}(b_i))$. There is a map of exact couples, given by $x \otimes z_i \mapsto p^*(x) \otimes x_i$, and it is going to be an isomorphism on E^1. So it is an isomorphism on E^∞, and $F^*(B) \cong E^*(X)$. □

There is also a relative version. If we define "fiberwise Thom class" to be $u \in E^*(X, X_\infty)$, such that the restriction to each fiber is a generator of it as a $E^*(\mathrm{pt})$-module, then the fiberwise Thom theorem gives a Thom isomorphism $E^*(B) \to E^*(X, X_\infty)$. So we get back the Gysin sequence in the case of a spherical fibration arising as a unit sphere bundle in a vector bundle.

We can also apply this to $\lambda_n \to \mathbb{R}P^n$ the tautological line bundle. Then we have $S^0 \to S^{2n+1} \to \mathbb{R}P^n$. Fiberwise Thom classes always exist for real vector bundles in $H^*(-;\mathbb{F}_2)$, so we have the Gysin sequence by the same argument as for $\mathbb{C}P^n$.

Proposition 15.7. *$H^*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2[w_1]/(w_1^{n+1})$ for $w_1 \in H^1(\mathbb{R}P^n;\mathbb{F}_2)$, and this is compatible with inclusions $\mathbb{R}P^m \hookrightarrow \mathbb{R}P^n$, and thus $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[[w_1]]$.*

We also have $\lambda_n \to \mathbb{C}P^n$ the tautological line bundle.

Theorem 15.8. *Suppose we have $E^*(-)$ such that $E^*(\mathbb{C}P^n) \cong E^*(\mathrm{pt})[c_1]/(c_1^{n+1})$ such that $c-1 \in E^2(\mathbb{C}P^n)$ is compatible with inclusions. (In particular $E^*(\mathbb{C}P^\infty) = E^*(\mathrm{pt})[[c_1]]$.) Then for each $\mathbb{C}$-vector bundle $\zeta = (E, B, p)$ there are classes $c_i(\zeta) \in E^{2i}(B)$ called **Chern classes**, uniquely determined by*

(i) $c_i(\zeta) = 0$ *for* $i > \dim_{\mathbb{C}} \zeta$,

(ii) $c_i(f^*\zeta) = f^* c_i(\zeta)$,

(iii) $c_0(\zeta) = 1$,

(iv) $c_1(\lambda_n) = c_1 \in E^2(\mathbb{C}P^n)$,

(v) if we define $c(\zeta) = \sum_i c_i(\zeta)$ *then* $c(\zeta \oplus \eta) = c(\zeta) \smile c(\eta)$.

The proof uses the **splitting principle**. If $\zeta = (E, B, p)$ is an n-dimensional $\mathbb{C}$-vector bundle, we can define the total space $P(\zeta)$ to be the projectivization of ζ, so that $\mathbb{C}P^{n-1} \to P(\zeta) \xrightarrow{\pi} B$. This is the space of (b, ℓ) with $b \in B$ and ℓ a $\mathbb{C}$-line in $p^{-1}(b)$. This has the property that $\pi^*\zeta$ splits off a line bundle $\lambda_{P(\zeta)} \subseteq \pi^*\zeta$ given by $((b, \ell), v)$ such that $v \in \ell$. Let us write

$$\pi^*(\zeta) = \lambda_{\pi(\zeta)} \oplus \mu_{\pi(\zeta)}.$$

Now we have a line bundle over $P(\zeta)$, so there is a classifying map $f : P(\zeta) \to \mathbb{C}P^\infty$ for $\lambda_{\pi(\zeta)}$.

$$\begin{array}{ccccc} \mathbb{C}P^{n-1} & \overset{j}{\hookrightarrow} & P(\zeta) & \xrightarrow{f} & \mathbb{C}P^\infty \\ & & \downarrow & & \\ & & B. & & \end{array}$$

The construction makes it clear that $j^*\lambda_{\pi(\zeta)} \cong \lambda_{n-1}$, so $f \circ j$ is homotopic to the standard inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^\infty$. If we define $y \in E^2(P(\zeta))$ to be f^*c_1, then $1, y, y^2, \ldots, y^{n-1}$ pull back along j to $1, c_1, \ldots, c_1^{n-1}$ on $E^*(\mathrm{pt})$-basis of $E^*(\mathbb{C}P^{n-1})$. Then Leray–Hirsch tells us that

$$E^*(P(\zeta)) \cong E^*(B) \otimes_{E^*(\mathrm{pt})} E^*(\mathrm{pt})\{1, y, \ldots, y^{n-1}\}.$$

In particular, π^* is injective.

Proof. Let us first show uniqueness, by induction over n the dimension of the vector bundles. For $n = 0$, (i) and (iii) determines it. For $n = 1$, (iv) says that $c_1(\lambda_n = c_1 \in E^*(\mathbb{C}P^{n-1})$ and (ii) shows that $c_1(\lambda_\infty) = c_1 \in E^*(\mathbb{C}P^\infty)$. Now any $\mathbb{C}$-line bundle is classified by a map $B \to \mathbb{C}P^\infty$ and then (ii) determines it.

Now let us do the induction step. If ζ is an n-dimensional bundle, we construct $P(\zeta)$. Then

$$\pi^*c(\zeta) = c(\pi^*\zeta) = c(\lambda_{P(\zeta)})c(\mu_{P(\zeta)}).$$

Then because π^* is injective, $c(\zeta)$ is uniquely determined. □

16 March 22, 2018

Today we are going to define $\Omega_*^0(*)$ and generalize it to $\Omega_*^0(X)$. Then we are going to show that this is isomorphic to $\pi_*(MO \wedge X_+)$, which is the Pontryagin–Thom theorem.

16.1 Computation of $H^*(BU(n))$ and $H^*(BO(n);\mathbb{F}_2)$

Last time we had the following theorem.

Theorem 16.1. *If E^* is such that $E^*(\mathbb{C}P^n) \cong E^*(\mathrm{pt})[c_1]/(c_1^{n+1})$ for $c_1 \in E^2(\mathbb{C}P^n)$ and is compatible with inclusion (so that $E^*(\mathbb{C}P^\infty) = E^*(\mathrm{pt})[[c_1]]$), then for each $\mathbb{C}$-vector bundle $\zeta = (E, B, p)$ there exist $c_i(\zeta) \in E^{2i}(B)$ uniquely determined by*

(i) $c_i(\zeta) = 0$ *if* $i > \dim_{\mathbb{C}} \zeta$,

(ii) $f^* c_i(\zeta) = c_i(f^*\zeta)$,

(iii) $c_0(\zeta) = 1$,

(iv) λ_n *over* $\mathbb{C}P^n$ *has* $c_1(\lambda_n) = c_1$,

(v) the total Chern class $c = \sum_i c_i(\zeta)$ *satisfies* $c(\zeta \oplus \eta) = c(\zeta) \smile c(\eta)$.

Last time we showed uniqueness, by using the splitting principle. If you take $P(\zeta) \to B$, the pullback of ζ splits out a line canonically. Now we need to show that these guys exist.

Proof. For $\dim_{\mathbb{C}} \zeta = 0$, we have this by (iii). For ζ an n-dimensional bundle, we have unique elements $c_1(\zeta), \dots, c_n(\zeta)$ such that

$$0 = \sum_{i=0}^{n} c_i(\zeta)(-y)^{n-i}.$$

This is because $E^*(P(\zeta)) = E^*(B)\{1, y, \dots, y^{n+1}\}$ by Leray–Hirsch, where y is the pullback of $c_1 \in E^2(\mathbb{C}P^\infty)$ along the map $P(\zeta) \to \mathbb{C}P^\infty$ classifying $\lambda_{P(\zeta)}$. (iv) is satisfied because our equation is $0 = c_1(\lambda_n) - c_1 = 0$. (i) is true by construction, and (ii) follows from naturality of the equation. The nontrivial thing is (v). We have that $P(\zeta \oplus \eta)$ is covered by $U \simeq P(\zeta)$ and $V \simeq P(\eta)$. So if you write down the element

$$x_\zeta = \sum_{i=0}^{n} c_i(\zeta)(-y)^{n}, \quad x_\eta = \sum_{i=0}^{m} c_i(\eta)(-y)^{m-i},$$

they might not be zero but $x_\xi x_\eta$ does. Then we only note that $c_j(\zeta \oplus \eta)$ is the coefficient of $(-y)^{n+m-j}$ of $0 = (\sum_i c_i(\zeta)(-y)^{n-i})(\sum_i c_i(\eta)(-y)^{m-i})$. □

Similarly, for real vector bundles and $H^*(-;\mathbb{F}_2)$, we have the following.

Theorem 16.2. *Let E^* be such that $E^*(\mathbb{R}P^n) \cong E^*(\mathrm{pt})[w_1]/(w_1^{n+1})$ with $w_1 \in E^1(\mathbb{F}P^n)$, satisfying compatibility with inclusions. Then there are $w_i(\xi) \in E^i(B)$ uniquely determined by (i), (ii), (iii), (v) and $w_1(\lambda_n) = w_1 \in E^1(\mathbb{R}P^n)$.*

Proposition 16.3. *$H^*(BU(n)) \cong \mathbb{Z}[[c_1, \dots, c_n]]$ and $H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[[w_1, \dots, w_n]]$, compatibly with inclusions.*

So $H^*(BU) = \mathbb{Z}[[c_1, c_2, \dots]]$ and $H^*(BO; \mathbb{F}_2) = \mathbb{F}[[w_1, w_2, \dots]]$.

Proof. Let us prove this by induction on n. We have the fiber sequence $S^{2n-1} \to BU(n-1) \to BU(n)$, and so we have the Gysin sequence

$$\cdots \to H^k(BU(n)) \xrightarrow{i^*} H^k(BU(n-1)) \to H^{k-2n+1}(BU(n)) \to H^{k+1}(BU(n)) \to \cdots.$$

But note that $H^k(BU(n)) \to H^k(BU(n-1))$ is surjective because given any vector bundle, you can add a trivial vector bundle. Sow we get short exact sequences

$$0 \to H^{k-2n}(BU(n)) \xrightarrow{e \smile -} H^k(BU(n)) \to H^k(BU(n-1)) \to 0.$$

This shows that the kernel of i^* is generated by e. So it suffices to show that $c_n = \pm e$. Clearly c_n is in the kernel of i^*, so $c_n = \alpha e$.

To compute this constant α, we look at the map

$$\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to BU(n); \quad \lambda_\infty \oplus \cdots \oplus \lambda_\infty.$$

The total Chern class of $(1 + c_1^{(1)}) \cdots (1 + c_1^{(n)})$ and so $c_n(\lambda_\infty \oplus \cdots \oplus \lambda_\infty) = c_1^{(1)} \cdots c_1^{(n)}$. This should be divisible by α, so we have $\alpha = \pm 1$. That is, $H^*(BU)) = H^*(BU(n-1))[[c_n]]$. ☐

16.2 Bordism groups

Definition 16.4. Let M_0, M_1 to be n-dimensional closed manifolds. Then a **cobordism** between them is an $(n+1)$-dimensional compact manifold W with the identification $\partial W \cong M_0 \amalg M_1$. If a cobordism exists, M_0 and M_1 are said to be **bordant**.

From a technical point of view, this is not a great definition.

Lemma 16.5. *This is an equivalence relation.*

Proof. $M_0 \sim M_0$ by $M_0 \times I$. If $M_0 \sim M_1$ by W, then you can reverse this to get $M_1 \sim M_0$. If $M_0 \sim M_1$ by W and $M_1 \sim M_2$ by W', we can take $W \cup_{M_1} W'$. Technically, you might want to include collar neighborhoods in the definition, in the case when the manifolds are required to be smooth. ☐

Definition 16.6. We define the group $\Omega_n^0(*)$ as the set of closed n-dimensional manifolds up to bordism.

Here, disjoint union makes this into an abelian group.

Example 16.7. What is $\Omega_n^0(*)$? If $n = 0$, this is just $\mathbb{Z}/2\mathbb{Z}$ because manifolds are points, and all 1-manifolds with boundary have an even number of boundary points. For $n = 1$, all 1-dimensional manifolds are disjoint unions of circle, so they are bordant to $\emptyset$. So $\Omega_1^0(*) = 0$.

For $n = 2$, classification of surfaces show that all closed surfaces are disjoint unions of connected sums of T^2 and $\mathbb{R}P^2$. Also, note that $\#$ is bordant to $\amalg$. Also, T^2 is null-bordant, so we only have to think about $\mathbb{R}P^2$. Clearly, $2[\mathbb{R}P^2] = 0$. But $\mathbb{R}P^2$ is not bound, because

$$\langle w_2(T\mathbb{R}P^2), [\mathbb{R}P^2]\rangle = 1$$

is invariant under bordism. If $\mathbb{R}P^2 = \partial W$, then $H_3(W, \partial W; \mathbb{F}_2) \to H_2(\partial W; \mathbb{F}_2)$ sends $[W, \partial W]$ to $[\partial W]$. Also, $T\partial W \oplus \epsilon = TW|_{\partial W}$. This tells you that

$$w(T\partial W) = i^* w(TW)$$

where $i : \partial W \hookrightarrow W$. Then

$$\langle w_2(T\partial W), [\partial W]\rangle = \langle i^* w_2(TW), \delta[W, \partial W]\rangle = \langle \bar{\delta} i^* w_2(TW), [W, \partial W]\rangle = \langle 0, [W, \partial W]\rangle = 0$$

because $\bar{\delta} i^*$ are two adjacent terms in the long exact sequence. Anyways, this shows that $\Omega_2^0(*) \cong \mathbb{Z}/2\mathbb{Z}$.

There are other versions of bordism:

- oriented bordism, where you have an orientation,
- framed bordism, where there is a trivialization on the tangent bundle,
- complex bordism, where there is an almost complex structure on the tangent bundle or that plus a trivial line bundle.

16.3 Bordism as a homology theory

Definition 16.8. We define $\Omega_n^0(X)$ as the set of closed n-dimensional manifolds M with $f : M \to X$, with bordism with maps to X.

Now note that $[M, f]$ only depends on the homotopy class of f, because if f_0, f_1 are homotopy via H, so that $H : M \times I \to X$, then this is a bordism from (M, f_0) to (M, f_1).

Example 16.9. Now, $\Omega_0^0(X)$ are points labeled by $\pi_0(X)$, and you can cancel only points in the same path component. So $\Omega_0^0(X) \cong \mathbb{Z}/2\mathbb{Z}\{\pi_0 X\}$.

This group is functorial in X. If you have a map $g : X \to Y$, then composition with g gives a homomorphism $f_* : \Omega_n^0(X) \to \Omega_n^0(Y)$ sending $[M, f] \mapsto [M, g \circ f]$. So this is a functor

$$\Omega_n^0(-); \quad \mathrm{Ho}(\mathsf{Top}) \to \mathsf{Ab}.$$

But we need a relative version to get a homology theory. We define

$$\Omega_n^0(X,A) = \left\{ \begin{matrix} n\text{-dimensional compact manifolds } M \\ \text{with } f: M \to X \text{ such that } f(\partial M) \subseteq A \end{matrix} \right\} \Bigg/ \begin{matrix} \text{bordism of manifolds with boundary with} \\ \text{map to } X \text{ such that boundary bordism maps to } A. \end{matrix}$$

Here, a this bordism of manifolds with boundary is going to be a manifold with corners. There is going to be a bordism between the boundaries. Now there is a natural transformation

$$\Omega_n^0(X,A) \xrightarrow{\partial_n} \Omega_{n-1}^0(A); \quad [M,f] \mapsto [\partial M, f|_{\partial M}].$$

Proposition 16.10. *The data of $\Omega_n^0(-) : \mathrm{Ho}(\mathsf{Top}) \to \mathsf{Ab}$ and ∂^n form a generalized homology theory satisfying the weak homotopy equivalence axioms and the wedge axiom.*

Proof. We have to check that if (X,A) is an excisive pair, we get a long exact sequence

$$\cdots \to \Omega_n^0(A) \to \Omega_n^0(X) \xrightarrow{h} \Omega_n^0(X,A) \xrightarrow{\partial} \Omega_{n-1}^0(A) \to \cdots.$$

Let's only check exactness at h and ∂. Clearly $\mathrm{im}(h) \subseteq \ker(\partial)$ because things in $\Omega_n^0(X)$ don't have boundary. For the converse, suppose that $[M,f] \in \ker\partial$. Then what you can do is to find some manifold that has $\bar{\partial}V$ as the boundary. Then we have $[M,f] = [M \cup_{\partial M} W, \tilde{f}]$ where W is this boundary.

For the weak homotopy equivalence axiom, it suffices to show that g induces a bijection on $[M,X] \xrightarrow{g \smile -} [M,Y]$. But this follows from Whitehead's theorem and Morse theory which says that every compact manifold has the homotopy type of a CW-complex. For the wedge axiom, it suffices to show that $M \to \bigvee_i X_i$ only hits finitely many X_i not in the base point. This follows from compactness of M. □

Next time, we are going to show that $\Omega_*^0(X) \cong \pi_*(MO \wedge X_+)$ although we don't have Brown representability for homology theories. Here, MO as a pre-spectrum can be described as

$$MO_k = \mathrm{Th}(\gamma_k)$$

where γ_k is the universal bundle over $BO(k) \cong \mathrm{Gr}_k(\mathbb{R}^\infty)$. Then we have

$$\mathrm{Gr}_k(\mathbb{R}^\infty) \to \mathrm{Gr}_{k+1}(\mathbb{R} \oplus \mathbb{R}^\infty) \cong \mathrm{Gr}_{k+1} 9\mathbb{R}^\infty); \quad P \mapsto \mathrm{span}(e_1) \oplus P,$$

and then $\Sigma MO_k = \Sigma\,\mathrm{Th}(\gamma_k) \cong \mathrm{Th}(\epsilon \oplus \gamma_k)$.

17 March 27, 2018

Last time we defined $\Omega_n^0(X)$ as n-dimensional closed unoriented M with $g : M \to X$, up to cobordism. We will show that $\Omega_n^0(X) \cong \pi_n(MO \wedge X_+)$, and compute $\pi_n(MO) \cong \Omega_n^0(X)$ using algebraic topology.

17.1 Pontryagin–Thom map

We are first going to describe a map $\Omega_n^0(X) \to \pi_n(MO \wedge X_+)$. Consider

$$BO(k) \simeq \mathrm{Gr}_k(\mathbb{R}^\infty) = \operatorname*{colim}_{N\to\infty} \mathrm{Gr}_k(\mathbb{R}^N).$$

This has a canonical universal bundle γ_k with total space given by $(P, \vec{v}) \in \mathrm{Gr}_k(\mathbb{R}^\infty) \times \mathbb{R}^\infty$ such that $\vec{v} \in P$. Using the restricted Riemannian metric, we can define a unit disk bundle $D(\gamma_k) = \{(P, \vec{v}) : \|\vec{v}\| \le 1\}$ and a unit sphere bundle $S(\gamma_k) = \{(P, \vec{v}) : \|\vec{v}\| = 1\}$. Then we can consider the pointed **Thom space**

$$\mathrm{Th}(\gamma_k) = D(\gamma_k)/S(\gamma_k).$$

To get a pre-spectrum, we need to give $\Sigma MO_k \to MO_{k+1}$. The idea is that if we take look at the map that is adding a trivial bundle, we get

$$\begin{array}{ccc} \epsilon \oplus \gamma_k & \longrightarrow & \gamma_{k+1} \\ \downarrow & & \downarrow \\ BO(k) & \xrightarrow{\epsilon\oplus -} & BO(k+1). \end{array}$$

Then we get

$$\Sigma\,\mathrm{Th}(\gamma_k) \cong \mathrm{Th}(\epsilon \oplus \gamma_k) \to \mathrm{Th}(\gamma_{k+1}).$$

This can be described also as $(t, (P, \vec{v})) \mapsto (\epsilon \oplus P, t\vec{e}_1 + \vec{v})$ where we view S^1 as $\mathbb{R}$.

Proposition 17.1. *Any closed manifold M can be embedded in some $\mathbb{R}^N$, unique up to isotopy if you're allowed to increase N.*

Proof. Because M is compact, there are finitely many charts $\varphi_i : \mathbb{R}^n \hookrightarrow M$. Pick a smooth partition of unity $\lambda_i : M \to [0, 1]$ for $1 \le i \le r$, subordinate to $\varphi_i(\mathbb{R}^n)$. Then we can explicitly write down the embedding as

$$m \mapsto (\lambda_1(m), \lambda_1(m)\varphi_1^{-1}(m), \dots, \lambda_r(m), \lambda_r(m)\varphi_r^{-1}(m)).$$

To get a isotopy, you want to extend a cylinder, and you can do the same thing on a cylinder. ☐

Using transversality and the Whitney trick, you can embed into $\mathbb{R}^{2n}$. Using Stiefel–Whitney classes, you can show that this is optimal. If $\mathbb{R}P^n$ immerses into $\mathbb{R}^{2n-2}$, then immersions have normal bundles, we we should get

$$\epsilon^{2n-2}|_{\mathbb{R}P^n} \cong T\mathbb{R}P^n \oplus \nu.$$

But if we take the Stiefel–Whitney class, we will get

$$1 = (1+w_1)^{n+1} w(v) = (1 + w_1 + w_1^n) w(v).$$

This contradicts that $w(v)$ has dimension $\leq n-2$.

Proposition 17.2 (tubular neighborhood theorem)**.** *If $M \hookrightarrow \mathbb{R}^N$ is closed, then there exists an $\epsilon > 0$ such that this extends to an embedding of $D_\epsilon(\nu_M) \hookrightarrow \mathbb{R}^N$.*

Proof. The normal bundle is $\nu_M \ni (m, \vec{v})$, so we may consider the map

$$\rho : (m, \vec{v}) \mapsto m + \vec{v} \in \mathbb{R}^N.$$

This has injective differential, and is an embedding on the 0-section. This means that it is locally an embedding, so that there is a $U \subseteq \nu_M$, an open neighborhood of the 0-section, such that $\rho|_U$ is an embedding. Then because M is compact we can do this. □

Now let us construct representatives. Take $[M, g] \in \Omega^0_n(X)$, and pick an embedding $e : M \hookrightarrow \mathbb{R}^{N+n}$. There is an ϵ such that $D_\epsilon(\nu_M) \hookrightarrow \mathbb{R}^{N+n}$. We have a Gauss map

$$M \to \mathrm{Gr}_N(\mathbb{R}^{N+n}) \hookrightarrow \mathrm{Gr}_N(\mathbb{R}^\infty); \quad m \mapsto \text{normal plane at } m.$$

So we get a map $D_\epsilon(\nu_M) \to D_\epsilon(\gamma_N)$, and combine this with $D_\epsilon(\nu_M) \to M \to X$ to get $D_\epsilon(\nu_M) \to D_\epsilon(\gamma_N) \times X$. If we quotient by the sphere bundles, we get the map

$$\mathbb{R}^{N+n} \to D_\epsilon(\nu_M)/S_\epsilon(\nu_M) \to D_\epsilon(\gamma_N)/S_\epsilon(\gamma_N) \wedge X_+ \cong \mathrm{Th}(\gamma_N) \wedge X_+.$$

This extends to a map

$$\begin{aligned} P_N(M, g, e, \epsilon) : S^{N+n} = \mathbb{R}^{N+n} \cup \{\infty\} &\to \mathrm{Th}(\gamma_N) \wedge X_+; \\ \vec{x} &\mapsto \begin{cases} [(f(m), \vec{v}/\epsilon), g(m)] & \vec{x} = (m, \vec{v}) \in D_\epsilon(v_M) \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

This gives an element

$$P_N(M, g, e, \epsilon) \in \pi_{N+n}(MO_N \wedge X_+) \to \operatorname*{colim}_k \pi_{k+n}(MO_k \wedge X_+) = \pi_n(MO \wedge X_+).$$

We need to check that this is independent of all the choices. First, if we change ϵ, this is only linearly interpolating. If we replacing e by an isotopic embedding, say (e_t) is the embeddings, then we may assume that the same ϵ works for all t. Then $t \mapsto P_N(M, g, e_t, \epsilon)$ is a homotopy. Next, we should check what happens when we change N. Without loss of generality assume $\epsilon = 1$. Here, we look at

$$\Sigma S^{N+n} \xrightarrow{\Sigma P(M,g,e,1)} \Sigma\,\mathrm{Th}(\gamma_N) \wedge X_+ \xrightarrow{\mathrm{str} \wedge \mathrm{id}} \mathrm{Th}(\gamma_{N+1}) \wedge X_+.$$

This is explicitly given by

$$(t, (m, \vec{v})) \mapsto (t, [(f(m), \vec{v}), g(m)]) \mapsto [(\epsilon \oplus f(m), (t\vec{e}_1 + \vec{v})), g(m)],$$

and $P_{N+1}(M, g, \tilde{e}, 1)$ is given by

$$\vec{x} \mapsto \begin{cases} [(\epsilon \oplus f(m), t\vec{e}_1 + \vec{v}), g(m)] & \text{if } \vec{x} = (m, t\vec{e}_1 + \vec{v}) \in D(\epsilon \oplus \nu_M) \\ \infty & \text{otherwise.} \end{cases}$$

These are the same maps. Going back to the different embeddings e, we can now increase N and isotope the embeddings.

Finally, we check that this does not depend on the choice of representative (M, g). If (W, h) is a cobordism from (M_0, g_0) to (M_1, g_1), then we can embed $W \hookrightarrow \mathbb{R}^{N+n} \times [0,1]$ with $W \cap (\mathbb{R}^{N+n} \times \{i\}) = M_i$. If we apply the Pontryagin–Thom construction in $\mathbb{R}^{N+n} \times [0,1]$, we get a map

$$S^{N+n} \wedge [0,1]_+ \to \mathrm{Th}(\gamma_N) \wedge X_+.$$

Then we get a homotopy between $P_N(M_0, g_0, e|_{M_0}, \epsilon)$ and $P_N(M_1, g_1, e|_{M_1}, \epsilon)$.

Lemma 17.3. *P is a homomorphism.*

Proof. We want to check that $P([M_0 \amalg M_1, g_0 \amalg g_1]) = P([M_0, g_0]) + P([M_1, g_1])$. We assume we have an embedding $e : M_0 \amalg M_1 \hookrightarrow \mathbb{R}^N$. By increasing the dimension if necessary, assume that $M_0 \subseteq \mathbb{R}^{N+n-1} \times (-\infty, 0)$ and $M_1 \subseteq \mathbb{R}^{N+n-1} \times (0, \infty)$. Take ϵ such that the disk bundles also lie in that region. Then the Pontryagin–Thom map factors as

$$S^{N+n} = (\mathbb{R}^{N+n})^+ \to (\mathbb{R}^{N+n-1} \times (-\infty, 0))^+ \vee (\mathbb{R}^{N+n-1} \times (0,\infty))^+ \to \mathrm{Th}(\gamma_N) \wedge X_+.$$

Each of the maps are just the same as $P([M_0, g_0])$ and $P([M_1, g_1])$, up to re-identifications of the spheres and the one-point compactification. □

17.2 The inverse map

Now we need a slightly harder fact in manifold theory.

Lemma 17.4. *Let N be a manifold, and $A \subseteq N$ be a closed subset. ($* \subseteq S^{N+n}$) Assume that Y is a space and U be an open subset which is a smooth manifold. ($\mathrm{int}(D) \subseteq \mathrm{Th}$) Let K be a closed submanifold with an open neighborhood V, such that $\overline{V} \subseteq U$. Then any $f : N \to Y$ such that $f(A) \subseteq Y \setminus U$ can be homotoped to $\tilde{f}$, relative to $f^{-1}(Y \setminus U)$, such that $\tilde{f}|_{\tilde{f}^{-1}(V)} : \tilde{f}^{-1}(V) \to V$ is smooth and transverse to K. (Then $\tilde{f}^{-1}(K)$ is a submanifold of N of codimension equal to the codimension of k in V.)*

Let us apply this to get a map $\mathcal{I} : \pi_n(MO \wedge X_+) \to \Omega_n^0(X)$. Any element $[f]$ can be represented by

$$\begin{array}{ccc} S^{N+n} & \longrightarrow & MO_N \wedge X_+ \\ & \searrow & \\ & & \mathrm{Th}(\gamma_{N,k}) \wedge X_+. \end{array}$$

By the lemma, we may assume that f is smooth on $f^{-1}(\mathrm{int}(D_{1/2}))$ and transverse to the 0-section, which is $\mathrm{Gr}_N(\mathbb{R}^{N+k})$. Then we define

$$\mathscr{I}([f]) = [f^{-1}(\text{0-section}), X\text{-component of } f|_{f^{-1}(\text{0-sec})}].$$

We need to check that this is well-defined. Increasing k and increasing N obviously does not change this. If we take homotopic $f_0, f_1 : S^{N+n} \to \mathrm{Th}(\gamma_{N,k}) \wedge X_+$, which is smooth on the inverse image of $D_{1/2}$ and transverse to the 0-section, we make the homotopy H smooth on the inverse image of $D_{1/2}$ and transverse to the 0-section, relative to $S^{N+n} \times \{0, 1\}$. Then $H^{-1}(\text{0-section})$ is a cobordism.

Now we want to check that $\mathcal{I}$ is really the inverse. If we take $\mathcal{I} \circ \mathcal{P}$ on $[M, g]$, note that $P([M, g])$ is represented by $P_N(M, g, e, \epsilon) : S^{N+n} \to \mathrm{Th}(\gamma_{N,n}) \wedge X_+$ is already transverse to the 0-section and the inverse image of the 0-section is M. So this is trivial.

For the other direction, note that we have made an additional choice of the classifying map $f : M \to \mathrm{Gr}_N(\mathbb{R}^\infty)$. If $h : S^{N+n} \to \mathrm{Th}(\gamma_{N,k}) \wedge X_+$ is transverse to the 0-section, then we get f by composing $f : h^{-1}(\text{0-sec}) \to \mathrm{Gr}_N(\mathbb{R}^{N+k}) \to \mathrm{Gr}_N(\mathbb{R}^\infty)$. Then we can use the embedding $e : f^{-1}(\text{0-sec}) \subseteq \mathbb{R}^{N+n}$. Then we claim that

$$P_N(M, g, e, \epsilon, f) : S^{N+n} \to \mathrm{Th}(\gamma_N) \wedge X_+$$

is homotopic to h. You can do a radial scaling in $D(\gamma_N)/S(\gamma_N)$ to push the difference between them to ∞.

18 March 29, 2018

We showed that $\Omega_n^0(*) \cong \pi_*(MO)$. So we know that

$$H^*(MO;\mathbb{F}_2) \cong H^*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[[w_1, w_2, \dots]].$$

But $\pi_{-*}\operatorname{Fun}(H\mathbb{F}_2, H\mathbb{F}_2)$ acts on $H^*(MO;\mathbb{F}_2) = \pi_{-*}\operatorname{Fun}(MO, H\mathbb{F}_2)$.

18.1 Cohomology operations

Having extra structure on cohomology is useful. For instance, it tells you that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ for $0 < m < n$ that is an isomorphism on π_1. This is because it can't send

$$\mathbb{F}_2[w_1]/(w_1^{n+1}) \cong H^*(\mathbb{R}P^n;\mathbb{F}_2) \leftarrow H^*(\mathbb{R}P^m;\mathbb{F}_2) \cong \mathbb{F}_2[w_1]/(w_1^{m+1}).$$

Question. *What are all the algebraic structure on cohomology?*

This is too hard, so maybe we want to look at linear ones first.

Definition 18.1. A **cohomology operation** is a natural transformation

$$\varpi : H^n(-;A) \to H^m(-;B)$$

of functors $\operatorname{Ho}(\mathsf{Top}^2) \to \mathsf{Ab}$.

Examples include $\mathrm{id} : H^n(-;A) \to H^n(-;A)$ or $(-)^p : H^n(-;\mathbb{F}_p) \to H^{pn}(-;\mathbb{F}_p)$ because $(x+y)^p = x^p + y^p$.

Proposition 18.2. *There are bijections between*

(i) natural transformations $H^n(-;A) \to H^m(-;B)$,

(ii) $\tilde{H}^m(K(A,n);B)$,

(iii) $[K(A,n), K(B,m)]_+$.

Proof. This is Yoneda. □

The upshot is that we just need to compute cohomology of Eilenberg–Mac Lane spaces. In principal you can do this using the spectral sequence coming from the fibration, but there will be too many differentials.

Remark. *Cohomology operations that are compatible with suspension isomorphisms are called* ***stable*** *and they are in bijection with*

$$H^m(HA;B) = \pi_{-m}\operatorname{Fun}(HA, HB).$$

18.2 Steenrod squares

Theorem 18.3 (Steenrod). *For $i \geq 0$ there exist natural transformations*

$$\mathrm{Sq}^i : H^n(-;\mathbb{F}_2) \to H^{n+i}(-;\mathbb{F}_2)$$

of functors $\mathrm{Ho}(\mathsf{Top}^2) \to \mathsf{Ab}$, *called **Steenrod squares**, uniquely determined by*

(i) $\mathrm{Sq}^0 = \mathrm{id}$,

(ii) $\mathrm{Sq}^i(x) = 0$ *for* $i > \deg(x)$,

(iii) $\mathrm{Sq}^n(x) = x^2$ *for* $\deg(x) = n$,

(iv) (Cartan formula) if we define $\mathrm{Sq}(x) = \sum_{i=0}^{\deg(x)} \mathrm{Sq}^i(x)$ *then* $\mathrm{Sq}(xy) = \mathrm{Sq}(x)\,\mathrm{Sq}(y)$.

These also satisfy

(v) (Adem relations) for $i < 2j$, we have

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_k \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k,$$

(vi) Sq^i *commutes with coboundaries.*

(vii) Sq^1 *is the mod 2 Bockstein coming from* $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

These are stable.

Lemma 18.4. Sq^i *commutes with suspension isomorphisms.*

Proof. Consider $SX = C_+X \cup_X C_-X$. We can consider the suspension isomorphism as

$$\begin{array}{ccc} \tilde{H}^*(X;\mathbb{F}_2) & \xrightarrow{\sigma} & \tilde{H}^{*+1}(SX;\mathbb{F}_2) \\ {\scriptstyle \mathrm{cob}}\downarrow\cong & & \cong\uparrow \\ H^{*+1}(C_+X, X;\mathbb{F}_2) & \xleftarrow[\cong]{\mathrm{exc}} & H^{*+1}(SX, C_-X;\mathbb{F}_2) \end{array}$$

Then all three maps are natural with respect to Sq^i, because two of them come from actual maps and the other is a coboundary map. □

Example 18.5. Consider $x \in H^1(X;\mathbb{F}_2)$. Then $\mathrm{Sq}(1) = 1$ and $\mathrm{Sq}(x) = x + x^2$. By Cartan's formula, we get

$$\mathrm{Sq}^i(x^r) = \binom{r}{i} x^{r+i}.$$

Example 18.6. If $f : S^{2n-1} \to S^n$ has Hopf invariant $H(f) \equiv 1 \pmod 2$, then all $\Sigma^k f$ are not null-homotopic.

18.3 Construction of Steenrod squares

Suppose G acts on $\{1, \dots, k\}$, and let X be a based (well-pointed) space. Then we can form
$$(X^{\wedge k})_{hG} = EG_+ \wedge_G X^{\wedge k}.$$
This is like a homotopy quotient. There is another construction, which is taking the quotient
$$EG \times_G X^k \to EG_+ \wedge_G X^{\wedge k}$$
by contracting $EG \times_G F_{k-1} X^k$ containing at least one basepoint. Then we can compose
$$i_X : X^{\wedge k} \to EG_+ \wedge X^{\wedge k} \twoheadrightarrow EG_+ \wedge_G X^{\wedge k}.$$
This is natural in pointed maps.

Lemma 18.7. *Assume $\tilde{H}^i(X; \mathbb{F}_2) = 0$ for $i < n$, and finite-dimensional for $i = n$. Then*
$$\tilde{H}^i((X^{\wedge k})_{hG}; \mathbb{F}_2) = \begin{cases} 0 & i < kn, \\ (\tilde{H}^n(X; \mathbb{F}_2)^{\otimes k})^G & i = kn. \end{cases}$$
Furthermore,
$$i_X^* : \tilde{H}^{kn}((X^{\wedge k})_{hG}; \mathbb{F}_2) \to \tilde{H}^{kn}(X^{\wedge k}; \mathbb{F}_2) \cong \tilde{H}^n(X; \mathbb{F}_2)^{\otimes k}$$
is the inclusion of invariants.

Proof. Consider the relative Serre spectral sequence for
$$(X^k, F_{k-1}X^k) \to EG \times_G (X^k, F_{k-1}, X^k) \to BG.$$
Then we get
$$H^p(BG, H^q(X^k, F_{k-1}X^k; \mathbb{F}_2)) \Rightarrow \tilde{H}^{p+q}((X^{\wedge k})_{hG}; \mathbb{F}_2).$$
This shows the first part, and for the second part, we use the fibration $(X^k, F_{k-1}X^k) \to (X^k, F_{k-1}X^k) \to *$ mapping into this spectral sequence □

Now we take $X = K_n = K(\mathbb{F}_2, n)$ and $G = C_2$. Then
$$\tilde{H}^i((K_n^{\wedge 2})_{hC_2}; \mathbb{F}_2) = \begin{cases} 0 & i < 2n \\ \mathbb{F}_2 & i = 2n. \end{cases}$$
We call this $u_{2n} \in \tilde{H}^{2n}((K_n^{\wedge 2})_{hC_2}; \mathbb{F}_2)$.

We also know that $i_{u_n}^* u_{2n} = i_n \wedge i_n$. Given $x \in \tilde{H}^n(X; \mathbb{F}_2)$, we know that it is classified by $f : X \to K_n$. We define
$$P(x) = ((f_X^{\wedge 2})_{hC_2})^* u_{2n} \in \tilde{H}^{2n}((X^{\wedge 2})_{C_2}; \mathbb{F}_2).$$

Then this will satisfy $i_X^* P(x) = x \wedge x$.

$$\begin{array}{ccccc} X^{\wedge 2} & \xrightarrow{f_*^{\wedge 2}} & K_n^{\wedge 2} & & \\ \downarrow & & \downarrow & \searrow^{i_n \wedge i_n} & \\ (X^{\wedge 2})_{hC_2} & \longrightarrow & (K_n^{\wedge 2})_{hC_2} & \xrightarrow{u_{2n}} & K_{2n} \end{array}$$

There is a diagonal map $\Delta : X \to X^{\wedge 2}$, that is C_2-equivariant with the trivial action on the left and the switching action on the right. Then we get a map

$$\Delta : (BC_2)_+ \wedge X = (EC_2)_+ \wedge_{C_2} X = (X)_{hC_2} \to (X^{\wedge 2})_{hC_2},$$

so

$$\Delta^* P(x) \in \tilde{H}^{2n}(\mathbb{R}P^\infty_+ \wedge X; \mathbb{F}_2) \cong \bigoplus_{i+j=2n} H^i(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes \tilde{H}^j(X; \mathbb{F}_2).$$

We can then uniquely write

$$\Delta^* P(x) = \sum_{i=-n}^{n} w_1^{n-i} \otimes \mathrm{Sq}^i(x).$$

Example 18.8. We claim that $\mathrm{Sq}^0(e) = e$. To check this, we look at $e \in \tilde{H}^1(S^1; \mathbb{F}_2)$. Then if we compute this, $\mathrm{Sq}^0(e) = e$.

18.4 Properties

Lemma 18.9. *$\mathrm{Sq}^i(x) = 0$ for $i < 0$, and $\mathrm{Sq}^i(x) = 0$ for $i > \deg(x)$.*

Proof. The second is trivial. For the first one, we only need to check in the universal case K_n, but this has no cohomology in degree $< n$. □

Lemma 18.10. *$\mathrm{Sq}^n(x) = x^2$ if $\deg(x) = n$.*

Proof. Consider

$$\begin{array}{ccc} S^0 \wedge X & \xrightarrow{\text{diag}} & X^{\wedge 2} \\ \downarrow & & \downarrow \\ (BC_2)_+ \wedge X & \longrightarrow & (X^{\wedge 2})_{hC_2}. \end{array}$$

If we pull back $P(x)$ along the bottom left, we get $\mathrm{Sq}^n(x)$. If we pull back along the top right, we get x^2. □

Let's now prove the Cartan formula. Consider the map

$$\delta : ((X \wedge Y)^{\wedge 2})_{hC_2} \to (X^{\wedge 2})_{hC_2} \wedge (Y^{\wedge 2})_{hC_2}; \quad (z, (x_1, y_1), (x_2, y_2)) \mapsto (z, (x_1, x_2), z, (y_1, y_2)).$$

Lemma 18.11. *We have $\delta^*(P(x) \wedge P(y)) = P(x \wedge y)$.*

Proof. We look at the universal case where $i_n \in \tilde{H}^n(K_n;\mathbb{F}_2)$ and $i_m \in \tilde{H}^m(K_m;\mathbb{F}_2)$. Write down

$$\begin{array}{ccc} (K_n \wedge K_m)^{\wedge 2} & \xrightarrow{i} & ((K_n \wedge K_m)^{\wedge 2})_{hC_2} \\ \downarrow & & \downarrow{\scriptstyle \delta} \\ K_n^{\wedge 2} \wedge K_m^{\wedge 2} & \longrightarrow & (K_n^{\wedge 2})_{hC_2} \wedge (K_m^{\wedge 2})_{hC_2}. \end{array}$$

If we pull back $P(i_n) \wedge P(i_m)$, then we get

$$i^*\delta^*(P(i_n) \wedge P(i_m)) = (i_n \wedge i_m)^{\wedge 2} = i^* P(i_n \wedge i_m).$$

But i^* is injective on where the first degree cohomology doesn't vanish. □

Corollary 18.12. $\mathrm{Sq}(xy) = \mathrm{Sq}(x)\,\mathrm{Sq}(y)$.

Proof. If we have $\mathrm{Sq}(x\wedge y) = \mathrm{Sq}(x)\wedge\mathrm{Sq}(y)$, this implies the statement by pulling back along the diagonal. WE have

$$\begin{array}{ccc} (BC_2)_+ \wedge X \wedge Y & \xrightarrow{\Delta} & ((X \wedge Y)^{\wedge 2})_{hC_2} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \delta} \\ (BC_2)_+ \wedge X \wedge (BC_2)_+ \wedge Y & \xrightarrow{\Delta\wedge\Delta} & (X^{\wedge 2})_{hC_2} \wedge (Y^{\wedge 2})_{hC_2}. \end{array}$$

Pulling back $P(x) \wedge P(y)$ implies that

$$\Delta P(x \wedge y) = \rho^*(\Delta^* P(x) \wedge \Delta^* P(y)).$$

Then we get

$$\sum_{k=0}^{\deg(x+y)} w_1^{\deg(x+y)-k} \otimes \mathrm{Sq}^k(x \wedge y) = \sum_{i+j=0}^{\deg(x+y)} w_1^{\deg(x+y)-i-j} \otimes \mathrm{Sq}^i(x) \wedge \mathrm{Sq}(y)^j(y).$$

This is exactly what we want. □

To show that Sq^i commutes with coboundary, we can use

$$\tilde{H}^1(I, \partial I; \mathbb{F}_2) \otimes H^*(A;\mathbb{F}_2) \to H^{*+1}(A \times I, A \times \partial I; \mathbb{F}_2).$$

Lemma 18.13. Sq^i *is a homomorphism.*

Proof. If we take the adjunction of the suspension isomorphism, we get

$$\begin{array}{ccc} K_n & \xrightarrow{\mathrm{Sq}^i(i_n)} & K_{n+i} \\ \downarrow & & \downarrow \\ \Omega K_{n+1} & \xrightarrow{\Omega\,\mathrm{Sq}^1(i_{n+1})} & \Omega K_{n+i+1}. \end{array}$$

This means that $\mathrm{Sq}^i(i_n)$ is homotopic to a loop map, so it should be a homomorphism. □

19 April 3, 2018

I still want to eventually compute $\Omega_*^0(*) = \pi_* MO$. We will realize that this free over the Steenrod algebra, so we are trying to understand more about these.

19.1 Adem relations

This is just the iterated version of taking a pointed space X to $(X^{\wedge 2})_{hC_2} = (EC_2)_+ \wedge_{C_2} X^{\wedge 2}$. If we write out, this is

$$([X^{\wedge 2}]^{\wedge 2}_{hC_2})_{hC_2} = (EC_2)_+ \wedge_{C_2} ((EC_2)_+ \wedge_{C_2} X^{\wedge 2})^{\wedge 2} = (EC_2 \times EC_2^2)_+ \wedge_{C_2 \wr C_2} X^{\wedge 4} = (X^{\wedge 4})_{hC_2 \wr C_2}.$$

Here $C_2 \wr C_2$ is $(C_2 \times C_2) \rtimes C_2$ which is isomorphic to D_8. This is a subset of Σ_4, so we have a map

$$(X^{\wedge 4})_{hC_2 \wr C_2} \to (X^{\wedge 4})_{h\Sigma_4}.$$

If we look at $X = K(\mathbb{F}_2, n)$, we can find a map on first cohomology groups

$$H^{4n}((X^{\wedge 4})_{hC_2 \wr C_2}; \mathbb{F}_2) \leftarrow H^{4n}((X^{\wedge 4})_{h\Sigma_4}; \mathbb{F}_2)$$

where both sides are $(\mathbb{F}_2^{\otimes 4})^{C_2 \wr C_2} \cong \mathbb{F}_2$ and $(\mathbb{F}_2^{\otimes 4})^{\Sigma_4} \cong \mathbb{F}_2$ and hence this is an isomorphism. This encodes additional symmetry.

We can look at

$$\begin{array}{ccc}
((X^{\wedge 2})^{\wedge 2}_{hC_2})_{hC_2} & \longrightarrow & (X^{\wedge 4})_{h\Sigma_4} \\
\uparrow {\scriptstyle (\Delta^{\wedge 2})_{hC_2}} & & \\
((BC_2)_+ \wedge X)^{\wedge 2}_{hC_2} & & \\
\uparrow {\scriptstyle \Delta} & & \\
(BC_2)_+ \wedge (BC_2)_+ \wedge X & \longrightarrow & (B\Sigma_4)_+ \wedge X
\end{array}$$

where the bottom map is induced by $\gamma : C_2 \times C_2 \hookrightarrow \Sigma_4$ given by the first C_2 factor being the action $\rtimes C_2$ part and the second C_2 being the first component of $C_2 \times C_2$.

You could also consider the map

$$C_2 \times C_2 \xrightarrow{\tau} C_2 \times C_2 \xrightarrow{\gamma} \Sigma_4$$

where τ is the flipping map. Here, γ and $\gamma \circ \tau$ are conjugate in Σ_4.

Lemma 19.1. *If $f_1, f_2 : G \to H$ are conjugate group homomorphisms, then $Bf_0, Bf_1 : BG \to BH$ are homotopic.*

Proof. We have two functors $*//G \to *//H$ and conjugacy gives a natural transformation between them. □

Now using this, let's look at the generator for $H^{4n}(-;\mathbb{F}_2)$ for $X = K(\mathbb{F}_2, n)$. We know that $P(P(\iota_n))$ is a generator of $H^{4n}(((X^{\wedge 2})^{\wedge 2}_{hC_2})_{hC_2};\mathbb{F}_2)$. Then if we pull back, we get

$$\Delta^*(\Delta^{\wedge 2}_{hC_2})^* P(P(\iota_n)) \in \tilde{H}^*((BC_2)_+ \wedge (BC_2)_+ \wedge X;\mathbb{F}_2) \cong \tilde{H}^*(X;\mathbb{F}_2)[[x,y]].$$

But because we can pull back along the other map, and we can swap them if we like, we see that this is symmetric in x and y.

Let us formally write

$$\mathrm{Sq}^{(}_x u) = \sum_{k\geq 0} x^{-k}\,\mathrm{Sq}^k(u), \quad \Delta^* P(u) = x^{\deg u}\,\mathrm{Sq}_x(u).$$

This is still a ring homomorphism by the Cartan formula. Then by naturality of $P(-)$, we have

$$\begin{aligned}\Delta^*(\Delta^{\wedge 2}_{hC_2})^* P(P(u)) &= \Delta^* P(\Delta^* P(\iota_n)) = \Delta^* P(y^n\,\mathrm{Sq}_y(\iota_n)) \\ &= x^{2n}\,\mathrm{Sq}_x(y^n\,\mathrm{Sq}_y(\iota_n)) = x^{2n}\,\mathrm{Sq}_x(y)^n\,\mathrm{Sq}_x(\mathrm{Sq}_y(\iota_n)) \\ &= x^{2n} y^{2n} (x^{-1}+y^{-1})^n\,\mathrm{Sq}_x(\mathrm{Sq}_y(n)).\end{aligned}$$

So $\mathrm{Sq}_x(\mathrm{Sq}_y(n))$ is symmetric in x and y. This

$$\mathrm{Sq}_x(\mathrm{Sq}_y) = \mathrm{Sq}_y(\mathrm{Sq}_x)$$

is the **Adem relations**.

Lemma 19.2. *The relation* $\mathrm{Sq}_x(\mathrm{Sq}_y) = \mathrm{Sq}_y(\mathrm{Sq}_x)$ *implies (v).*

Proof. You can rewrite this as generating functions for $\mathrm{Sq}^i\,\mathrm{Sq}^j$ by substituting s, y in terms of x, y, and rewrite the right side in terms of s and t. The left hand side is

$$\mathrm{Sq}_x(\mathrm{Sq}_y) = \sum_{j\geq 0} \mathrm{Sq}_x(y^{-j}\,\mathrm{Sq}^j) = \sum_{j\geq 0} \mathrm{Sq}_x(y)^{-j}\,\mathrm{Sq}_x(\mathrm{Sq}^j) = \sum_{j\geq 0} (y^2(x^{-1}+y^{-1}))^{-j}\,\mathrm{Sq}_x(\mathrm{Sq}^j).$$

Here, take $t = y^{-2}(x^{-1}+y^{-1})$ and $s = x^{-1}$. Then we get

$$\mathrm{Sq}_x(\mathrm{Sq}_y) = \sum_{j\geq 0} t^j\,\mathrm{Sq}_x(\mathrm{Sq}^j) = \sum_{i,j\geq 0} s^i t^j\,\mathrm{Sq}^i\,\mathrm{Sq}^j.$$

On the other hand,

$$\begin{aligned}\mathrm{Sq}_y(\mathrm{Sq}_x) &= \sum_{j\geq 0} (x^2(x^{-1}+y^{-1})^{-j}\,\mathrm{Sq}_y(\mathrm{Sq}^j) = \sum_{j\geq 0} x^{-2j} y^{2j} t^j\,\mathrm{Sq}_y(\mathrm{Sq}^j) \\ &= \sum_{i,j\geq 0} x^{-2j} y^{2j} t^j y^{-i}\,\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{i,j\geq 0} s^{2j} t^j y^{2j-1}\,\mathrm{Sq}^i(\mathrm{Sq}^j).\end{aligned}$$

If you solve for y, you get

$$y^m = \sum_{k\geq m} \binom{-k-1}{k-m} s^{k-m} t^{-k}.$$

Then you can just plug this in compare both sides. $\square$

19.2 The Steenrod algebra

This is just the algebra

$$\mathcal{A} = T[\mathrm{Sq}^0, \mathrm{Sq}^1, \dots]/(\mathrm{Sq}^0 = 1, \text{Adem relations}).$$

There is an augmentation $\epsilon : \mathcal{A} \to \mathbb{F}_2$ by sending Sq^0 to 1 and Sq^1 to 0. Then we can define the indecomposables $\mathcal{A}/\overline{\mathcal{A}} \cdot \overline{\mathcal{A}}$ for $\overline{\mathcal{A}} = \ker \epsilon$. For instance,

$$\mathrm{Sq}^1 \mathrm{Sq}^2 = \binom{1}{1} \mathrm{Sq}^3 \mathrm{Sq}^0 = \mathrm{Sq}^3$$

and so Sq^3 is indecomposable.

Lemma 19.3. *$\mathcal{A}$ is generated by Sq^{2^k} and these give a basis of indecomposables.*

Proof. By induction we prove that Sq^i is the sum of products of Sq^{2^k}. For $i = 1$, this is clear. For the induction step, we consider $2^k < i < 2^{k+1}$. Then

$$\mathrm{Sq}^{i-2^k} \mathrm{Sq}^{2^k} = \sum_{j=0}^{\frac{i-2^k}{2}} \binom{2^k - j - 1}{i - 2^k - 2j} \mathrm{Sq}^{i-j} \mathrm{Sq}^j = \binom{2^k - 1}{i - 2^k} \mathrm{Sq}^i + \cdots .$$

But the coefficient is nonzero because $(1+z)^{2^k - 1} = 1 + z + z^2 + \cdots + z^{2^k - 1}$ and all the coefficients are 1.

To show that Sq^{2^k} are indecomposable, recall that on $\mathbb{R}P^\infty$, we have $\mathrm{Sq}^i(w_1^r) = \binom{r}{i} w_1^{r+i}$. Then only Sq^0 and Sq^{2^k} don't vanish on $w_1^{2^k}$. So if Sq^{2^k} were decomposable, then Sq^{2^k} should vanish. □

If you have a map $f : S^{2n-1} \to S^n$ that has Hopf invariant 1, then $n = 2^k$. This is because if $D(f) = S^n \cup_f D^{2n}$, then additively

$$H^*(D(f)) = \mathbb{F}_2\{1, a_n, b_{2n}\}$$

and by definition $\mathrm{Sq}^n a_n = a_n^2 = H(f) b_{2n}$. If Sq^n were decomposable, it would vanish. This shows that Sq^n has to be indecomposable, and n is a power of 2.

Still this is not very useful for computations. We want an additive basis for the Steenrod algebra. For $I = (i_1, \dots, i_r \geq 1$, we define

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}_{i_r} .$$

If $i < 2j$, you can use the Adem relations to simplify things. So we may assume that I is **admissible**, that is, $i_j \geq 2 i_{j+1}$.

Lemma 19.4. *Sq^I for I admissible are an additive basis for $\mathcal{A}$.*

Proof. We want to show that they are linearly independent. To show this, we act on a particular cohomology class

$$H^*((\mathbb{R}P^\infty)^n; \mathbb{F}_2) = \mathbb{F}_2[[x_1, \dots, x_n]].$$

Let us write

$$\prod(1+x_i) = 1+\sigma_1+\sigma_2+\cdots+\sigma_n.$$

If we take $\operatorname{Sq}\sigma_n$, we get

$$\operatorname{Sq}\sigma_n = \operatorname{Sq}(x_1)\cdots\operatorname{Sq}(x_n) = (x_1+x_1^2)\cdots(x_n+x_n^2) = \sigma_n(1+\sigma_1+\cdots+\sigma_n).$$

From this we can read off $\operatorname{Sq}^i(\sigma_n) = \sigma_n\sigma_1$. We claim that $d(I) = i_1+\cdots+i_r \leq n$, with I admissible, the claim is that

$$\operatorname{Sq}^I(\sigma_n) = \sigma_n\sigma_{i_1}\cdots\sigma_{i_r} + \text{lower order terms}$$

where lower order terms means that $m \prec m'$ if you take σ_N largest N such that $\sigma_N \mid m$ and $\sigma_{N'}$ such that $\sigma_{N'} \mid m'$, then either $N < N'$ or $N = N'$ and $m/\sigma_N \prec m'/\sigma_{N'}$. One you have established this, it immediately follows that Sq^I are linearly independent. □

Note that this also shows that we didn't miss any relation here. Any missed relation will give one in the universal case, but we showed that here are no any other relations.

19.3 Cohomology of $K(\mathbb{F}_2, n)$

Now we have many natural transformations and all the relations between them, but we still need to show that these are all the natural transformations. Define

$$e(I) = 2i_1 - d(I) = i_1 - i_2 - i_3 - \cdots - i_r = (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_r).$$

This we call the **excess**. For example, admissible I with excess 0 is necessarily $\operatorname{Sq}^\emptyset$. Admissible I with excess 1 is $I = (2^k, 2^{k-1}, \dots, 2, 1)$.

Theorem 19.5 (Serre)**.** *$H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ is a free polynomial over $\mathbb{F}_2$ on $\operatorname{Sq}^I(i_n)$ with I admissible and $e(I) < n$.*

This is by induction on n. It is clear for $n = 1$. For the induction step, you need to know about transgressions. Here, there is some action of the Steenrod algebra on the spectral sequence, so that if something has maximal length differential, its squares also have maximal length differentials. After using this to compute the differentials, you need some algebra lemma that relates the cohomology of base and the cohomology of the fiber.

Anyways, the point is that Sq^i generates all the natural transformations as $n \to \infty$.

20 April 5, 2018

The goal for today is to prove the following theorem:

$$\pi_*(MO) \cong \mathbb{F}_2[x_i : i \neq 2^k - 1].$$

This is true additively, but it is true as an algebra as well.

20.1 BO and MO revisited

Recall that $BO = \operatorname{colim}_k BO(k)$ and MO the Thom spectrum with kth level $\mathrm{Th}(\gamma_k \to BO(k))$.

Here is another way to define Stiefel–Whitney classes. If you have a n-dimensional bundle $\xi = (E, B, p)$, we saw that there is a Thom isomorphism

$$p^*(-) \smile u : H^*(B; \mathbb{F}_2) \to \tilde{H}^{*+n}(\mathrm{Th}(\xi); \mathbb{F}_2)$$

where u is the Thom class. You can define $w_i(\xi)$ by

$$\mathrm{Sq}^i(u) = w_i(\xi) \smile u.$$

We can check the defining properties.

(i) naturality: Observe that if u is a Thom class for ξ, then $\mathrm{Th}(f) : \mathrm{Th}(f^*\xi) \to \mathrm{Th}(\xi)$ pulls back u to the Thom class for $f^*\xi$. So by naturality of Steenrod squares,

$$w_i(f^*\xi) \smile \mathrm{Th}(f)^*(u) = \mathrm{Th}(f)^*(w_i(\xi) \smile u) = f^* w_i(\xi) \smile \mathrm{Th}(f)^* u.$$

(ii) normalization: $w_0(\xi) = 1$ follows from $\mathrm{Sq}^0 = \mathrm{id}$. For λ_n the canonical bundle over $\mathbb{R}P^n$, we observe the $\mathrm{Th}(\gamma_n) = \mathbb{R}P^{n+1}$ with Thom class x. Now because $\mathrm{Sq}^1 x = x^2$, we see that $w_1(\lambda_n) = x \in H^1(\mathbb{R}P^\infty; \mathbb{F}_2)$.

(iv) product formula: This follows from the external Cartan formula.

Because $\mathrm{Th}(\xi)$ only depends on the sphere bundle as a fibration, w_i don't depend on the smooth structure.

We proved that

$$H^*(BO(n); \mathbb{F}_2) = \mathbb{F}_2[[w_1, \ldots, w_n]]$$

and to see that, we used the map

$$H^*(BO(n); \mathbb{F}_2) \to H^*((\mathbb{R}P^\infty)^n; \mathbb{F}_2) \cong \mathbb{F}[[x_1, \ldots, x_n]]$$

induced by the classifying map for $\lambda_\infty \oplus \cdots \oplus \lambda^\infty$. Then w_i was sent to the ith symmetric polynomial σ_i in $x_1, \ldots, x_n$.

Letting $n \to \infty$, we get

$$H^*(BO; \mathbb{F}_2) \cong \mathbb{F}_2[[w_1, w_2, \ldots]]$$

as $\mathbb{F}_2$-algebras, and also $H^*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[[w_1, w_2, \dots]]$ as $\mathbb{F}_2$-vector spaces by Thom isomorphism. Then we can dualize this and say

$$H_*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[[w_1^*, w_2^*, \dots]], \quad H_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[[w_1^*, w_2^*, \dots]].$$

But the homologies also have an algebra structure. Note that BO is a colimit of $BO(k) = \mathrm{Gr}_k(\mathbb{R}^\infty)$. So we have a map

$$\mathrm{Isom}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty) \times \mathrm{Gr}_k(\mathbb{R}^\infty) \times \mathrm{Gr}_l(\mathbb{R}^\infty) \to \mathrm{Gr}_{k+l}(\mathbb{R}^\infty).$$

The space of isometries $\mathrm{Isom}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty)$ is contractible, so we get a map that is commutative up to homotopy. Using $\mathrm{Isom}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty)$, you prove that these maps are associative up to homotopy. This extends to Thom spaces

$$\mathrm{Isom}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty)_+ \wedge \mathrm{Th}(\gamma_k) \wedge \mathrm{Th}(\gamma_k) \to \mathrm{Th}(\gamma_{k+l}).$$

Letting $k, l \to \infty$, we get

$$BO \times BO \to BO, \quad MO \wedge MO \to MO$$

that is unital, commutative, associated up to homotopy. Taking the $\mathbb{F}_2$-homology, we get unital commutative $\mathbb{F}_2$-algebra structures on $H_*(BO;\mathbb{F}_2)$ and $H_*(MO;\mathbb{F}_2)$.

Lemma 20.1. $H_*(BO;\mathbb{F}_2) \cong H_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[[x_1, x_2, \dots]]$.

Proof. We have a map $H_*(BO(1);\mathbb{F}_2) \cong \mathbb{F}_2\{1, x_1, x_2, \dots\} \to H_*(BO;\mathbb{F}_2)$. Then there is a map $\mathbb{F}_2[[x_1, x_2, \dots]] \to H_*(BO;\mathbb{F}_2)$. They have the same dimension in each degree, so we check that this is surjective. You dualize and then check that this is surjective. □

20.2 Dual Steenrod algebra

We now want a coproduct structure on the Steenrod algebra. The Cartan formula says that

$$\mathrm{Sq}^k(x \otimes y) = \sum_{i+j=k} \mathrm{Sq}^i(x) \otimes \mathrm{Sq}^j(y).$$

So if you define $\Delta(\mathrm{Sq}^k) = \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$, then $\mathrm{Sq}^k(x \otimes y) = \Delta(\mathrm{Sq}^k)(x \otimes y)$. So we can at least define

$$T(\mathrm{Sq}^0, \mathrm{Sq}^1, \dots) \xrightarrow{\Delta} T(\mathrm{Sq}^0, \mathrm{Sq}^1, \dots) \otimes T(\mathrm{Sq}^0, \mathrm{Sq}^1, \dots)$$

as a coproduct. The question is whether this is compatible with the Adem relations.

Lemma 20.2. *Δ descends to a homomorphism $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.*

Proof. We have

$$\begin{array}{ccc}
T(\mathrm{Sq}^0, \mathrm{Sq}^1, \dots) & \longrightarrow & T(\mathrm{Sq}^0, \mathrm{Sq}^1, \dots)^{\otimes 2} \\
\downarrow & & \downarrow \\
\mathcal{A} & \dashrightarrow & \mathcal{A} \otimes \mathcal{A} \\
\downarrow{\scriptstyle i_n \times i_n} & & \downarrow{\scriptstyle i_n \otimes i_n} \\
H^*(K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n); \mathbb{F}_2) & \longrightarrow & H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)^{\otimes 2}
\end{array}$$

and Serre shows that acting on $i_n \times i_n$ and acting on $i_n \otimes i_n$ are isomorphism for $d(I) < n$. This shows that there is a dashed arrow for low degrees. But then we can take n arbitrarily large. □

Then you can check that Δ is cocommutative and coassociative. Then $\mathcal{A}$ becomes a cocommutative comonoid in $\mathrm{Alg}_{\mathbb{F}_2}$. This shows that $\mathcal{A}$ is a cocommutative Hopf algebra.

Now if you have two spaces X and Y, you have

$$\begin{array}{ccc}
\mathcal{A} \otimes H^*(X; \mathbb{F}_2) \otimes H^*(Y; \mathbb{F}_2) & \longrightarrow & \mathcal{A} \otimes H^*(X \times Y; \mathbb{F}_2) \\
\downarrow{\scriptstyle \Delta} & & \\
\mathcal{A} \otimes \mathcal{A} \otimes H^*(X; \mathbb{F}_2) \otimes H^*(Y; \mathbb{F}_2) & & \\
\downarrow{\scriptstyle 1 \otimes \tau \otimes 1} & & \downarrow \\
\mathcal{A} \otimes H^*(X; \mathbb{F}_2) \otimes \mathcal{A} \otimes H^*(Y; \mathbb{F}_2) & \longrightarrow & H^*(X \times Y; \mathbb{F}_2).
\end{array}$$

Then $H^*(X; \mathbb{F}_2)$ is a commutative monoid object in $\mathcal{A}$-modules.

We have $\mathcal{A}$ as a cocommutative Hopf algebra over $\mathbb{F}_2$. So if we take the dual, $\mathcal{A}^*$ is going to be a commutative Hopf algebra. Consider $x \in H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$. Take ξ_i to be the functional such that $\xi_i(\theta)$ is the coefficient of x^{2^i} in $\theta(x)$. Then we have a map

$$\lambda : \mathbb{F}_2[\xi_1, \xi_2, \dots] \to \mathcal{A}^*.$$

Theorem 20.3 (Milnor)**.** *We have that λ is an isomorphism of $\mathbb{F}_2$-algebras, and $\Delta(\xi_k) = \sum_{i+j=k} \xi_1^{2^i} \otimes \xi_j$.*

20.3 Thom's theorem

Our goal is to compute $\pi_*(MO)$. First we will show that $MO \simeq \bigwedge_\alpha \Sigma^\alpha H\mathbb{F}_2$ by showing that $H^*(MO; \mathbb{F}_2)$ is a free $\mathcal{A}$-module.

Theorem 20.4 (Milnor–Moore)**.** *If $\mathcal{A}$ is a connected Hopf algebra over a field k, and M is a connected $\mathcal{A}$-module algebra, and $A \to M$ acting on $1 \in M$ is injective, then M is a free A-module.*

We will apply this to $A = \mathcal{A}$ and $M = H^*(MO; \mathbb{F}_2)$.

Proof. Indecomposables of M as an A-module is $\overline{M} = M/\overline{A}M$ where $\overline{A} = \ker(\epsilon : A \to k)$. Then there is a map $\pi : M \to \overline{M}$. Because we are over a field, we can take a section σ. These sections should look like generators, so we have

$$\phi : A \otimes \overline{M} \xrightarrow{1\otimes\sigma} A \otimes M \to M.$$

Here, $A \otimes \overline{M}$ is a free A-module because A acts trivially on $\overline{M}$. We want to show that this is an isomorphism.

Surjectivity is proved by induction over the degree of M. By construction, we can write $m = \phi(1 \otimes \pi m) + \sum a_i \otimes m_i$ where all m_i have lower degree by connectedness. But then by induction you can write these m_i by lower degree terms as ϕ of something.

For injectivity, consider the map

$$A \otimes \overline{M} \xrightarrow{\phi} M \xrightarrow{\Delta} M \otimes M \xrightarrow{1\otimes\pi} M \otimes \overline{M}.$$

This is A-linear, and sends $1 \otimes \overline{m}$ to $\sigma\overline{m} \otimes 1 + \cdots + 1 \otimes \overline{m}$. If we filter by the degree of $\overline{M}$, this is going to be

$$A \otimes F_j\overline{M}/F_{j-1}\overline{M} \to M \otimes F \otimes F_j\overline{M}/F_{j-1}\overline{M}; \quad a \otimes \overline{m} \to a1 \otimes \overline{m}.$$

But we know that $A \to M$ given by $a \mapsto a1$ is injective, and then by induction over k, we see that this map $A \otimes \overline{M} \to M \otimes \overline{M}$ is injective as well. □

To show that $H^*(MO; \mathbb{F}_2)$ is a free $\mathcal{A}$-module, it suffices to show that $a \mapsto a{\cdot}u$ is injective. But we have

$$\mathrm{Th}(\gamma_k) \leftarrow \mathrm{Th}(\lambda_\infty \oplus \cdots \oplus \lambda_\infty) \cong (\mathbb{R}P^\infty)^n$$

and u is pulled back to $x_1 \cdots x_n$. To show that the action on u is injective, it suffices to show that the action on $x_1 \cdots x_n$ is injective. But this we know for degree $\leq n$. Take n arbitrary now.

So we have a map

$$MO \to \bigvee_\alpha H\mathbb{F}_2$$

is an $\mathbb{F}_2$-cohomology isomorphism. To see that it is a weak equivalence, we can use the $\pi_*(MO)$ is 2-torsion. Then you can use some Postnikov tower argument to show that this is a weak equivalence. Anyways, this shows that

$$\pi_*(MO) = \pi_*(\textstyle\bigvee_\alpha H\mathbb{F}_2) = \displaystyle\bigoplus_\alpha \mathbb{F}_2[\alpha].$$

So the question is where the generators are.

To find this, we use that

$$H_*(MO; \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, \ldots] \cong \bigoplus_\alpha \mathbb{F}_2[\xi_1, \xi_2, \ldots].$$

So we additively get

$$\pi_*(MO) \cong \mathbb{F}_2[x_i : i \neq 2^k - 1].$$

21 April 10, 2018

We showed that $\pi_* MO \cong \Omega^0_*(*)$. We will show that $\Omega^{\infty-1}MTO(d) \simeq BCob(d)$, and along the way, prove $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$.

21.1 Geometric realization revisited

Theorem 21.1 (May)**.** *Suppose $X_1, X_2 \subseteq X$ and $B_1, B_2 \subseteq B$ be excisive triads, and consider a map f between them. If $(X_1, X_1 \cap X_2) \to (B_1, B_1 \cap B_2)$ and $(X_2, X_1 \cap X_2) \to (B_2, B_1 \cap B_2)$ are n-connected, then $(X, X_1) \to (B, B_1)$ and $(X, X_2) \to (B, B_2)$ are also n-connected.*

The hypothesis is implied by $X_1 \to B_1$ and $X_2 \to B_2$ being n-connected and $X_1 \cap X_2 \to B_1 \cap B_2$ being $(n-1)$-connected. The conclusion implies that $X \to B$ is n-connected.

Corollary 21.2. *If $U = \{U_i\}$ is an open cover of B that is closed under finite intersection. If $f : X \to B$ is such that $f|_{f^{-1}(U)} : f^{-1}(U_i) \to U_i$ is n-connected, then f is n-connected.*

You can replace this hypothesis $X_1, X_2 \subseteq X$ and $B_1, B_2 \subseteq B$ being excisive triads with $X_1 \cap X_2 \hookrightarrow X_1$ and $B_1 \cap B_2 \hookrightarrow B_1$ are Hurewicz cofibrations. The reason is that you can replace $X_1, X_2 \subseteq X$ with the double mapping cylinder

$$X_1 \cup (X_1 \cap X_2 \times [0, 1]) \cup X_2.$$

Consider a **semi-simplicial space**, which is a functor

$$X_\bullet : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathsf{Top}$$

where Δ_{inj} are the category with injective maps only. Then we can take a **thick geometric realization**

$$\|X_\bullet\| = \int^{\Delta^{\mathrm{op}}_{\mathrm{inj}}} \Delta^\bullet \times X_\bullet = \left(\coprod_{p \geq 0} \Delta^p \times X_p\right) / (\delta_i t, x) \sim (t, d_i x).$$

Proposition 21.3. *If $f_\bullet : X_\bullet \to Y_\bullet$ is a map of semi-simplicial spaces that has $f_p : X_p \to Y_p$ levelwise n-connected, then $\|f_\bullet\| : \|X_\bullet\| \to \|Y_\bullet\|$ is n-connected.*

Proof. The proof uses the skeletal filtration

$$F_j \|X_\bullet\| = \left(\coprod_{0 \leq p \leq j} \Delta^p \times X_p\right) / \sim .$$

Then we have a pushout

$$\begin{array}{ccc} \partial\Delta^j \times X_j & \longrightarrow & F_{j-1}\|X_\bullet\| \\ \big\downarrow & & \big\downarrow \\ \Delta^j \times X_j & \longrightarrow & F_j\|X_\bullet\|. \end{array}$$

Then we can prove by induction over j that $F_j\|X_\bullet\| \to F_j\|Y_\bullet\|$ is n-connected. For $j = 0$, this is $f_0 : X_0 \to Y_0$ which is n-connected by assumption. For the induction step, we can just apply the lemma. □

This is not optimal, because there are other highly connected maps. To get $\|f_\bullet\|$ n-connected, you only need f_p to be $(n-p)$-connected.

21.2 Quasi-fibrations

Gluing together fibrations don't necessarily give fibrations. For instance, if you map $[0,1] \to \{0\}$ and $[0,1] \to [0,1]$ together on the base, you don't get a fibration. But you might expect some nice properties.

Definition 21.4. A map $f : X \to B$ is a **quasi-fibration** at $A \subseteq B$ if $f : (X, X^{-1}(A)) \to (B, A)$ is a weak equivalence.

Lemma 21.5. *If $f : X \to B$ is a Serre fibration, it is a quasi-fibration at all nonempty $A \subseteq B$.*

Proof. We need to prove that

(i) $\mathrm{im}(\pi_0(f^{-1}(A)) \to \pi_0(X)) = f^{-1}(\mathrm{im}(\pi_0(A) \to \pi_0(B)))$,
(ii) for each $x_0 \in f^{-1}(A)$, $\pi_i(X, f^{-1}(A), x_0) \to \pi_i(B, A, f(x_0))$ is a bijection.

For (i), this just is saying that $f(x)$ can be connected to A by a path if and only if x can be connected to $f^{-1}(A)$ by a path. This is clear. For surjectivity of (ii), we can just lift the map $D^i \to B$, and for injectivity of (ii), we can lift homotopies. □

Definition 21.6. A **quasi-fibration** is a map $X \to B$ which is a quasi-fibration at all $b_0 \in B$.

These were defined by Dold–Thom to prove that if X is a pointed path-connected space then

$$\mathrm{Sym}^\infty(X) = \operatorname*{colim}_{k\to\infty} X^k/\Sigma_k$$

then $\pi_i \,\mathrm{Sym}^\infty(X) \cong \tilde{H}_i(X)$. We can compare $f : X \to B$ to $\pi : Pf \to B$ given by $\pi(x,\gamma) = \gamma(1)$. Because $(Pf, \pi^{-1}(b_0)) \simeq (B, b_0)$, we see that $(X, f^{-1}(b_0)) \to (B, b_0)$ is a weak equivalence if and only if $(X, f^{-1}(b_0)) \to (Pf, \pi^{-1}(b_0))$ is a weak equivalence. Then because $X \to Pf$ is a weak equivalence, this is if and only if $f^{-1}(b_0) \to \pi^{-1}(b_0)$ is a weak equivalence. This shows that f is a quasi-fibration if and only if

$$\mathrm{fib}_{b_0}(f) \to \mathrm{hofib}_{b_0}(f)$$

is a weak equivalence for all $b_0 \in B$.

Corollary 21.7. *Fibers of a quasi-fibration over a path-connected B are all weakly equivalent.*

The next goal is to construct them.

Definition 21.8. We say that $A \subseteq B$ is **distinguished** if $f|_{f^{-1}(A)} : f^{-1}(A) \to A$ is a quasi-fibration.

We use a long exact seqeuence of homotopy groups of triples.

Lemma 21.9. *If $A \subseteq B$ is distinguished, then f is a quasi-fibration at A (i.e., $(X, f^{-1}(A)) \to (B, A)$ is a weak equivalence) if and only if f is a quasi-fibration at all $a \in A$ (i.e., $(X, f^{-1}(a)) \to (B, a)$ is a weak equivalence).*

Theorem 21.10. *If $A_1, A_2 \subseteq B$ are such that $A_1, A_2 \subseteq A_1 \cup A_2$ is an excisive triad and $A_1, A_2, A_1 \cap A_2$ are distinguished, then $A = A_1 \cup A_2$ is also distinguished.*

Proof. By applying the lemma, we get that

$$(f^{-1}(A_i), f^{-1}(A_1 \cap A_{)})) \to (A_1, A_1 \cap A_2)$$

is a weak equivalence. Then May's theorem applies and we get that $(f^{-1}(A_1 \cup A_2), f^{-1}(A_i)) \to (A_1 \cup A_2, A_i)$ is an weak equivalence. Then the lemma in the opposite direction implies that $(f^{-1}(A_1 \cup A_2), f^{-1}(a)) \to (A_1 \cup A_2, a)$ is distinguished. □

Corollary 21.11. *If B has a cover $\mathcal{U} = \{U_i\}$, closed under finite intersections, such that each U_i is distinguished, then B is distinguished (i.e., $X \to B$ is a quasi-fibration).*

Lemma 21.12. *If $A \subseteq C \subseteq B$ where A is distinguished and there is a deformation h_t of C into A and H_t of $f^{-1}(C)$ into $f^{-1}(A)$ covering h_t, such that $H_1 : f^{-1}(c) \to f^{-1}(h_1(c))$ is a weak equivalence for all C, then C is distinguished.*

Proof. We have

$$\begin{array}{ccc} (f^{-1}(C), f^{-1}(c)) & \xrightarrow{H_1} & (f^{-1}(A), f^{-1}(h_1(c))) \\ \downarrow & & \downarrow \simeq \\ (C, c) & \xrightarrow{\quad h_1 \quad} & (A, h_1(c)). \end{array}$$

We see that the horizontal maps are homotopy equivalences by the deformation, and the right vertical map is a weak equivalence because A is distinguished. So the left vertical map is also a weak equivalence. □

Proposition 21.13 (Hordic)**.** *If we have*

$$\begin{array}{ccccc} X' & \hookleftarrow & X' \times_{B'} A & \xrightarrow{g} & X \\ \text{Serre fib.} \downarrow f' & & \downarrow q & & f \downarrow \text{quasifib.} \\ B' & \hookleftarrow & A & \xrightarrow{j} & B, \end{array}$$

and if $g : q^{-1}(a) \to f^{-1}(j(a))$ is a weak equivalence for all $a \in A$, then $X' \cup_{X' \times_{B'} A} X \to B' \cup_A B$ is a quasi-fibration.

Proof. The map is already distinguished over $B' \setminus A$ and B. But they don't cover, so we need to make B larger. To do this, we find an NDR-pair structure on (B', A) and lift to a compatible one $(X', X' \times_{B'} A)$. Then we get a U of A with a deformation retraction h_t of U onto A and a deformation H_t of $f^{-1}(U)$ onto $X' \times_{B'} A$ covering h_t. This moves the fibers over f' over $b' \in U$ to fibers f' over $h_1(b')$ and induces a homotopy equivalence. Then the glued space $X' \cup_{X' \times_{B'} A} X$ gets mapped to $f^{-1}(h_1(b'))$ using g. This is a weak equivalence by assumption. So $U \cup B$ is distinguished and so $U \cup B, U' \setminus A, B' \setminus A$ are distinguished and cover $B' \cup_A B$. □

22 April 12, 2018

Today we will prove Bott periodicity $\Omega^2(\mathbb{Z}\times BU)\simeq \mathbb{Z}\times BU$.

22.1 A lemma of Segal

Definition 22.1. We say that a diagram

$$\begin{array}{ccc} E & \xrightarrow{g'} & E' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \longrightarrow & B' \end{array}$$

is a **homotopy cartesian** if

- $\mathrm{hofib}_b(f)\to \mathrm{hofib}_{g(b)}(f')$ is a weak equivalence for all $b\in B$,
- $\mathrm{hofib}_{e'}(g')\to \mathrm{hofib}_{f'(e')}(g)$ for all $e'\in E'$,
- $E\to E'\times_{B'} B$ is a weak equivalence.

Lemma 22.2. *Let $f_\bullet : E_\bullet \to B_\bullet$ be a map of semi-simplicial spaces such that*

$$\begin{array}{ccc} E_p & \xrightarrow{d_i} & E_{p-1} \\ \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p-1}} \\ B_p & \xrightarrow{d_i} & B_{p-1} \end{array}$$

is homotopy cartesian for all $p\geq 0$. Then

$$\begin{array}{ccc} E_\bullet & \longrightarrow & \|E_\bullet\| \\ \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & \|B_\bullet\| \end{array}$$

is also homotopy cartesian.

Proof. Without loss of generality, we may assume that each f_p is a Hurewicz fibration, after replacing E_p by Pf_P. This is fine because we proved that level-wise weak equivalence gives a weak equivalence on geometric realization. Now the homotopy cartesian condition becomes that $f_p^{-1}(b)\to f_{p-1}^{-1}(d_i(b))$ is a weak equivalence for all $b\in B_p$. We prove by induction that

$$F_j\|E_\bullet\|\to F_j\|B_\bullet\|$$

is a quasi-fibration. For $j=0$, this is clear. For the inductive step, we need to look at the pushout of

$$\begin{array}{ccccc} \Delta^j E_p & \hookleftarrow & \partial\Delta^j\times E_p & \longrightarrow & F_{j-1}\|E_\bullet\| \\ \downarrow{\scriptstyle \mathrm{fib}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{q-fib}} \\ \Delta^j\times B_p & \hookleftarrow & \partial\Delta^j B_p & \longrightarrow & F_{j-1}\|B_\bullet\|. \end{array}$$

But this is the setting in the proposition of Hordic, and so the map $F_j\|E_\bullet\| \to F_j\|B_\bullet\|$ is a quasi-fibration if $(\mathrm{id} \times f_p)^{-1}(t,b) \to (F_{j-1}\|f_\bullet\|)^{-1}(g(f,b))$ are weak equivalences. But the first one is weak equivalent to $f_p^{-1}(b)$ and the second when is weak equivalent to $f_{p-1}^{-1}(d_i(b))$, but they are weakly equivalent by the hypothesis. As we set $j \to \infty$, we get that $\|E_\bullet\| \to \|B_\bullet\|$ is a quasi-fibration.

To show that the diagram is homotopy cartesian, we only need to check weak equivalence on homotopy fibers, but this is true because we have a quasi-fibration. □

The main application is to $EM \to BM$ where M is a topological monoid and $EM = |B_\bullet(x, M, M)|$ and $BM = |B_\bullet(*, M, *)|$. We say that M is **group-like** if $\pi_0 M$ is a group.

Corollary 22.3. *If M is group-like and well-pointed, then $M \simeq \Omega BM$.*

Proof. Note that $EM \simeq *$. Since M is well-pointed, we can replace geometric realization by thick geometric realization. Then we can look at

$$\begin{array}{ccc} M & \longrightarrow & \|B_\bullet(*, M, M)\| \simeq * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \|B_\bullet(*, M, *)\| \simeq BM. \end{array}$$

If this is homotopy cartesian, then the vertical homotopy fibers M and ΩBM will be weakly equivalent.

So we want to check that the diagram

$$\begin{array}{ccc} M^p \times M & \xrightarrow{d_i} & M^{p-1} \times M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ M^p & \xrightarrow{d_i} & M^{p-1} \end{array}$$

is homotopy cartesian. If $i < p$, the diagram looks like

$$\begin{array}{ccc} X \times M & \xrightarrow{f \times \mathrm{id}} & Y \times M \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

so this is clear. For $i = p$, the diagram looks like

$$\begin{array}{ccc} Z \times M^2 & \xrightarrow{\mathrm{id} \times \mu} & Z \times M \\ \downarrow{\scriptstyle \mathrm{id} \times \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ Z \times M & \xrightarrow{\pi_1} & Z. \end{array}$$

This is homotopy cartesian if $\mu \times \pi_1 : M^2 \to M^2$ is a weak equivalence. But this we did in one of the homework. By the result of Suguwara, this is a weak equivalence for M path-connected. If M is not path-connected, at least this shows this for the identity path component. Because M is group-like, all path components are weakly equivalent. □

22.2 A lemma of McDuff

We haven't used a lot about weak equivalences in the theory of quasi-fibrations. So we should be able to replace weak equivalences with homology equivalences not much will change. The main part where homotopy was used was May's lemma. By excision, we have the following.

Lemma 22.4. *Let $X_1, X_2 \subseteq X$ and $B_1, B_2 \subseteq B$ be excisive triples with a map f between them. If $(X_i, X_1 \cap X_2) \to (B_i, B_1 \cap B_2)$ is n-homology connected for $i = 1, 2$ then $(X, X_i) \to (B, B_i)$ is n-homology connected for $i = 1, 2$.*

We say that

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \big\downarrow f & & \big\downarrow f' \\ B & \xrightarrow{g} & B' \end{array}$$

is a **homology cartesian** if $\mathrm{hofib}_b(f) \to \mathrm{hofib}_{g(b)}(f')$ is a homology equivalence. Then a version of Segal's lemma with homology cartesian is true.

Proposition 22.5 (group completion theorem)**.** *If M is homotopy commutative, then*

$$H_*(M)[\pi_0^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

22.3 Bott periodicity

We now prove that $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$. Once you have proven this, you can compute

$$\pi_i(\mathbb{Z} \times BU) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

The strategy is to consider the homotopy commutative topological monoid

$$M = \coprod_{n \geq 0} BU(n).$$

Lemma 22.6. $\Omega BM \simeq \mathbb{Z} \times BU$*.*

Proof. The lemma of Segal–McDuff shows that

$$\Omega BM \simeq_{H_*} \mathrm{hocolim}_x\, M = \mathbb{Z} \times BU.$$

But both spaces are simply connected. The right hand side is clearly simply connected, and the left hand side is because $M \to \mathbb{N}$ is 2-connected, and then $BM \to B\mathbb{N}$ is 3-connected, and so $\Omega BM \to \Omega B\mathbb{N} \simeq \mathbb{Z}$ is 2-connected. □

Since $\Omega(\mathbb{Z} \times BU) \simeq U$, it suffices to show that $BM \simeq U$. Then we will get

$$\Omega(\mathbb{Z} \times BU) \simeq \Omega U \simeq \Omega BM \simeq \mathbb{Z} \times BU.$$

The intuitive idea is that BM is built out of M^p. Well M is roughly the moduli space of finite-dimensional complex vector spaces. So M^p is the moduli space of p tuples tuples of $\mathbb{C}$-vector spaces, or p orthogonal subspaces of $\mathbb{C}^\infty$. These can be thought of as eigenspaeces of some hermitian operator.

Proposition 22.7. *$BM \simeq U$.*

Proof. Let's first think how you built BM. You can do this from a simplicial topological groupoid. From $[p] \mapsto G_p$, we can take $[p] \mapsto BG_p$ and then take geometric realization. Consider

$$[p] \mapsto U_p$$

where U_p has objects $(n_1, \dots, n_p)$ nonnegative integers with morphisms only automorphisms

$$U_p(\vec{n}, \vec{m}) = \begin{cases} \emptyset & \vec{n} \neq \vec{m} \\ U(n_1) \times \cdots \times U(n_p) & \vec{n} = \vec{m}. \end{cases}$$

Then $BU_p = \coprod_{\vec{n}} BU(n_1) \times \cdots \times BU(n_p) = M^p$. Then if we take geometric realization, we should get BM.

Now we note that for proper spaces we can change levelwise the spaces up to weak homotopy. Consider now the category V_p with objects $\coprod_{\vec{n}} V_{\vec{n}}(\mathbb{C}^\infty)$, where $V_{\vec{n}}(\mathbb{C}^\infty)$ is the space of isometrically injective maps $\rho : \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_p} \hookrightarrow \mathbb{C}^\infty$. (This space is contractible.) Morphisms are given by, if there exists a $U(\vec{n}) \ni (A_1, \dots, A_p)$ such that $\rho' \circ (A_1 \oplus \cdots \oplus A_p) = \rho$ then this is the unique morphism $\rho \to \rho'$. This is set up so that BV_p is described as

$$BV_p = \coprod_{\vec{n}} B(*, U(\vec{n}), V_{\vec{n}}(\mathbb{C}^\infty)) \to BU_p = \coprod_{\vec{n}} B(*, U(\vec{n}), *).$$

Because $V_{\vec{n}}(\mathbb{C}^\infty) \to *$ is a weak equivalence, this map is a weak equivalence.

We compare this with a different simplicial topological category P_p. An object $\Pi(p)$ is the space of p-tuples $(E_1, \dots, E_p)$ of finite rank Hermitian projections such that $E_i E_j = 0 = E_j E_i$ if $i \neq j$, and morphisms only identities. There is a functor

$$V_p \to P_p; \quad \rho \mapsto (\text{projection onto } \rho(\mathbb{C}^{n_i})).$$

This gives a map $|[p] \mapsto BV_p| \to |[p] \mapsto BP_p| = |[p] \mapsto \Pi(p)|$. Levelwise, $BV_p \to \Pi(p)$ is a Serre fibration, and the fibers are $B(*, U(\vec{n}), U(\vec{n})) \simeq *$ by extra degeneracy. So $BV_p \simeq \Pi(p)$ and then then $BM \simeq |[p] \mapsto \Pi(p)|$.

The claim now is that this is homeomorphic to $\mathcal{U}$. We can send

$$\begin{aligned}(\vec{t}, (E_1, \dots, E_*)) \mapsto (\mathrm{id} - E_1 - \cdots - E_p) \\ + e^{2\pi i t_0} E_1 + e^{2\pi i (t_0 + t_1)} E_2 + \cdots + e^{2\pi i}(t_0 + \cdots + t_{p-1}) E_p.\end{aligned}$$

This is a homeomorphism by the spectral theorem. ∎

Once we have this, we can just define K-theory as the spectrum with even levels $\mathbb{Z} \times BU$, and odd levels U.

23 April 17, 2018

The last topic is going to be the homotopy type of the cobordism category.

23.1 Cobordism category

The definition $\Omega_n^0(*)$ take π_0 twice:

$$\Omega_n^0(*) = \left\{ \begin{matrix} \text{diffeomorphism classes of} \\ \text{closed smooth } n\text{-manifolds} \end{matrix} \right\} /\text{cobordism}.$$

Definition 23.1. $\overline{\mathsf{Cob}}^0(n)$ is the category with objects diffeomorphism classes of closed smooth $(n-1)$-dimensional manifolds and morphisms diffeomorphism classes of cobordisms.

Lemma 23.2. $\Omega_{n-1}^0(*) = \pi_0 B\overline{\mathsf{Cob}}^0(n)$.

Proof. The 0-cells are objects and 1-cells are the non-identity morphisms. So if we mod out by the equivalence relation generated by the 1-cells, it is going to be $\Omega_{n-1}^0(*)$. ☐

To think of diffeomorphism classes as π_0, we need to discuss moduli spaces of manifolds. There is a weak homotopy type $\mathcal{M}_{n-1}$ determined by, for nice X,

$$[X, \mathcal{M}_{n-1}] = \left\{ \begin{matrix} \text{manifold bundles over } X \text{ with} \\ (n-1)\text{-dimensional closed fibers} \end{matrix} \right\} /\text{iso}.$$

We might take manifold bundle to be a submersion, and require that X is smooth. But we will think as nice being paracompact and manifold bundling meaning locally trivial, numerable with fiber a manifold M and transition functions in $\operatorname{Diff}(M)$ (with the C^∞-topology). Then

$$\mathcal{M}_{n-1} \simeq \coprod_{[M]} B \operatorname{Diff}(M).$$

Then $\pi_0 \mathcal{M}_{n-1}$ is going to be the diffeomorphism classes of $n-$-dimensional closed manifolds. One can similarly define this for cobordisms.

We will use a different model which is better for defining a category.

Lemma 23.3. $\operatorname{Emb}(M, \mathbb{R}^\infty)$ *is weakly contractible and the map*

$$\operatorname{Emb}(M, \mathbb{R}^\infty) \to \operatorname{Emb}(M, \mathbb{R}^\infty)/\operatorname{Diff}(M)$$

has local section with respect to a numerable open cover.

So $\operatorname{Emb}(M, \mathbb{R}^\infty)/\operatorname{Diff}(M) \simeq_w B \operatorname{Diff}(M)$. Now we can use this model and define

$$\operatorname{ob}(\mathsf{Cob}^0(n)) = \mathbb{R} \times \coprod_{[M]} \operatorname{Emb}(M, \mathbb{R}^\infty)/\operatorname{Diff}(M).$$

You should really think of this a manifold lying in $\{s\} \times \mathbb{R}^\infty \subseteq \mathbb{R} \times \mathbb{R}^\infty$. For W a cobordism from M_0 to M_1, we write

$$\tilde{W} = (-\infty, 0] \times M_0 \cup W \cup [1, \infty) \times M_1.$$

Now look at

$$\tilde{\mathrm{Emb}}(W, [0,1] \times \mathbb{R}^\infty) = \left\{ \begin{matrix} \text{embeddings } \varphi : \tilde{W} \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty \text{ that are} \\ \text{of the form } \varphi(m,t) = (\varphi(m), t) \text{ on semi-infinite strips} \end{matrix} \right\}.$$

Consider the group $\tilde{\mathrm{Diff}}(W)$ which consists of diffeomorphisms of $\tilde{W}$ which look like a diffeomorphism $\mathrm{id} \times f_0$ on the $(-\infty, 0] \times W_0$ part and likewise on the other part. Then $\tilde{\mathrm{Diff}}(W)$ acts on $\tilde{\mathrm{Emb}}(W, [0,1] \times \mathbb{R}^\infty)$. We can now define

$$\mathrm{mor}(\mathsf{Cob}^0(n)) \subseteq \mathbb{R}^2 \times \tilde{\mathrm{Emb}}(W, [s,t] \times \mathbb{R}^\infty)/\tilde{\mathrm{Diff}}(W)$$

for $s < t$. You should think of this as a cobordism from $\{s\} \times W_0$ to $\{t\} \times W_1$.

This is almost a category, but it doesn't have units. We can think of this as a non-unital topological category $\mathsf{Cob}^0(n)$. Still we can take the nerve as a semi-simplicial set, and we can take the thick geometric realization. The functor $\mathsf{Cob}^0(n) \to \overline{\mathsf{Cob}}^0(n)$ is 1-connected on objects and 0-connected on morphisms. So N_p is $(1-p)$-connected an B is 1-connected. This shows that the functor is a π_0-isomorphism. Therefore

$$\pi_0 B\mathsf{Cob}^0(n) \cong \Omega^0_{n-1}(*).$$

23.2 Galatius–Madsen–Tillmann–Weiss theorem

We can map

$$B\mathsf{Cob}^0(n) \subseteq \left\{ \begin{matrix} \text{always manifold} \\ \text{near } \{0\} \times \mathbb{R}^\infty \end{matrix} \right\} = \operatorname*{colim}_k \left\{ \begin{matrix} \text{always manifold near} \\ \{0\} \times \mathbb{R}^\infty \text{ contained in } \mathbb{R}^{n+k} \end{matrix} \right\}$$
$$\to \operatorname*{colim}_k \Omega^{n+k} \left\{ \begin{matrix} \text{affine } n\text{-planes} \\ \text{in } \mathbb{R}^{n+k} + \emptyset \end{matrix} \right\}.$$

So we have a map

$$B\mathsf{Cob}^0(n) \to \operatorname*{colim}_{k\to\infty} \Omega^{n+k-1} \mathrm{Th}(\gamma^\perp_{n,k}).$$

Definition 23.4. Define the nth **Madsen–Tillmann spectrum** as $MTO(n)$ having $(n+k)$th space $\mathrm{Th}(\gamma^\perp_{n,k})$ and structure maps as above.

Theorem 23.5 (GMTW). $B\mathsf{Cob}^0(n) \simeq \Omega^{\infty-1} MTO(n)$.

If we look at $H_*(MTO(n); \mathbb{F}_2)$, we have

$$H_*(MTO(n); \mathbb{F}_2) = \operatorname*{colim}_{k\to\infty} H_{*+n+k}(\mathrm{Th}(\gamma^\perp_{n,k}; \mathbb{F}_2) = \operatorname*{colim}_{k\to\infty} H_{*+n}(\mathrm{Gr}(\mathbb{R}^{n+k}); \mathbb{F}_2) = H_{*+n}(BO(n); \mathbb{F}_2).$$

So the first homology appears in degree $-n$, and in fact $MTO(n)$ is $(-n)$-connective.

If you define an n-category $\operatorname{Bord}^0(n)$, then $B\operatorname{Bord}^0(n) \simeq \Omega^{\infty-n}MTO(n)$. So

$$\Omega^0_{n-1}(*) = \pi_0 B\operatorname{Cob}^0(n) = \pi_{-1}MTO(n).$$

Proposition 23.6. *$\pi_{-1}MTO(n) \cong \pi_{n-1}MO$.*

This will prove the Pontryagin–Thom theorem. We are going to construct a map $\Sigma^n MTO(n) \to MO$. On the k-th level, we have

$$(\Sigma^n MTO(n))_k = MTO(n)_{n+k} = \operatorname{Th}(\gamma^\perp_{n+k}) \cong \operatorname{Th}(\gamma_{k,n}) \to \operatorname{Th}(\gamma_{k,\infty}) = (MO)_k.$$

So if we let $n \to \infty$, we get a map of spectrum after checking compatibility. But this map $\Sigma^n MOT(n) \to MO$ factors through $\Sigma^{n+1}MTO(n+1)$ because on kth spaces,

$$\begin{array}{ccccc} \operatorname{Th}(\gamma^\perp_{n,k}) & \xrightarrow{\cong} & \operatorname{Th}(\gamma_{k,n}) & & \\ \downarrow & & \downarrow & \searrow & \\ \operatorname{Th}(\gamma^\perp_{n+1,k}) & \xrightarrow{\cong} & \operatorname{Th}(\gamma_{k,n+1}) & \longrightarrow & \operatorname{Th}(\gamma_{k,\infty}). \end{array}$$

So we have a fibration

$$MTO(0) \to \Sigma MTO(1) \to \Sigma^2 MTO(2) \to \cdots \to MO.$$

Lemma 23.7. *The cofiber of $\Sigma^{n-1}MTO(n-1) \to \Sigma^n MTO(n)$ is $\Sigma^\infty BO(n)_+$.*

Proof. If ξ and η is over B, we have a cofiber sequence

$$\operatorname{Th}(p^*\xi) \to \operatorname{Th}(\xi) \to \operatorname{Th}(\xi \oplus \eta)$$

where $p : S(\eta) \to B$. Now we take $B = \operatorname{Gr}_n(\mathbb{R}^{n+k})$, $\xi = \gamma^\perp_{n+k}$, $\eta = \gamma_{n,k}$. Then

$$\operatorname{Th}(p^*\gamma^\perp_{n+k}) \to \operatorname{Th}(\gamma^\perp_{n,k}) \to \Sigma^{k+n}\operatorname{Gr}_n(\mathbb{R}^{k+n})_+.$$

Here, $S(\eta) = \mathrm{O}(n+k)/\mathrm{O}(k) \times \mathrm{O}(n-1)$ and there is a map $\operatorname{Gr}_{n-1}(\mathbb{R}^{n-1+k}) = \mathrm{O}(n+k-1)/\mathrm{O}(k) \times \mathrm{O}(n-1) \to S(\eta)$ and it is $(n+k-2)$-connected. Then $i^*p^*\gamma^\perp_{n,k} = \gamma^\perp_{n,k-1}$ and so $\operatorname{Th}(\gamma^\perp_{n-1,k}) \to \operatorname{Th}(p^*\gamma^\perp_{n,k})$ that is $(2n+k-1)$-connected. So the first space becomes $\Sigma^{-1}MTO(n)$. It is clear that the second and third spaces become $MTO(n)$ and $\Sigma^\infty BO(n)+$. □

Now we have that $\Sigma^{\infty+n}BO(n)_+$ are n-connected. So the map $\Sigma^n MTO(n) \to MO$ is n-connected and in particular

$$\Sigma^{-1}MTO(n) = \pi_{n-1}\Sigma^n MTO(n) \to \pi_{n-1}MO$$

is an isomorphism. So we have the Pontryagin–Thom theorem.

23.3 Tangential structures and $K(\mathrm{FinSet})$

We have

$$B\mathsf{Cob}^{\mathrm{Fr}}(n) \simeq \Omega^{\infty-1}(\Sigma^{-n}\mathbb{S}).$$

In the case $n = 0$, we have that 0-manifolds are points, so

$$\mathrm{ob}(\mathsf{Cob}^0(0)) \simeq *, \quad \mathrm{mor}(\mathsf{Cob}^0(0)) \simeq \coprod_{k\geq 0} B\Sigma_k.$$

So as a semi-simplicial set,

$$N_\bullet \mathsf{Cob}^0(0) \simeq N_\bullet(\text{topological manifold } \textstyle\coprod_{k\geq 0} B\Sigma_k).$$

Because we call $\Omega B(\coprod_{k\geq 0} BU(k)) = \mathbb{Z} \times BU$ just K-theory, it is not that far fetched to call $\Omega B(\coprod_{k\geq 0} B\Sigma_k)$ the K-theory of finite set.

Theorem 23.8 (Barratt–Priddy–Quillen–Segal)**.** $K(\mathrm{FinSet}) \simeq \Omega^\infty \mathbb{S}$.

Then by the group completion theorem,

$$H_*(\Omega B(\textstyle\coprod_{k\geq 0} B\Sigma_k)) \cong H_*(\coprod_{k\geq 0} B\Sigma_k)[\pi_0^{-1}].$$

Then we get

$$H_*(\Omega^\infty_0 \mathbb{S}) \cong H_*(\Omega_0 B(\textstyle\coprod_{k\geq 0} B\Sigma_k)) \cong \mathrm{colim}_{k\to\infty} H_*(B\Sigma_k).$$

So in particular $\pi_1(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$.

24 April 19, 2018

I said there is a map like $B\mathsf{Cob}(n)^0 \simeq \Omega^{\infty-1}MTO(n)$. Today we will try to construct this map, as a zig-zag of weak equivalences.

24.1 Spaces of submanifold

We will write

$$\mathsf{Cob}^0(n) = \operatorname*{colim}_{K\to\infty} \mathsf{Cob}^0(n,k)$$

where $\mathsf{Cob}^0(n,k)$ is manifolds embedded in $\mathbb{R}^{n+k}$.

Definition 24.1. We define $\Psi_n(\mathbb{R}^{n+k})$ the set of closed subsets $X \subseteq \mathbb{R}^{n+k}$ which are smooth submanifolds of dimension n, with topology as follows:

(1) Pick a tubular neighborhood $\varphi_X : \nu_X \hookrightarrow \mathbb{R}^{n+k}$ and let $\Gamma_c(\nu_X)$ denote the compactly supported smooth sections with the C^∞-topology. Then there is a map $\Gamma_c(\nu_X) \hookrightarrow \Psi_n(\mathbb{R}^{n+k})$ and this should be a homeomorphism onto its image.

(2) For $N \subseteq \mathbb{R}^{n+k}$, we have a quotient space $\Psi_n(\mathbb{R}^{n+k}, \mathbb{R}^{n+k}\backslash N)$ as $\Psi_n(\mathbb{R}^{n+k})/\sim$ with $X \sim X'$ if $X \cap N = X' \cap N$.

(3) We have $\Psi_n(\mathbb{R}^{n+k}) = \operatorname{colim}_{r\to\infty} \Psi_n(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus B_r(0))$.

If $U \subseteq \mathbb{R}^{n+k}$, we call $\Psi_n(U \subseteq \mathbb{R}^{n+k})$ the subspace of $\Psi_n(\mathbb{R}^{n+k})$ with $X \subseteq U$. If U is bounded, then

$$\Psi_n(U \subseteq \mathbb{R}^{n+k}) = \coprod_{[M] \text{ closed}} \operatorname{Emb}(M, U)/\operatorname{Diff}(M).$$

Example 24.2. If $X \in \Psi_n(\mathbb{R}^{n+k})$ is compact, then there exists a map $\mathbb{R}^{n+k} \to \Psi_n(\mathbb{R}^{n+k})$ given by $\vec{v} \mapsto X + \vec{v}$. This extends to $S^{n+k} \to \Psi_n(\mathbb{R}^{n+k})$ with $\infty \mapsto \emptyset$.

If $X \in \Psi_n(\mathbb{R}^{n+k})$ and $p \in \mathbb{R}^{n+k}$, there is a zooming map

$$[1,\infty) \to \Psi_n(\mathbb{R}^{n+k}); \quad s \mapsto s(X-p)+p.$$

This extends to $[1,\infty]$ as well.

Lemma 24.3. *If $U_0, U_1 \subseteq X$ is an open cover, then*

$$\begin{array}{ccc} U_0 \cap U_1 & \longrightarrow & U_1 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & X \end{array}$$

is homotopy cocartesian. This means that the map from the homotopy pushout to X is a weak equivalence.

You can do this explicitly, but let's use a new technique.

Definition 24.4. A **Serre microfibration** is a map $f : X \to B$ there is an $\epsilon > 0$ and a partial lift $D^i \times [0, \epsilon] \to X$.

$$\begin{array}{ccc} D^i & \longrightarrow & X \\ \downarrow & \nearrow & \\ D^i \times [0,\epsilon] & & \downarrow \\ \downarrow & & \\ D^i \times [0,1] & \longrightarrow & B \end{array}$$

Lemma 24.5. *If $f : X \to B$ is a Serre microfibration and has weakly contractible fibers, then it is a Serre fibration (and hence a weak equivalence).*

Proof. Let us do this for $i = 0$. Consider γ a path in B. For each $s \in [0, 1]$, pick a point $h_s \in f^{-1}(\gamma(s))$. Then we get an extension of γ to $\tilde{h}_s : [s-\epsilon_s, s+\epsilon_s] \to X$. By compactness, a finite number of intervals cover $[0, 1]$, so we have lifts

$$\tilde{h}_j : [j/N, (j + 1)/N] \to X.$$

These don't connect well, but the fibers are path-connected, so we can connect the endpoints to get a path $\tilde{H} : [0, 1] \to X$. This path $\tilde{H}$ does not cover γ, but covers $\gamma \circ \eta$ for some step-function $\gamma \circ \eta$. Now consider

$$\eta_t = (1 - t)\eta + t \operatorname{id}.$$

This is a family starting at η and a homeomorphism for $t > 0$. So we apply the microfibration property again to

$$\begin{array}{ccc} [0,1] & \xrightarrow{\tilde{H}} & X \\ \downarrow & & \downarrow \\ [0,1] \times [0,1] & \xrightarrow{(s,t) \mapsto \gamma(\eta_t(s))} & B \end{array}$$

to get a lift $\bar{H} : [0, 1] \times [0, \epsilon] \to X$. Then $\bar{H} \circ \eta_\epsilon^{-1}$ is the desired lift.

In the general case, we reduce to $i = 0$ by taking adjoints. You can show that $X^{D^i} \to B^{D_i}$ is a microfibration with weakly contractible fibers. $\square$

Now to show the lemma, it suffices to show that the map

$$U_0 \cup ([0, 1] \times U_0 \cap U_1) \cup U_1 \to X$$

is a microfibration, because the fibers are clearly contractible. You can do this using that U_0, U_1 are opens and D^i is compact.

Let us now look at $\Psi_n(\mathbb{R}^{n+k})$. There is a map

$$\mathrm{Th}(\gamma_{n,k}^\perp) \to \Psi_n(\mathbb{R}^{n+k}),$$

because $\mathrm{Th}(\gamma_{n,k}^\perp)$ is the space of affine planes.

Lemma 24.6. $\mathrm{Th}(\gamma_{n,k}^{\perp}) \hookrightarrow \Psi_n(\mathbb{R}^{n+k})$ *is a weak equivalence.*

Proof. We will take two open subset $U_0, U_1 \subseteq \Psi_n(\mathbb{R}^{n+k})$, by U_0 the set of X with $0 \notin X$, and U_1 the set of X such that there is a unique $p \in X$ closest to the origin. Then we can look at

$$\begin{array}{ccccc} V_0 = \{\infty\} \cup \{(V, \vec{v}) : \vec{v} \neq 0\} & \longleftarrow & V_0 \cap V_1 & \longrightarrow & V_1 = \{(V, \vec{v})\} \\ \downarrow & & \downarrow & & \downarrow \\ U_0 & \longleftarrow & U_0 \cap U_1 & \longrightarrow & U_1. \end{array}$$

Now it suffices to show that all the vertical maps are weak equivalences, because we can take the homotopy pushout. The left two spaces are both weakly contractible, because we can zoom in at the origin. For the right vertical map, they are both weakly equivalent to $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ by zooming in and translating the unique point to the origin at the same time. For the middle one, we can zoom in while making the affine plane distance 1. So this shows that all the vertical maps are weak equivalences. ☐

Now we can take suspension and write

$$\begin{array}{ccc} \Sigma\,\mathrm{Th}(\gamma_{n,k}^{\perp}) & \xrightarrow{\Sigma i_k} & \Psi_n(\mathbb{R}^{n+k}) \\ \big\downarrow{\scriptstyle (t,(V,\vec{v})) \mapsto (V, \vec{v} + t\vec{e}_1)} & & \big\downarrow{\scriptstyle (t,X) \mapsto X + te_1} \\ \mathrm{Th}(\gamma_{n,k+1}^{\perp}) & \xrightarrow{i_{i+1}} & \Psi_n(\mathbb{R}^{n+k+1}). \end{array}$$

Then we get a map of spectrum

$$MTO(n) \xrightarrow{\simeq} \Psi_n$$

where Ψ_n is the spectrum of smooth n-dimensional submanifolds of Euclidean space.

24.2 Scanning map

Definition 24.7. We define $\mathsf{Cob}^0(n,k)$ as the topological non-unital category with objects $(s, X) \in \mathbb{R} \times \Psi^{n-1}(\mathbb{R} \times I^{n+k-1})$ such that $X \subseteq \{s\} \times I^{n+k-1}$. Morphisms are given by $(s, t, W) \in \mathbb{R}^2 \times \Psi_n(\mathbb{R} \times I^{n+k-1})$ such that $s < t$ and $W \cap ((\infty, s] \times I^{n+k-1})$ and $W \cap ([t, \infty) \times I^{n+k-1})$ are products.

Lemma 24.8. *There is a zigzag of weak equivalences*

$$B\mathsf{Cob}^0(n,k) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \Psi_n(\mathbb{R} \times I^{n+k-1})$$

that is natural in k.

Proof. We first look at the map

$$\|X_\bullet^\pitchfork\| \to \Psi_n(\mathbb{R} \times I^{n+k-1})$$

where $X_p^\pitchfork = \{(X, (t_0 < \cdots < t_p))\}$ where $W \pitchfork \{t_i\} \times I^{n+k-1}$. The claim is that ht is a microfibration with weakly contractible fibers. Now we look at the subset $X_\bullet^\pitchfork \supseteq X_\bullet^+$ such that W intersects with each $\{t_i\} \times I^{n+k-1}$ orthogonally. Levelwise weak equivalences by linear interpolation gives

$$\|X_\bullet^+\| \xrightarrow{\simeq} \|X_\bullet^+\|.$$

Finally, we can push away the end parts off to infinity and get $\|X_\bullet^+\| \xrightarrow{\simeq} \|N_\bullet \mathsf{Cob}^0(n,k)\|$. ∎

Now we get

$$\begin{array}{ccc}
B\mathsf{Cob}^0(n,k) & & \Omega^{n+k-1}\,\mathrm{Th}(\gamma_{n,k}^\perp) \\
\uparrow & & \\
\vdots & & \downarrow \\
\downarrow & & \\
\Psi_n(\mathbb{R} \times I^{n+k-1}) & \longrightarrow & \Omega^{n+k-1}\Psi_n(\mathbb{R}^{n+k}).
\end{array}$$

As we take colimit as $k \to \infty$, we get

$$\begin{array}{ccc}
B\mathsf{Cob}^0(n) & & \Omega^{n-1}MTO(n) \\
\uparrow \simeq & & \\
\vdots & & \downarrow \simeq \\
\downarrow \simeq & & \\
\Psi_n(\mathbb{R} \times I^{n+\infty-1}) & \longrightarrow & \Omega^{\infty-1}\Psi_n.
\end{array}$$

The theorem is that the bottom map is also a weak equivalence.

25 April 24, 2018

25.1 Overview

We started with elementary homotopy theory. We showed that we can extend any $X \to Y$ to a coexact sequence $X \to Y \to Cf$ and an exact sequence $Qf \to X \to Y$. These are useful when f is a Hurewicz cofibration (in which case Cf can by replaced with Y/X) or when f is a Hurewicz fibration (in which case Qf can be replaced with the fiber F). This can be used to get a long exact sequence of homotopy group. For CW-complexes, weak equivalences are homotopy equivalences.

At this point, we started talking about spectra. These are extraordinary (co)homology theories, and they are more computable, and detect connectivity when the X is 1-connected. Cohomology theories are represented by $\mathrm{Ho}(\mathsf{Sp})$, and we discussed examples HA, $\mathbb{S}$, $\Sigma^\infty X$, KU, KO, MU, MO, $MTO(n)$, etc.

The main tool for studying cohomology is spectral sequences. They can be used to compute cohomology of filtered spaces. The Atiyah–Hirzebruch–Serre spectral sequence says that

$$E^2_{p,q} = H_p(B; E_q(F)) \Longrightarrow E_{p+q}(X).$$

Then we started looking at classifying spaces. If G is a topological group, BG is a homotopy type determined by

$$[-, BG] \cong \{\text{numerable principal } G\text{-bundles over } -\}/\text{isomorphism}.$$

This can be recognized by BG carrying a numerable principal G-bundle with contractible total space. Using this characterization, we constructed BG by the bar construction $|[p] \mapsto G^p|$ and used this as a model. In the case $G = \mathrm{O}(n)$ and $G = \mathrm{U}(n)$, we also had geometric models for BG in terms of the Grassmannian.

To compute the cohomology of the classifying spaces, it was useful to have a lower bound on the dimension. Using Leray–Hirsch, we constructed Stiefel–Whitney classes $w_i \in H^i(BO; \mathbb{F}_2)$ and Chern classes $c_i \in H^{2i}(BU; \mathbb{Z})$. This allowed us to do the full computation of cohomology.

Then we defined the unoriented bordism groups $\Omega^0_n(*)$. These turned out to be isomorphic to $\pi_n MO$ where MO is the Thom spectrum with kth level $\mathrm{Th}(\gamma_k)$.

To compute $\pi_n MO$, we defined the Steenrod algebra $\mathcal{A} = \pi_{-ast}\mathsf{Fun}(H\mathbb{F}_2, H\mathbb{F}_2)$. We constructed the Sq^i and proved their properties. These squares generated the entire Steenrod algebra, and so we completely determined $\mathcal{A}$.

Using Milnor–Moore, we showed that $H^*(MO, \mathbb{F}_2)$ is free over $\mathcal{A}$. Moreover, we showed that

$$\pi_* MO = \mathbb{F}_2[x_i : i \neq 2^i - 1].$$

By studying quasi-fibrations and homology fibrations, and proving Segal's lemma/group completion theorem, we are able to show Bott periodicity

$$\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU, \quad \Omega^\infty(\mathbb{Z} \times BO) \simeq \mathbb{Z} \times BO.$$

The final thing we did was to give another proof of Thom's theorem. We can think of the unoriented cobordism group as

$$\Omega_n^0(*) \cong \pi_0 B\mathsf{Cob}^0(n).$$

This can be used to deduce the Pontryagin–Thom. Here, we showed that $B\mathsf{Cob}^0(n) \simeq \Omega^{\infty-1} MTO(n)$.

25.2 Outlook

Often you only care about information at $\mathbb{F}_p$ or $\mathbb{Q}$. You can do this at the space level, and this is called localization and completion of spaces. Rationally, there are are a lot of computational tools, if X is 1-connected and finite type. The homotopy theory of these spaces are equivalent to

$$\begin{array}{ccc} & \left\{\begin{matrix}\text{1-connected finite type}\\ \text{rational spaces}\end{matrix}\right\} & \\ \overset{\text{Quillen}}{\swarrow\!\!\nearrow} & & \overset{\text{Sullivan}}{\nwarrow\!\!\searrow} \\ \left\{\begin{matrix}\text{0-connected finite type}\\ \text{DG Lie algebras}\end{matrix}\right\} & \xleftrightarrow{\text{Koszul duality}} & \left\{\begin{matrix}\text{1-connected finite type}\\ \text{CDGAs}\end{matrix}\right\} \end{array}.$$

We could have also talked more about categorical homotopy theory. There is a way of taking homotopy colimit and limit that is invariant under weak equivalence of diagrams. You can define a model structure on functor categories, and for computational purposes you can even build models in terms of geometric realizations of simplicial spaces (or totalization of cosimplicial spaces). This will allow us to use spectral sequences and so on. In the modern approach, you set up a theory of $(\infty, 1)$ categories so that limits and colimits are already homotopy limits and homotopy colimits.

In the computational stable homotopy theory side, the Adams spectral sequence computes $\pi_i \mathsf{Fun}(E, F)$. Classically, the Adams spectral sequence computes the homotopy groups of the sphere spectrum. You first approximate $\mathbb{S}$ from the right by R-module spectra. If we look at

$$\mathbb{S} \to \operatorname{Tot}\Big(R \leftleftarrows\!\!\!\!\!\rightrightarrows R \wedge R \to \cdots\Big),$$

This is not necessarily an equivalence, but if R is connective and $\pi_0 R = \mathbb{F}_p$ then the right hand side is $\mathbb{S}_p^\wedge$. Then the Bousfield–Kan spectral sequence identifies the E^2-page as

$$E_{s,t}^2 = \operatorname{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{t-s}\mathbb{S}_p^\wedge.$$

You maybe want to change R to something nicer. Quillen tells you how to do this for (MU_*, MU_*MU). At each prime p, the spectrum MU splits to pieces, so we get (BP_*, BP_*BP). This is called the Adams–Novikov spectral sequence.

When you are using this pair (MU_*, MU_*MU), you are using the structure of a Hopf algebroid. This is the structure you would get by looking at the ring of functions on an affine stack. Quillen showed that this is a presentation of $\mathcal{M}_{fgl}$. Here, (BP_*, BP_*BP) are the p-typical formal group laws. They are stratified by height.

We didn't to anything with ordinary K-theory. But topological K-theory can be used to study things, like vector fields on spheres or the image of J. This map is given by $O(n) \to \Omega^n S^n$ and passing to $O \to \Omega^n \mathbb{S}$. If you look at its homotopy groups, we get $J : \pi_* O \to \pi_* \mathbb{S}$. Adams computed the image.

You could also replace topological vector bundles by algebraic vector bundles on a scheme. Then we have

$$K_0(R) = \left\{ \begin{matrix} \text{group completion of isomorphism classes of} \\ \text{finitely generated projective modules} \end{matrix} \right\}.$$

This is going to be π_0 of some spectrum $K(R)$. There are lots of applications in geometric topology where $K(\Sigma^\infty_+ \Omega X)$ is important, and also number theory where $K(\mathbb{Z})$ is important.

K-theory has two properties:

- $K(-)$ is "homotopy-invariant" in the algebraic geometry sense. That is, $K(R) \simeq K(R[x])$.
- $K(-)$ satisfies Nisnevich descent.

Motivic homotopy theory is a homotopy theory of $\mathbb{A}^1$-invariant Nisnevich simplicial sheaves on Sm/k. If we take $k = \mathbb{C}$, there is something called Betti realization $\mathrm{Ho}(\mathsf{Sp}_{\mathrm{Mat}(\mathbb{C})}) \to \mathrm{Ho}(\mathsf{Sp})$.

If the spaces have group actions, you can define a genuine homotopy equivalence, which is not only a homotopy equivalence on the underlying space. Here, you can say things like $(KU)^{hC_2} \cong KO$.

Originally the motivation for looking at the cobordism category was not to prove the Pontryagin–Thom theorem. There is something called Mumford's conjecture. Let $\Gamma_{g,1}$ be the isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$ fixing γ. Then the conjecture states that

$$H^*(B\Gamma_{g,1}; \mathbb{Q}) \cong \mathbb{Q}[k_1, k_2, \ldots]$$

for $* \leq \frac{2}{3} g$, where $|k_i| = 2i$. Here,

$$\begin{aligned} \operatorname*{colim}_{g \to \infty} H_*(B\Gamma_{g,1}; \mathbb{Q}) &= H_*(\Omega_0 B(\textstyle\coprod B\Gamma_{g,1}); \mathbb{Q}) \\ &\cong H_*(\Omega B \mathsf{Cob}^{\mathrm{So}}(2); \mathbb{Q}) \cong H_*(\Omega^\infty MTSO(2); \mathbb{Q}). \end{aligned}$$

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