Math 233a - Theory of Schemes I

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This course was taught by Arnav Tripathy, at TTh 2:30-4 in Science Center 507. The course covered roughly the first half of Vakil's *Foundations of Algebraic Geometry*. There were 6 undergraduates and 3 graduate students enrolled. There were two exercise sheets, each around 10 problems for the entire semester, and no course assistants.

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We will use Vakil's *Foundations of Algebraic Geometry*.

1.1 Introduction

Scheme theory is a modern language for algebraic geometry, which is the study of geometry of solutions of systems of polynomial equations. You can ask about the zero locus of $x^2 + y^2 - 1$, and the subject becomes close to geometry or topology, or you can ask about integer solutions of $x^n + y^n - z^n = 0$. If you are doing classical algebraic geometry over $\mathbb{C}$, then you don't need much of a language to get to the front end. But if you want to prove things, or see the connection between the geometric and algebraic side.

I want to talk about the Weil conjectures, which we won't prove because it is quite hard. If we have r equations $f_i(x_1, \dots, x_n) = 0$ then we expect their intersection to have dimension $n - r$. Let us look at a smooth curve X, i.e., a 1-dimensional set of solutions of $y^2 = x^3 + x$. Consider X over $\mathbb{C}$. These are 1-dimensional complex manifolds, and they are parametrized by the genus (or the number of holes).

Here is another thing you can do. How many solutions are there in integers? This is a bit tricky, so let us count the number of solutions over $\mathbb{F}_q$, a finite field. There will be some number of solutions $|X(\mathbb{F}_q)|$. Heuristically, this is a 1-dimensional manifold, and so we expect $|X(\mathbb{F}_q)| \sim q$.

Theorem 1.1. *If g is the genus of X, then*

$$||X(\mathbb{F}_q)| - q| < 2g\sqrt{q}.$$

This is interesting, because the error term somehow depends on the topology of the surface. This is algebraic geometry.

1.2 Category theory

This is basically a language.

Definition 1.2. A **category** $\mathscr{C}$ is some fellow with a collection of objects $\operatorname{ob}\mathscr{C}$ and morphisms $\operatorname{Mor}(A, B)$ for every $A, B \in \mathscr{C}$, with

(i) a composition law: given $A, B, C \in \operatorname{ob}\mathscr{C}$ and $f \in \operatorname{Mor}(A, B)$ and $g \in \operatorname{Mor}(B, C)$ a morphism $g \circ f \in \operatorname{Mor}(A, C)$ such that the composition law is associative,

(ii) identity morphisms: for every $A \in \operatorname{ob}\mathscr{C}$ and element $\operatorname{id}_A \in \operatorname{Mor}(A, A)$ which is a two-sided identity for composition.

Example 1.3. The category Set of sets have objects sets and morphisms maps. The category InjSet have object sets and morphisms injective maps. There are categories Grp, AbGrp, Top, Mfld, Ring, CRing, etc.

Definition 1.4. A morphism $f : A \to B$ is an **isomorphism** if it is invertible, i.e., there exists a $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Definition 1.5. A **poset** is a set P with a binary relation $\leq$ satisfying

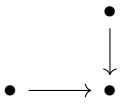
(i) reflexivity: $x \leq x$ for all $x \in P$

(ii) transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$

(iii) antisymmetry: if $x \leq y$ and $y \leq x$ then $x = y$

Proposition 1.6. *A poset is the same thing as a category where all morphism spaces have cardinality at most* 1.

Let us draw an example.



There are also things called diagram categories, which allow more than one arrows.

Definition 1.7. A **subcategory** $\mathscr{D} \subseteq \mathscr{C}$ is the following:

(i) $\mathrm{ob}\,\mathscr{D} \subseteq \mathrm{ob}\,\mathscr{C}$

(ii) for all $A, B \in \mathrm{ob}\,\mathscr{D}$, $\mathrm{Mor}_{\mathscr{D}}(A, B) \subseteq \mathrm{Mor}_{\mathscr{C}}(A, B)$

A **full subcategory** is one where if $A, B \in \mathrm{ob}\,\mathscr{D}$ then $\mathrm{Mor}_{\mathscr{D}}(A, B) = \mathrm{Mor}_{\mathscr{C}}(A, B)$.

Definition 1.8. A (covariant) **functor** $F : \mathscr{C} \to \mathscr{D}$ consists of the following:

(i) $F : \mathrm{ob}\,\mathscr{C} \to \mathrm{ob}\,\mathscr{D}$,

(ii) given $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$ a $F(f) \in \mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$, preserving identity maps and composition.

Example 1.9. There are the forgetful functors. There is a technical definition, but colloquially these are just functors that forget structure. For example, consider $F : \mathsf{AbGrp} \to \mathsf{Set}$ given by $A \mapsto A$ and $(f : A \to B) \mapsto (f : A \to B)$.

Example 1.10. The whole business of algebraic topology is to find interesting functors. The fundamental group is a functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$, or fancier, $\pi_1 : \mathsf{Top} \to \mathsf{Grpd}$. Likewise $H_i(-, \mathbb{Z}) : \mathsf{Top} \to \mathsf{AbGrp}$ and $K_* : \mathsf{Top} \to \mathsf{GrAbGrp}$.

Definition 1.11. A functor is called **faithful** if the associated maps of morphisms are injective, i.e., $\mathrm{Mor}_{\mathscr{C}}(A, B) \hookrightarrow \mathrm{Mor}_{\mathscr{D}}(A, B)$. A functor is **full** if the associated maps of morphisms are surjective.

Example 1.12. The forgetful functor $\mathsf{AbGrp} \to \mathsf{Set}$ is faithful but not full. The forgetful functor $\mathsf{AbGrp} \to \mathsf{Grp}$ is fully faithful.

Example 1.13. For S a set, we define $\mathrm{Subset}(S)$ as the category where objects are subsets $T \subseteq S$ and the morphisms are inclusions. More interestingly, if X is a topological space, we can think of a category $O_P(X)$ of open sets of X with morphisms inclusions. This *is* the topology of X in some sense. You can do this for other objects, for instance, for a manifold X, consider the category of open sets inside submersions to X.

Definition 1.14. Let $\mathscr{C}$ be a (locally small) category and $A \in \mathrm{ob}\,\mathscr{C}$. Define a functor $h^A : \mathscr{C} \to \mathsf{Set}$ in the following way.

- For $B \in \mathrm{ob}\,\mathscr{C}$, let $h^A(B) = \mathrm{Mor}_{\mathscr{C}}(A, B)$.
- For $B_1, B_2 \in \mathrm{ob}\,\mathscr{C}$, we define $\mathrm{Mor}_{\mathscr{C}}(A, B_1) \to \mathrm{Mor}_{\mathscr{C}}(A, B_2)$ by $g \mapsto f \circ g$.

This whole setup gives a **Yoneda embedding**:

$$\mathscr{C} \to \mathrm{Funct}(\mathscr{C}, \mathsf{Set}); \quad A \mapsto h^A$$

Here, $\mathrm{Funct}(\mathscr{C}, \mathscr{D})$ is a category with objects functors and morphisms natural transformations.

Proposition 1.15. *The Yoneda embedding is faithful.*

Why are we doing this thing? Sometimes it is easier to construct stuff in $\mathrm{Funct}(\mathscr{C}, \mathsf{Set})$. Then we only need to show that that thing actually lives in $\mathscr{C}$.

Definition 1.16. For $\mathscr{C}$ a category, the **opposite category** $\mathscr{C}^{\mathrm{op}}$ has the same objects but all the morphism spaces are reversed: $\mathrm{Mor}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Mor}_{\mathscr{C}}(B, A)$.

Definition 1.17. A **contravariant functor** from $\mathscr{C}$ to $\mathscr{D}$ is a (covariant) functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ (or from $\mathscr{C}$ to $\mathscr{D}^{\mathrm{op}}$).

For example, $H^*(-, \mathbb{Z})$ is a functor $\mathsf{Top}^{\mathrm{op}} \to \mathsf{GrAbGrp}$. The dual vector space or the Pontryagin dual also gives a contravariant functor.

Definition 1.18. The (contravariant) **Yoneda embedding** is constructed in the following way. For $A \in \mathrm{ob}\,\mathscr{C}$, there is a contravariant functor $h_A : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$ by $B \mapsto \mathrm{Mor}_{\mathscr{C}}(B, A)$. Then this gives and embedding $\mathscr{C} \hookrightarrow \mathrm{Funct}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$.

Definition 1.19. A **natural transformation** η between two functors $f_1, f_2 \in \mathrm{Funct}(\mathscr{C}, \mathscr{D})$ is the data: for every $A \in \mathrm{ob}\,\mathscr{C}$ such that $\eta(A) \in \mathrm{Mor}_{\mathscr{D}}(f_1(A), f_2(A))$ such that for any $g : A \to B$ in $\mathscr{C}$, the following diagram commutes.

$$\begin{array}{ccc} f_1(A) & \xrightarrow{f_1(g)} & f_1(B) \\ \downarrow{\scriptstyle \eta(A)} & & \downarrow{\scriptstyle \eta(B)} \\ f_2(A) & \xrightarrow{f_2(g)} & f_2(B) \end{array}$$

Definition 1.20. A **representable functor** for $\mathscr{C}$ is an object $F \in \mathrm{Funct}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$ that is in the (essential) image of the Yoneda embedding.

2 January 26, 2017

I figured out most people know about categories.

2.1 Sheaves

In geometry, shapes are good to study by their ring of functions. Here is functor $F : \mathsf{Top}^{\mathrm{op}} \to \mathsf{CRing}$ defined as $X \mapsto \{\text{continuous functions}\}$. Likewise there can be a functor $\mathsf{Diff}^{\mathrm{op}} \to \mathsf{CRing}$ by $X \mapsto C^\infty(X)$. Usually rings are easier, so this functor kind of embeds our category to the category of rings. This is a special case of a Yoneda functor $h_A : X \mapsto \mathrm{Mor}_{\mathscr{C}}(X, A)$, in this case $A = \mathbb{R}$.

Why do we use $\mathbb{R}$? This is because of the Whitney embedding. The basic statement is that there is an embedding $X \hookrightarrow \mathbb{R}^N$. This gives a set of functions that contains everything about X. But there is a problem with this motivation. The holomorphic category $\mathsf{CplxMfld}$ does not have a Whitney embedding. The maximum principle tells us that compact complex manifolds only have constant functions. The solution to this problem is to study functions on open sets on the manifold.

Now we have to keep track of a lot of data, namely for each $U \subseteq X$ the ring of functions $C(U)$ on U. There are going to be restriction maps $\mathrm{res} : C(U) \to C(V)$ and so this is actually a functor $C : \mathrm{Op}(X)^{\mathrm{op}} \to \mathsf{CRing}$.

Definition 2.1. A **presheaf** on a topological space X valued in a category $\mathscr{C}$ is a functor $\mathscr{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathscr{C}$.

Note that you can define analogues of sheaves by using other things than open sets.

Example 2.2. Continuous functions on a topological space, smooth functions on manifolds, holomorphic functions on complex manifolds, polynomials on schemes are presheaves.

Note that if $\mathscr{C}$ is abelian, then the category $\mathsf{PShv}_{\mathscr{C}}(X)$ of all presheaves is also abelian. Also, all constructions of kernels and cokernels are pointwise.

Definition 2.3. A **sheaf** is a presheaf $\mathscr{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathscr{C}$ that satisfies the following: for any open cover $\{U_i\}_{i \in I}$,

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \cap U_j)$$

is exact.

You can try to define categories to other things than commutative rings. One of the basic examples of adjoint functors is the $\otimes$-Hom adjunction. In topological spaces, we have

$$\mathrm{Mor}_{\mathsf{Top}_*}(X \wedge Y, Z) \cong \mathrm{Mor}(X, \mathrm{Mor}(Y, Z)).$$

So you look at it and hmm. Now there are many cohomologies in topological spaces, like $H^*(-;\mathbb{Z})$ or K^* or MU^*. Are these representable? Not quite, but there are these things called "spectra", that are like abelian group objects in topology. You can replace CRing by other categories, and derived algebraic geometry is one of these things. The point is that this is a useful framework.

2.2 Stalks and sheafification

There is a forgetful functor $\mathsf{Shv}(X) \to \mathsf{PShv}(X)$. Now the question is, is there a preferred way to make a sheaf from a presheaf? To restate the question, is there a left adjoint (a free/minimal way to make a presheaf into a sheaf)? The answer is yes, and it is called sheafification, usually denoted s.

Tensor products and cokernels in the category of shaves naïvely lands in the category of presheaves and it needs to be fixed. Because the condition for something being a sheaf is left exact, limt constructions are going to be fine but colimit constructions don't work.

Example 2.4. Take the example on $X = \mathbb{C} \setminus \{0\}$

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \longrightarrow 0,$$

where $\mathbb{Z}$ is the constant sheaf, the sheaf of locally constant functions with values in $\mathbb{Z}$, $\mathscr{O}$ is the sheaf of holomorphic functions, $\mathscr{O}^*$ is the sheaf of nonvanishing holomorphic functions. The map $\mathscr{O}(U) \to \mathscr{O}^*(U)$ is not surjective for $U = X$, because the function $f(z) = z$ is not the exponential of a globally defined function. But this is a surjection in the category of sheaves, because there exists an open cover $\{U_i\}_{i\in I}$ of $\mathbb{C}\setminus\{0\}$ such that $z|_{U_i}$ does have a preimage in $\Gamma(U_i, \mathscr{O})$.

Definition 2.5. The **sheafification functor** is defined by the following.

$$s\mathscr{F}(U) = \varinjlim_{\{U_i\} \text{ open cover}} \ker\Big(\prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \cap U_j) \Big)$$

Proposition 2.6. *If $\mathscr{C}$ is an abelian category, then $\mathsf{Shv}_{\mathscr{C}}(X)$ is also abelian where kernels and as in $\mathsf{PShv}_{\mathscr{C}}(X)$ and*

$$\operatorname{coker}_{\mathsf{Shv}}(\mathscr{F} \to \mathscr{H}) = s(\operatorname{coker}_{\mathsf{Pshv}}(\mathscr{F} \to \mathscr{H})).$$

The **structure sheaf** $\mathscr{O}_X$ on a space is the sheaf of functions on X. Let $\mathscr{F}$ be a (pre)sheaf, and let $p \in X$ be a point. I'd like sections of $\mathscr{F}$ in a very small neighborhood of p.

Definition 2.7. The **stalk** at p is defined as

$$\mathscr{F}_p = \varinjlim_{p\in U \subseteq X} \mathscr{F}(U).$$

The elements of $\mathscr{F}_p$ are **germs** at p.

Example 2.8. Let us look at two sheaves $C^\infty(\mathbb{R})$ and $C^{\text{anal}}(\mathbb{R})$. The ring $C^{\text{anal}}(\mathbb{R})$ is a very large ring, but the stalk at 0 is quite small. It is a subring of $\mathbb{R}[[x]]$ because it is determined by the Taylor series.

2.3 Functoriality of sheaves

Let $f : X \to Y$ be a continuous map of topological spaces. This give functors between the two categories $\mathsf{Shv}(X)$ and $\mathsf{Shv}(Y)$.

Definition 2.9. The **direct image sheaf** is defined as

$$f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V)).$$

Definition 2.10. The **inverse image sheaf** is defined as

$$f^{-1}\mathscr{F}(U) = \varinjlim_{f^{-1}(V) \supseteq U} \mathscr{F}(V).$$

If you remember from category theory, filtered colimits are both left and right exact. This is why this construction works.

3 January 31, 2017

3.1 Idea of schemes

We want to define geometric spaces in terms of the ring of functions. There are two approaches to define the category of schemes. The first one is the functor-of-points, which is basically the Yoneda functor, and the second approach is using locally-ringed-space.

We know how to define a topological space in terms of continuous functions, smooth manifolds as smooth functions, complex manifolds as holomorphic functions. So we want to define a "algebraic space" by regular(polynomial) functions.

For instance, we want to from $\mathbb{A}^n_{\mathbb{C}}$ get $\mathbb{C}[x_1, \dots, x_n]$. In the case of smooth manifolds, we have C^∞-rings, which are commutative rings over $\mathbb{R}$ with some notion of differentiation. So for schemes, we want this ring, which we want to be commutative. We might add some conditions like it is over $\mathbb{C}$, but we want to do this. In fact, here are some additions that algebraic geometry has over other geometries:

- don't need to work over $\mathbb{C}$ or $\mathbb{F}_p$ or $\mathbb{Z}$
- nilpotents (why not?)
- patch things together (which is stupid I think)

Definition 3.1. The category of schemes is defined as $\mathsf{Sch} = \mathsf{CRing}^{\mathrm{op}}$.

This is actually a joke. The reason this doesn't work is because this is the ring of global functions. Instead, we want a sheaf of functions on all open sets. We also need a notion of a "small enough"(affine) scheme that is determined by its global functions.

Definition 3.2. The category of **affine schemes** is defined as $\mathsf{AffSch} = \mathsf{CRing}^{\mathrm{op}}$.

The functor $\mathsf{CRing}^{\mathrm{op}} \to \mathsf{AffSch}$ is called Spec.

Now our focus is the patch together affine schemes to global schemes. The first approach is to use the Yoneda embedding. For a scheme $X \in \mathsf{Sch}$ we have a functor $h_X : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$. This should be determined by $h_X : \mathsf{AffSch}^{\mathrm{op}} = \mathsf{CRing} \to \mathsf{Set}$. So a scheme should be a functor $\mathsf{CRing} \to \mathsf{Set}$. And it we want it to "locally" take the shape of $h_{\operatorname{Spec} B} : A \mapsto \operatorname{Mor}_{\mathsf{CRing}}(B, A)$. This is the functor of points approach, but we are not going to use this.

Definition 3.3. The category $\mathsf{RingSpc}$ of **ringed spaces** is defined as

- objects $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of commutative rings on X,
- morphisms $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ defined by a continuous map $f : X \to Y$ and a map $\phi : \mathscr{O}_Y \to f_* \mathscr{O}_X$ giving the data of how to pull functions.

Definition 3.4. The category $\mathsf{LocRingSpc}$ of **locally ringed spaces** has objects as above, but all stalks $\mathscr{O}_{X,x}$ must be local for all $x \in X$. The morphisms are as above, but we require

$$\phi : \mathcal{O}_{Y,f(x)} \to f_*(\mathscr{O}_{X,x}), \quad \mathfrak{m}_{Y,f(x)} \to \mathfrak{m}_{X,x}.$$

In other words, we want functions that vanish on $f(x)$ to pull back to functions that vanish on x.

3.2 Affine schemes

We want to define $\operatorname{Spec} A$ as a locally ringed space.

Definition 3.5. We define the underlying set of $\operatorname{Spec} A$ as

$$\operatorname{Spec} A = \{\mathfrak{p} \subseteq A \text{ prime ideals}\}.$$

For example, if $A = \mathbb{C}[x]$ then $\operatorname{Spec} A = \{(0), (x-a) \text{ for } a \in \mathbb{C}\}$. So we have the usual complex plane $\mathbb{C}$ and some weird lurker (0).

Definition 3.6. We are going to give the **Zariski topology** on $\operatorname{Spec} A$ so that the closed are going to be of the form

$$V(S) = \{\mathfrak{p} \supseteq S\} \subseteq \operatorname{Spec} A$$

for some subset $S \subseteq A$. You can check that this is a topology.

The affine space is defined as

$$\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n].$$

We have a bunch of maximal ideals $(x_1 - a_1, \dots, x_n - a_n)$. We also have this one (0) ideals. But there are also some intermediate prime ideals $(x_1x_2 - 2)$. So there is some deeper structure in algebraic geometry.

The reason there aren't more maximal ideals is because of the **Nullstellensatz**.

Theorem 3.7 (Weak Nullstellensatz)**.** *If $k = \overline{k}$, then maximal ideals in $k[x_1, \dots, x_n]$ are of the form $(x - a_1, \dots, x - a_n)$.*

Theorem 3.8 (Strong Nullstellensatz)**.** *If k is a field and A is an algbera over k, then A is finitely generated as a k-algebra if and only if A is finite as a k-module.*

Obviously open sets are given as

$$\operatorname{Spec} A \setminus V(S) = \{\mathfrak{p} : \mathfrak{p} \not\supseteq S\}.$$

This is a bit weird, so let us take one function at a time and let

$$D(f) = \operatorname{Spec} A \setminus V(f) = \{\mathfrak{p} : \mathfrak{p} \not\ni f\}.$$

We see that $D(f) = \operatorname{Spec} A_f$.

Definition 3.9. The structure sheaf on $\operatorname{Spec} A$ is defined as $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$.

There are so many things to check. One is about sheaves on a base for a topology. So far I've told you what to get for a base. You need to check that this extends uniquely to all open sets. We also need to check that $D(f) = D(g)$ then $A_f = A_g$. You also need to check the sheaf axioms. Finally you need to check that stalks are local rings. This is actually easy because

$$\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = \varinjlim_{D(f) \ni \mathfrak{p}} A_f = A_{\mathfrak{p}}$$

is a local ring. Let me also check the sheaf axiom.

Proposition 3.10. *$\mathscr{O}_{\operatorname{Spec} A}$ satisfies the sheaf axiom.*

Proof. It suffices to check for basic open covers of basic open sets. Now $U = D(f) = \operatorname{Spec} A_f$ and so we can rename $A_f = A$. So $U = \operatorname{Spec} A$. Suppose I have an open cover of $\operatorname{Spec} A$ by basic opens $\{D(f_i)\}_{i \in I}$. When does $\bigcup D(f_i) = \operatorname{Spec} A$? This means that $(f_i)_{i \in I} = A$, which is equivalent to $1 = \sum_i c_i f_i$ for $c_i \in A$.[1] Now we are trying to show the exactness of

$$1 \to A \to \prod_{i \in I} A_i \to \prod_{i,j \in I} A_{f_i f_j}.$$

Let us show injectivity. We can assume I is finite. We have $a \in A$ such that $a \in A_{f_i}$ is 0 for all i. Then $a \cdot f_i^{n_i} = 0$ for every i. Then

$$a = a \cdot 1^{\sum n_i} = a\Big(\sum c_i f_i\Big)^{\sum n_i} = 0.$$

We now have some finite stuff that is like a_i/f_i. By the condition we have $a_1/f_1 = \cdots = a_n/f_n$. Then we have that this is equal to $(\sum c_i a_i)/(\sum c_i f_i) = \sum c_i a_i \in A$. □

3.3 Schemes

Definition 3.11. A **scheme** $(X, \mathscr{O}_X)$ is a locally ringed space that's "locally isomorphic" to affine schemes. In other words, for every $x \in X$ there exists a neighborhood $U \ni x$ such that $(X, \mathscr{O}_X)|_U = (U, \mathscr{O}|_U)$ is isomorphic (as a locally rined space) to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for $A \in \mathsf{CRing}$.

Example 3.12. Let us try to glue two $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ together. Here is a stupid example to mess around with people. You can try to glue them at every point except for the origin. Then you get an affine line with two points at the origin. The topological space is

$$\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus \{(x) \subseteq \mathbb{C}[x]\}} \mathbb{A}^1.$$

The structure sheaf is defined via base for the topology on one or the other factor. This is an example of a non-separated scheme.

[1] Note that this means that there exists a finite subcover. So $\operatorname{Spec} A$ is always (quasi)compact in the Zariski topology.

Example 3.13. Let us glue two $\mathbb{A}^1$ and this time do something cool. We are going to glue x with $1/x$. This gives $\mathbb{P}^1_{\mathbb{C}}$, which is a "compact" geometric object. The right word is proper.

4 February 2, 2017

First I defined affine schemes AffSch as $\mathsf{CRing}^{\mathrm{op}}$ and defined Sch as a subcategory of $\mathsf{LocRingSpc}$ that looks locally like affine schemes. A **morphism** of schemes is a map $f : X \to Y$ of locally ringed spaces, i.e., $\mathscr{O}_X \to f_*\mathscr{O}_Y$.

4.1 Morphism of schemes

One thing is, we want AffSch to be a full subcategory of Sch. That is, we want to check

$$\mathrm{Mor}_{\mathsf{Sch}}(\operatorname{Spec} B, \operatorname{Spec} A) \cong \mathrm{Mor}_{\mathsf{CRing}}(A, B).$$

Suppose we have a map $\phi : B \to A$. Given a prime ideal $\mathfrak{q} \subseteq B$ we get a prime ideal $\phi^{-1}(\mathfrak{q})$ of A. This is continuous because the inverse image of $D(f)$ is $D(\phi(f))$. Giving the map of sheaves is easy. We need to give a map $\mathscr{O}_{\operatorname{Spec} A} \to \phi_*\mathscr{O}_{\operatorname{Spec} B}$. This is giving a map $\Gamma(D(f), \mathscr{O}_{\operatorname{Spec} A}) \to \Gamma(D(\phi(f)), \mathscr{O}_{\operatorname{Spec} B})$. This is same as giving a map $A_f \to B_{\phi(f)}$. You can do this by using the universal property or something. Finally you need to check that maximal ideals of stalks map to maximal ideals. On stalks, maps are given by $A_{\phi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$. You can check that maximal ideals are sent to maximal ideals.

Conversely, given a map $\operatorname{Spec} B \to \operatorname{Spec} A$ we can look at the global sections and get a map $A \to B$.

For any scheme Y, note that $\mathrm{Mor}_{\mathsf{Sch}}(-, Y)$ is a sheaf. This is because maps are determined locally. What if $Y = \operatorname{Spec} A$? Take $U_i = \operatorname{Spec} B_i$ to be an affine cover of X, and let us assume that $U_i \cap U_j$ are also affine. Then

$$1 \to \mathrm{Mor}(X, \operatorname{Spec} A) \to \prod_i \mathrm{Mor}_{\mathsf{CRing}}(A, B_i) \rightrightarrows \prod \mathrm{Mor}_{\mathsf{CRing}}(A, B_{ij}).$$

We can also use the sheaf property of $\mathscr{O}_X$ and see that

$$1 \to \Gamma(X, \mathscr{O}_X) \to \prod B_i \rightrightarrows \prod B_{ij}.$$

If you think about this, you get an isomorphism

$$\mathrm{Mor}_{\mathsf{CRing}}(A, \Gamma(X, \mathscr{O}_X)) \cong \mathrm{Mor}(X, \operatorname{Spec} A).$$

There is a forgetful functor $\mathsf{Sch} \to \mathsf{AffSch}$. Now what I'm saying is that the global section is the adjoint functor.

Proposition 4.1. *X is affine if and only if $X \to \operatorname{Spec} \Gamma(X, \mathscr{O}_X)$ is an isomorphism.*

Example 4.2. Let's look at $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. There are points like $(x-2, y-4)$ and (0). There are also points like $(y - x^2)$.

What's the use of generic points? You want to sometimes show that something is true generically. In this case, you can show that it is true at the generic point.

Example 4.3. What does $\operatorname{Spec} \mathbb{C}[x,y]_{(x,y)}$ look like? It is first a subspace of $\operatorname{Spec} \mathbb{C}[x,y]$. It only zooms in $(0,0)$ and look at the points (curves, generic points) that passes through $(0,0)$.

Example 4.4. What does $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$ look like? There are ideals generated real irreducible polynomials, $(x-r), (x-\gamma)(x-\bar{\gamma})$ for $\gamma \in \mathbb{C} \setminus \mathbb{R}$. You can think this as $\mathbb{A}^1_{\mathbb{C}}/\operatorname{Gal}(\mathbb{C}/\mathbb{R})$. Same thing for $\operatorname{Spec} \mathbb{F}_q[x]$ or $\operatorname{Spec} \mathbb{Q}[x]$.

Example 4.5. The ring $k[[t]]$ is a discrete valuation ring and so there are two points. This is a more localized version of $\mathbb{C}[x,y]_{(x,y)}$.

4.2 Non-affine schemes

If $(X, \mathscr{O}_X)$ is a scheme, and $U \subseteq X$ is a scheme, then I claim that $(U, \mathscr{O}_X|_U)$ is a scheme. As a locally ringed space, it is obvious. For any point $p \notin U$ take an affine neighborhood of p. This intersected with U is also locally affine because we have basic open sets $D(f)$ contained in U.

Example 4.6. Let's take $U = \mathbb{A}_2 \setminus \{(0,0)\}$. I claim that this is not affine. Let us compute its global functions. We can take $U_1 = \operatorname{Spec} \mathbb{C}[x,y]_x$ and $U_2 = \operatorname{Spec} \mathbb{C}[x,y]_y$. Then the equalizer sequence shows that

$$\Gamma(U) = \mathbb{C}[x,y]_x \cap \mathbb{C}[x,y]_y = \mathbb{C}[x,y].$$

Now if U is an affine scheme, then $U = \operatorname{Spec} \mathbb{C}[x,y]$. But how do we know that it's not? We know that $U \subseteq \mathbb{A}$ which is not an isomorphism gives an isomorphism $\operatorname{Spec} \Gamma(U, \mathscr{O}_U) \to \operatorname{Spec} \Gamma(\mathbb{A}^2, \mathscr{O}_{\mathbb{A}^2})$. By functoriality, this cannot be the case.

This is actually Hartog's lemma. If you have a codimension two thing then any function defined outside that extends well over that codimension two set.

Example 4.7. We defined $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1 = \mathbb{P}^1$. What is the ring of global sections? We have an open cover and compute it as $\mathbb{C}[x] \cap \mathbb{C}[x^{-1}] = \mathbb{C}$.

5 February 7, 2017

5.1 Open and closed subsets

Schemes are basically topological spaces, and open subsets of a scheme is a scheme. Later we will see that "closed subset of a scheme is also a scheme". Closed subsets in $\operatorname{Spec} A$ looks like $V(I) = \operatorname{Spec} A/I$. So closed subsets of affine schemes are affine schemes.

Abstractly $\mathsf{AffSch} = \mathsf{CRing}^{\mathrm{op}}$. Typically we care about rings A that are finitely generated over a field k, i.e., of finite type. In this case, we have a surjective map $k[x_1, \dots, x_n] \to A$ and so $\operatorname{Spec} A = V(I) \subseteq \mathbb{A}^n_k$. So we will typically look at closed subsets of affine space, or them patched together.

5.2 Nilpotents

There is this guy $\operatorname{Spec} k[x]/(x^2)$. What is the set? It is just $\operatorname{Spec} A = \{(x)\}$. In general if we define the **nilradical**

$$\mathfrak{N}(A) = \{x \in A : x^n = 0\},$$

then this has to be contained in any prime ideal. That is, if $\mathfrak{p} \subseteq A$ is prime, then $\mathfrak{N}(A) \subseteq \mathfrak{p}$.

So that's it. We have one point. There is a typical example of a scheme of one point, $\operatorname{Spec} k$. How are these related? Visibly we have

$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[x]/(x^2) \hookrightarrow \mathbb{A}^1_k.$$

We can also have a sequence of embeddings

$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[x]/(x^2) \hookrightarrow \cdots \hookrightarrow \operatorname{Spec} k[x]/(x^n) \hookrightarrow \cdots \hookrightarrow \operatorname{Spec} k[[x]] \hookrightarrow \mathbb{A}^1_k.$$

This $\operatorname{Spec} k[[x]]$ is the algebraic geometers' local disc around a point. Note that has two points. You can think the sequence as taking thicker and thicker neighborhoods around the point $0 \in \mathbb{A}^1_k$. These are sometimes called "fat points".

You can also look at points in $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. I can look at $k[x, y]/(x^2, y)$. This kills everything in the y-direction, but has some fatness in the x-direction. Compare $k[x, y]/(x, y^2)$ that has some fatness in the y-direction. There can be $k[x, y]/(x^2, xy, y^2)$ or $k[x, y]/(x^2, y^3)$.

Why does one introduce all these? There is a subject called intersection theory. Suppose you have some curves and you ask, how many times do they intersect? For example, how often do two conics intersect? Generically, there are 4 intersections. We want to make this really true. First we take algebraically closed fields to resolve this. Then because there are intersections at ∞, we have to work in projective space. We also may have points of tangency, so we have to take multiplicities into account. In this case, we want the intersection to be the three points, but one of them having multiplicity 2. We want this to be a fat point with fuzz in the tangent direction.

5.3 Projective space

Our goal is to compactify $\mathbb{A}^n$ via "directions in which we can go off to ∞". The solution to this is using **projective space** $\mathbb{P}^n_k$, lines through the origin in $\operatorname{Spec} k[x_0, \dots, x_n]$. There are coordinates on this space, identified with

$$[a_0, \dots, a_n] \sim [\lambda a_0, \dots, \lambda a_n]$$

for $\lambda \in k \setminus \{0\}$.

But points are not good enough for us. So we are now going to glue affine charts to $\mathbb{P}^n_k$. Let us look at the coordinates for which $a_n = 1$. This is going to give an affine chart

$$U_n = \operatorname{Spec} k[x_{0,n}, \dots, x_{n-1,n}].$$

Likewise we can take $U_i = \operatorname{Spec} k[x_{0,i}, \dots, x_{i-1,i}, x_{i+1,i}, \dots, x_{n,i}]$. In order to glue them, I need to provide isomorphisms between open sets $U_{i,j} \subseteq U_i$ and opens in $U_{j,i} \subseteq U_j$. Looking at the picture, we want

$$U_{i,j} = \operatorname{Spec} k[x_{0,i}, \dots, x_{n,i}]_{x_{j,i}} = \operatorname{Spec} k[x_{0,i}, \dots, x_{n,i}, x_{j,i}^{-1}].$$

Then we give isomorphisms

$$\Phi_{i,j} : k[x_{0,i}, \dots, x_{n,i}, x_{j,i}^{-1}] \to k[x_{0,j}, \dots, x_{n,j}, x_{i,j}^{-1}].$$

We want $x_{i,j} = a_i/a_j$. This gives ways of identifying, and it will be $x_{t,i} \mapsto x_{t,j}x_{i,j}^{-1}$ and $x_{i,j} \mapsto x_{j,i}^{-1}$. We need the cocycle relation, and you can check this.

5.4 Connectedness

Definition 5.1. A topological space **connected** if it is not a disjoint union of two open subsets. A **connected component** is a maximal connected subset. (General connected components are closed, but not necessarily open.)

Definition 5.2. A scheme X is **connected** if its underlying space is connected.

Proposition 5.3. $\operatorname{Spec} A$ *is disconnected if and only if* $A = A_1 \times A_2$.

Proof. If $A = A_1 \times A_2$ then $\operatorname{Spec} A = \operatorname{Spec} A_1 \amalg \operatorname{Spec} A_2$. Now suppose $\operatorname{Spec} A = V(I_1) \amalg V(I_2)$. This means that every $\mathfrak{p}$ contains one of I_1 or I_2. We are going to look for idempotents in A. We first see $I_1 + I_2 = A$. Then the other condition says $\sqrt{I_1} \cap \sqrt{I_2} = \mathfrak{N}(A)$. Then $i_1 + i_2 = 1$ with $i_1 \in I_1$ and $i_2 \in I_2$. We also know that $(i_1 i_2)^{\text{big}} = 0$. So you can raise to a power $(i_1 + i_2)^N = 1$ and define so that $i_1' + i_2' = 1$ and $i_1' i_2' = 0$. □

Here we are using

Theorem 5.4. $\mathfrak{N}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$.

Proof. One direction is obvious. For the other direction, take $x \notin \mathfrak{N}(A)$. Then A_x is not the zero ring, and so there is a prime ideal. Then this prime does not contain x. □

Example 5.5. Consider $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. This is something like $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-many copies of $\overline{\overline{Q}}$. But this doesn't have the discrete topology. In fact, the Zariski topology is the profinite topology of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The basis is given by the inverse image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\overline{Q}) \to \mathrm{Gal}(E/\mathbb{Q})$.

6 February 9, 2017

6.1 Projective schemes

You should read about the Proj construction, but for me a **projective scheme** is a closed subset of $\mathbb{P}^n_k$.

What do the closed subsets of $\mathbb{P}^n_k$ look like? They are the vanishing set of some homogeneous polynomials. Scheme theoretically, $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, k_n]$ is defined by gluing some $\mathbb{A}^n_k$ together. For some ideal I, the closure $\overline{V(I)}$ is a closed subset of $\mathbb{P}^n_k$.

Example 6.1. Take $V(I) \subseteq \mathbb{A}^2_{k,z} = \operatorname{Spec} k[x/z, y/z]$ for $I = (x^2 - 4y)$. What is the closure of $V(I)$? The only way I can answer this is by going to the other charts, because that is how I defined $\mathbb{P}^2_k$. Restricting to the intersection $\mathbb{A}^2_{k,z} \cap \mathbb{A}^2_{k,x} = \operatorname{Spec} k[y/x, z/x, x/z]$, we will get $V((x/z)^2 - 4(y/z)^2)$. Now what ideal $J \subseteq k[y/x, z/x]$ will restrict to this set? In other words, I am looking for the maximal ideal J such that any prime $\mathfrak{p}$ containing J also contains $(x/z)^2 - 4(y/z)$. This prime $\mathfrak{p}$ will also contain $1 - (y/x)(z/x)$. So J has to exactly be the radical of $(1 - 4yz/x^2)$.

The point is that we have to naturally work with homogeneous ideals. So closed subset of $\mathbb{P}^n_k$ (radical) homogeneous ideals of $k[x_0, \dots, x_n]$.

6.2 Irreducibility reducedness

Definition 6.2. If X is a topological space, X is said to be **irreducible** if X can't be written as the union of two proper closed subsets. Equivalently, any nonempty open subset is dense. A scheme is **irreducible** if its underlying space is.

Example 6.3. $\mathbb{A}^1_k$ is irreducible because every nonempty open contains the generic point. $\operatorname{Spec} k[x, y]/(xy)$ is not irreducible.

Proposition 6.4. *If A is an integral domain, then* $\operatorname{Spec} A$ *is irreducible.*

Proof. Any nonempty open set contains the generic point. □

Definition 6.5. A ring is **reduced** if $\mathcal{N}(A) = 0$. A scheme X is **reduced** if it satisfies the following equivalent properties:

(i) For every open U, $\Gamma(U, \mathcal{O}_X)$ is reduced.

(ii) There exist an affine cover $\{\operatorname{Spec} A_i\}$ such that each A_i is reduced.

(iii) Each stalk $\mathcal{O}_{X,p}$ is reduced.

Example 6.6. The fat point is not reduced. A fat line, $\operatorname{Spec} k[x, y]/(y^2)$ is not reduced. A line with a fat point, $\operatorname{Spec} k[x, y]/(y, xy)$, is not reduced.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) First note that any affine refinement of a reduced affine cover is again a reduced affine cover. This is because A reduced implies A_S reduced. Then $U = \bigcup \operatorname{Spec} A_{i,f_i}$ and so we get an injection

$$0 \to \Gamma(U, \mathcal{O}_X) \hookrightarrow \prod_i A_{i,f_i}.$$

Equivalence with (iii) can be done similarly. □

Theorem 6.7. *A is an integral domain if and only if* $\operatorname{Spec} A$ *is irreducible and reduced.*

So if something is not an integral domain, you can think about why it fails.

Proof. One direction is trivial. For the other direction, assume $xy = 0$ with $x, y \neq 0$. Then $\operatorname{Spec} A_{xy} = \operatorname{Spec} A_x \cap \operatorname{Spec} A_y$ with $\operatorname{Spec} A_{xy} = \emptyset$ but $\operatorname{Spec} A_x$, $\operatorname{Spec} A_y \neq \emptyset$. □

Here is the way algebraic geometers think:

$$\begin{Bmatrix}\text{closed subsets}\\ \text{of } \operatorname{Spec} A\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{radical ideals}\\ I \subseteq A\end{Bmatrix}$$
$$\begin{Bmatrix}\text{irreducible subsets}\\ \text{of } \operatorname{Spec} A\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{prime ideals}\\ \mathfrak{p} \subseteq A\end{Bmatrix} = \operatorname{Spec} A$$
$$\begin{Bmatrix}\text{irreducible components}\\ \text{of } \operatorname{Spec} A\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{minimal primes}\\ \mathfrak{p} \subseteq A\end{Bmatrix}.$$

6.3 Quasicompactness

Definition 6.8. A scheme is **quasicompact** is its underlying space is compact.

Proposition 6.9. *X is quasicompact if and only if it has a finite affine open cover.*

This is good for ruling out various pathologies. Here is a question. Does any scheme have closed points? The answer is no, but that sucks. If the scheme is quasicompact, then the answer is yes.

Here is an counterexample. Recall that the DVR $k[[t]]$ has two points, one $\operatorname{Spec} k$ and $\operatorname{Spec} k((t))$. Then we can try to stack these schemes together like letting $k_1 = k((t))$ and looking at $\operatorname{Spec} k_1[[t_2]]$. You can make this work, and this scheme has no closed points.

Proposition 6.10. *A quasicompact scheme has a closed point.*

Proof. Use the finite intersection property. You can prove that if V is closed and has at least two points, then there exists a proper closed subset $W \subseteq V$. □

7 February 14, 2017

Recall that quasicompact schemes have closed points. In fact, schemes of finite type over k have a dense set of closed points. (This should be some form of Nullstellensatz.)

7.1 Affine communication lemma

We were defining basic properties of schemes. We wish the property P be such that the following are equivalent:

(1) P holds for any affine open subset of X.

(2) P holds for some affine open cover of X.

Lemma 7.1. *Suppose P is a property of affine opens in X. Suppose:*

(1) If $\operatorname{Spec} A \subseteq X$ has property P, then $\operatorname{Spec} A_f \subset X$ has property P.

(2) If $\operatorname{Spec} A \subseteq X$ if affine open and has a basic open cover $\{\operatorname{Spec} A_{f_i}\}$ each of one having property P, then $\operatorname{Spec} A$ has property P.

Then if there exists an open affine cover of X having P, every affine has P.

Proof. I need to check that $\operatorname{Spec} B$ has P. We have an open cover $\{\operatorname{Spec} B \cap \operatorname{Spec} A_i\}$ of $\operatorname{Spec} B$. You can see that it suffices to show that $\operatorname{Spec} A \cap \operatorname{Spec} B$ can be covered by sets that are basic open in both.

So let us prove this. For any $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we want to find a neighborhood of p that is basic open in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $p \in \operatorname{Spec} A_f \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B$ and let $p \in \operatorname{Spec} B_g \subseteq \operatorname{Spec} A_f$. Then we get a restriction map $B \to A_f$, and so the image of g gives a $\tilde{g} \in A_f$. Then $\operatorname{Spec} B_g = \operatorname{Spec}(A_f)_{\tilde{g}}$. □

Example 7.2. Being reduced is an example of P. Being Noetherian is has this property. If A is Noetherian, then A_f is Noetherian. Suppose A_{f_i} are Noetherian. Take any ascending chain $I_1 \subseteq I_2 \subseteq \cdots$. Then in A_{f_i} it stabilizes, and so after the maximum of the place they stabilizer, I_i stabilizes.

Definition 7.3. A scheme X is **locally Noetherian** if it satisfies the following equivalent two conditions:

(i) there exists an affine open cover by Noetherian affines.

(ii) every affine open is Noetherian.

Definition 7.4. A topological space X is **Noetherian** is it satisfies the descending chain condition for closed sets.

So if A is Noetherian, then $\operatorname{Spec} A$ is Noetherian. The reason we care about this is because there is Noetherian induction, which is quite useful. We want Noetherian schemes to have this property. So we need some more finite condition that being locally Noetherian. This is why we have "locally".

Definition 7.5. A scheme X is **Noetherian** if it is locally Noetherian and quasicompact.

We can do it with other P.

Definition 7.6. A scheme X is **locally of finite type over** R if there exists an open affine cover by finitely generated R-algebras. It is called **of finite type over** R if it is locally of finite type over R and quasicompact.

7.2 Properties of morphisms

We don't definite more properties of schemes. This is mostly the influence of Grothendieck: only define properties of morphisms. Then how can we talk about properties of schemes? Every scheme X comes with a map like $X \to \operatorname{Spec} k$, or at least $X \to \operatorname{Spec} \mathbb{Z}$. This is called the **structure map**. Note that $\operatorname{Spec} \mathbb{Z}$ is the terminal object in the category of schemes. So this makes sense.

But why? Let us think in the category of topological spaces. When is a map compact? This means that the inverse image of compact sets is compact. Then a space X is compact if and only if $X \to *$ is a compact map, trivially.

Definition 7.7. A morphism $\pi : X \to Y$ is **quasicompact** if it satisfies the following equivalent conditions:

(i) the inverse image of some affine open cover is quasicompact.

(ii) the inverse image of any affine open is quasicompact.

(iii) the inverse image of any open quasicompact is quasicompact.

Definition 7.8. A scheme X is **quasi-separated** if the intersection of any two affine opens is quasicompact.

Note that if X is quasicompact and quasi-separated, then there exists a finite affine open cover $\{\operatorname{Spec} A_i\}_{i \in I}$ of X such that each pairwise intersection $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ has a finite open cover by $\{\operatorname{Spec} A_{ijk}\}$ that are both basic and open in both A_i and A_j.

The reason this is important is because the products in the sheaf condition

$$1 \to \Gamma(X, \mathscr{O}_X) \to \prod A_i \rightrightarrows \prod A_{ijk}$$

are finite.

Definition 7.9. A morphism $\pi : X \to Y$ is **quasi-separated** if it satisfies the following equivalent conditions:

(i) the inverse image of any affine open is quasi-separated.

(ii) the inverse image of some affine open cover is quasi-separated.

Definition 7.10. A morphism $\pi : X \to Y$ is **affine** if satisfies the following equivalent conditions:

(i) the inverse image of any affine open is affine.

(ii) the inverse image of some affine open is quasi-separated.

Proof. Let P to be "affine preimage". The first condition of the affine communication is easy. The second condition is hard. You want to show that $X \to \Gamma(X, \mathscr{O}_X)$ is an isomorphism. You need to use the fact that isomorphisms can be checked affine locally. Look at the diagram

$$\begin{array}{ccc} \operatorname{Spec} B_i & \hookrightarrow & X \\ \downarrow & & \downarrow \eta \\ \operatorname{Spec} \Gamma(X, \mathscr{O}_X)_{\hat{\pi}(f_i)} & \hookrightarrow & \operatorname{Spec} \Gamma(X, \mathscr{O}_X) \\ \downarrow & & \downarrow \hat{\pi} \\ \operatorname{Spec} A_{f_i} & \hookrightarrow & \operatorname{Spec} A. \end{array}$$

We want to check that $\Gamma(X, \mathscr{O}_X)_{\hat{\pi}(f_i)} \to B_i$ is an isomorphism. The diagram means that the locus in X of where $\hat{\pi}(f_i)$ doesn't vanish is $\operatorname{Spec} B_i$. Then $\Gamma(\operatorname{Spec} B_i, \mathscr{O}_X) = \Gamma(X, \mathscr{O}_X)_{\hat{\pi}(f_i)}$. (This works because X is quasicompact and quasi-separated.) $\square$

8 February 16, 2017

We were defining properties of morphisms. We certainly want the property to be target-space local.

8.1 Affineness is affine local

We were trying to show that if $\pi : X \to \operatorname{Spec} A$ is a morphism, $\operatorname{Spec} A$ has an open cover $\operatorname{Spec} A_{f_i}$, and the inverse image of $\operatorname{Spec} A_{f_i}$ is $\operatorname{Spec} B_i$, then X is affine. We factored π to $X \to \operatorname{Spec} \Gamma(X, \mathscr{O}_X)$ and $\hat{\pi} : \operatorname{Spec} \Gamma(X, \mathscr{O}_X) \to \operatorname{Spec} A$. The inverse image of $\operatorname{Spec} A_{f_i}$ under $\hat{\pi}$ is

$$\hat{\pi}^{-1}(\operatorname{Spec} A_{f_i}) = \operatorname{Spec} \Gamma(X, \mathscr{O})_{\hat{\pi}(f_i)}.$$

We claim that isomorphisms can be checked locally:

Proposition 8.1. *If $f : X \to Y$ is a morphism of schemes such that $\{U_\alpha\}$ is an open cover of Y and $f^{-1}(U_\alpha) \to U_\alpha$ is an isomorphism for all α, then f is an isomorphism.*

So we can just check that $\operatorname{Spec} B \to \operatorname{Spec} \Gamma(X, \mathscr{O}_X)_f$ is an isomorphism. But we really need to know what B is. We see that

$$B = \Gamma(X \times_{\operatorname{Spec} A} \operatorname{Spec} A_f = X_f, \mathscr{O}_X).$$

Here X_f is the open subscheme of where f does not vanish on X, where f is considered as in $\Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$. So what we are trying to show is that

$$\Gamma(X, \mathscr{O}_X)_f \to \Gamma(X_f, \mathscr{O}_X)$$

is an isomorphism.

This is now a general question. When is this map an isomorphism? We have to use the sheaf condition. Cover the X by affines $\{\operatorname{Spec} R_\alpha\}$. Cover the intersections $\operatorname{Spec} R_\alpha \cap \operatorname{Spec} R_\beta$ by affines $\operatorname{Spec} R_{\alpha\beta\gamma}$ that are affine in both affines. Then we have

$$\Gamma(X, \mathcal{O}_X) = \operatorname{equalizer}\Big(\prod R_\alpha \rightrightarrows \prod R_{\alpha\beta\gamma}\Big).$$

Localizations commutes with finite products. So assuming that all stuff are finite, we get that

$$(\Gamma(X, \mathcal{O}_X))_f = \operatorname{equalizer}\Big(\Big(\prod R_\alpha\Big)_f \rightrightarrows \Big(\prod R_{\alpha\beta\gamma}\Big)_f\Big),$$

$$\Gamma(X_f, \mathcal{O}_X) = \operatorname{equalizer}\Big(\prod (R_\alpha)_f \rightrightarrows \prod (R_{\alpha\beta\gamma})_f\Big).$$

So this is true if X is quasicompact and quasi-separated.

Now is X quasicompact and quasi-separated? These notions are local, and there is an affine cover of $\operatorname{Spec} A$ with affine (and thus quasicompact and quasi-separated) preimage. So we see that the inverse image of $\operatorname{Spec} A$, which is X is quasicompact and quasi-separated.

8.2 Quasicoherent sheaves

We want to talk about closed embeddings. This is not just a topological property, because there is something happening at the structure sheaf. There is a map $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$. This is not determined by the underlying set. But for a general scheme X, how do we talk about its ideals?

Let us talk about modules. This is something of the philosophy that if you want to look at groups, then you look at representations, if you want to look at rings R, then you look at R-modules on which R act on. It's somehow easier to look at linear structures. In the case of schemes, we might want to look at sheaves of modules.

Definition 8.2. Let X be a topological space and $\mathscr{O}_X$ be a sheaf of rings on X. We define the category

$$\mathscr{O}_X\text{–}\mathsf{Mod} = \{\text{category of sheaves of modules for } \mathscr{O}_X\},$$

which are the sheaves $\mathcal{F}$, such that for any open $U \subseteq X$, $\mathcal{F}(U)$ is naturally a module over $\mathscr{O}_X(U)$ such that the structure morphisms $\mathscr{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ commute with restriction. For X a scheme, we define $\mathscr{O}_X\text{–}\mathsf{Mod}$ as above.

Let M be a A-module. This gives a $\mathscr{O}_{\operatorname{Spec} A}$-module $\tilde{M}$, define by

$$\tilde{M}(\operatorname{Spec} A_f) = M_f = M \otimes_A A_f.$$

We need to check the sheaf condition, but that is the same as checking the sheaf condition for rings. So we get a functor

$$F_A : A\text{-}\mathsf{Mod} \to \mathscr{O}_{\operatorname{Spec} A}\text{-}\mathsf{Mod}.$$

This is not essentially surjective. Take $\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$, and extend this sheaf by zero as $j_!(\mathscr{O}_{\mathbb{A}^1 \setminus \{0\}})$, because the global section is 0.

Definition 8.3. For X a scheme, a **quasicoherent sheaf** $\mathcal{F}$ is a sheaf of $\mathscr{O}_X$-modules such that the following equivalent conditions hold:

(i) There exists an affine open cover such that on any $\operatorname{Spec} A$, $\mathcal{F}|_{\operatorname{Spec} A}$ is in the essential image of F_A.

(ii) For any affine open $\operatorname{Spec} A$, the restriction is in the essential image of F_A.

9 February 28, 2017

We were talking about quasicoherent sheaves, which form an abelian category $\mathsf{QCoh}(X)$. These are just locally modules over the rings over open affine sets. Then what are coherent sheaves? Coherent sheaves are supposed to be an analogue of finitely presented modules. A finitely generated module is something like $A^n \to M \to 0$. Sometimes you want the kernel also to be finitely generated, and this is $A^m \to A^n \to M \to 0$, which is also called finitely presented. But this does not glue well.

Definition 9.1. A **coherent sheaf** on X is

(1) a quasicoherent sheaf locally modeled on finitely generated modules *on Noetherian schemes*. (This is then automatically finitely presented.)

(2) There is a general definition, but we don't care.

The abelian category is denoted by $\mathsf{Coh}(X)$.

We have $\mathsf{QCoh}(X) \subseteq \mathscr{O}_X\text{−}\mathsf{Mod}$, and you can check that both are abelian, i.e., closed under direct sum, kernel, cokernel, and image. The functor $F_A : A\text{−}\mathsf{Mod} \to \mathsf{QCoh}(\operatorname{Spec} A)$ is exact, i.e.,

$$0 \to \widetilde{\ker} \to \tilde{M} \to \tilde{N} \to \widetilde{\operatorname{coker}} \to 0.$$

Tensor products exist. In general, $\mathscr{F} \otimes \mathscr{G}$ needs to be sheafified. But you can check that $\tilde{M} \otimes \tilde{N} \cong \widetilde{M \otimes_A N}$ and so nothing terrible happen.

In general, there is a Hom-set of sheaves. We define the **sheaf Hom**

$$(\mathscr{H}om(\mathscr{F}, \mathscr{G}))(U) = \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U).$$

This can be checked to be a sheaf. On quasicoherent sheaves, you can check that $\mathscr{H}om(\tilde{M}, \tilde{N}) = \widetilde{\operatorname{Hom}_A(M, N)}$.

If you want to do talk about derived functors, there is the problem of whether there are enough projectives or enough injectives. For instance, you might want to take the derived functor of Γ, which is left-exact. This gives rise to the notion of sheaf cohomology:

$$0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}) \to \Gamma(\mathscr{H}) \to H^1(\mathscr{F}) \to H^1(\mathscr{G}) \to \cdots.$$

9.1 Geometry of quasicoherent sheaves

I am going to talk about sheaves again. I have some topological space, and there is are some sections over open sets. The most motivating example is the sheaves of functions (not necessarily in the context of algebraic geometry). Let's generalize this example. I can look at sheaves of maps to some other object. We can define a sheaf defined by $\Gamma(U, \mathscr{F}) = \operatorname{Mor}(U, Y)$. Here is a slightly bigger example. Suppose we have some $\mathscr{X} \to X$. Then we can think of the sheaf of sections: $\Gamma(U, \mathscr{F}) = \operatorname{Sect}(\pi^{-1}(U) \to U)$. This is indeed good enough, as any sheaf is of this form. (Look up the espace étalé construction.)

Example 9.2. If X is a scheme with structure $\mathscr{O}_X$, consider $\mathscr{O}_X^{\oplus n}$. This is a free sheaf. In the context of other geometry, this is like the sections of $X \times \mathbb{R}^n \to X$. This is quite boring.

Example 9.3. We can twist this trivial bundle and think about rank n vector bundles. In C^∞ manifolds, a **vector bundle** V on a manifold X is a morphism $\pi : V \to X$ such that there exists an open cover (U_α) of X such that on each U_α the restriction $\pi^{-1}(U_\alpha)$ is isomorphic to $U_\alpha \times \mathbb{R}^n$. These correspond to locally free shaves.

Definition 9.4. A **vector bundle** of rank n on a scheme X is a sheaf $\mathscr{F}$ such that there exists an open affine cover U_α of X such that $\mathscr{F}|_{U_\alpha} \cong \mathscr{O}_{U_\alpha}^{\oplus n}$. This is the same thing as a **locally free sheaf**.

It is clear that vector bundles are quasicoherent sheaves. Given a locally free sheaf, you can actually recover the total space of the vector bundle by taking the sheafy symmetric algebra and then taking the Spec and gluing them: $\mathrm{Tot}(\mathscr{F}) = \mathscr{S}pec(\mathscr{S}ym\mathscr{F}^\vee)$.

What are vector bundles on affine schemes? They are quasicoherent sheaves so they correspond to A-modules. Let us all these $\{M\}$. Then clearly $M_\mathfrak{p}$ need to be free. Over local Noetherian rings, finitely generated and flat is equivalent to free, and is equivalent to finitely generated and projective. Since being projective and being flat are both local properties, we get that M is globally projective and flat. So over a Noetherian scheme, these are just finitely generated projective modules.

Theorem 9.5 (Serre–Swan)**.** *In C^∞-manifolds, the category of vector bundles is equivalent to the finitely generated projective $C^\infty(M)$-modules.*

Let us go back to general quasicoherent sheaves. Here is a general intuition. Suppose $\mathscr{F}$ is a sheaf. Coherent sheaves for me are going to be vector bundles on some stratification. On the other side, there is are skyscraper sheaves. Let us try to make this precise.

9.2 Generic freeness

Theorem 9.6 (Generic freeness, Grothendieck)**.** *If M is a finitely generated module on a Noetherian integral domain A, then there exists $f \neq 0$ such that M_f is a free A_f-module.*

Proof. Let us go to the generic point of A. Then we have $M \otimes_A K(A)$ is free over $K(A)$ of let's say dimension n. Pick n basis elements $m_1/a_1, \ldots, m_n/a_n$. Pick $f = a_1 \cdots a_n$ and look at A_f. Now in M_f all these elements make sense. This gives us a map $A_f^n \to M_f$. Tensoring with $K(A)$ with yield the isomorphism $K(A)^n \to M \otimes K(A)$. Let

$$0 \to N_1 \to A_f^n \to M_f \to N_2 \to 0.$$

Localizing gives $N_1 \otimes K(A) = N_2 \otimes K(A) = 0$. Now N_1 and N_2 are finitely generated by the Noetherian hypothesis, and they are torsion elements. Since there are a finite number of them, you can enlarge f sufficiently large so that they all die out. □

Corollary 9.7. *For $\mathscr{F} \in \mathsf{Coh}(X)$ where X is Noetherian and integral, there exists a $U_1, U_2, \ldots, U_n$ where U_1 is dense in X, U_2 dense in $X \setminus U_1$, ..., $U_n = X \setminus U_1 \setminus \cdots \setminus U_{n-1}$, such that $\mathscr{F}|_{U_i}$ is a vector bundle.*

There is something called mirror symmetry. Given two varities X and Y, the algebraic geometric properties of X match the symplectic geometry of Y in a very nontrivial way. The coherent sheaves of X match the Lagrangians on Y. For instance, for $Y = T^2$ we have X an elliptic curve.

Because the theorem is due to Grothendieck, there is going to be a ridiculous generalization:

Theorem 9.8. *If M is a finitely generated algebra B over a Notherian integral domain A, then there exists a $f \neq 0 \in A$ such that M_f is a free A_f-module.*

10 March 2, 2017

We were talking about quasicoherent sheaves and generic freeness.

Theorem 10.1 (Grothendieck generic freeness)**.** *If M is a finite module over B and B is a finitely generated algebra over A, and A is a Noetherian domain, then there exists an $f \neq 0$ in A such that M_f is a free A_f-module.*

Proof. We start reducing the problem. We have a surjection $A[x_1, \dots, x_n] \to B$. M is a module over B, and so we can consider M as a finitely generated module over $A[x_1, \dots, x_n]$. That is, we may assume $B = A[x_1, \dots, x_n]$ without loss of generality.

You can also check that it suffices to check for $B = A[x]$, because we can use induction to get more generators.

Now we need some new ideas. Let M be a finite module over $A[x]$, we want to show that there exists a $f \in A$ such that M_f is a free A_f module. What we did last time actually use the fact that M is finite over A. So the idea is to break M up (using the x-action) into pieces that are finite over A.

Suppose M is generated by $m_1, \dots, m_n$ as a module over $A[x]$. Let $M_1 \subseteq M$ be the A-submodule generated by $m_1, \dots, m_n$. Define M_i as the A-submodule of M given by $M_i = M_{i-1} + xM_{i-1}$. By definition, $M = \bigcup_{i=1}^{\infty} M_i$. Look at the successive M_i/M_{i-1} (with $M_0 = 0$). These are all finite as A-modules, because they are generated by $x^i m_j$.

We claim that $\{M_i/M_{i-1}\}_{i=1}^{\infty}$ stabilizes. We have maps

$$M_{i-1}/M_{i-2} \to M_i/M_{i-1}$$

that are surjective. Since A is Noetherian and they are abelian, this must eventually must isomorphisms.

Now by the theorem we proved last time, there exists a $f \neq 0$ such that all $(M_i/M_{i-1})_f$ are finite and free over A_f. We have exact sequences

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0,$$

and so each M_i are (non-canonically) isomorphic to A^{n_i}. Then taking the colimit presents M as a free module. □

10.1 Chevalley's theorem

This has some consequences. Let us work in $\mathbb{C}$, and suppose I want to understand set defined by polynomial equations. This is of interest to logicians. We are trying to describe the subset of $\mathbb{C}^n$ that satisfies some equations. For instance, take

$$S = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^2 + x_2^2 = 3\}.$$

This is boring, and we've done this when dealing with varieties. But consider

$$S = \{(x_1, x_2) \in \mathbb{C}^2 : \exists y,\ x_1 y + x_2^2 + x_1 x_2 y^2 = 0\}.$$

As an algebraic geometer, I would consider it as a projection of something in $\mathbb{C}^3$. Then this can be thought of as the question, What are the images of "nice" morphisms of varieties? The answer is that they are the finite (disjoint) union of locally sets. These are called **constructible sets**.

Proposition 10.2. *For X a topological space and $S \subseteq X$, the following are equivalent:*

(i) $S \subseteq \overline{S}$ *is open.*

(ii) $S = U \cap C$ *for a open U and closed C.*

(iii) S *is open inside a closed set.*

(iv) S *is closed inside and open set.*

*In this case, we say that S is **locally closed**.*

Theorem 10.3 (Chevalley)**.** *Suppose $\pi : X \to Y$ is a finite type morphism of Noetherian schemes. Then π sends constructible sets to constructible sets.*

Proof. Let us first do some reduction. Clearly we can just check that the image of a locally closed set is constructible. Then we can just assume that X is that locally closed set and so we can just show that $\operatorname{im} \pi$ is constructible. Since Y is quasicompact, we may assume that the base Y is affine. Again by quasicompactness of π (or equivalently X in this situation), we may assume that X is also affine.

So we want to show that if B is finitely generated over A then the image of $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ is constructible. We are actually interested in the "support" of B as an A-module. Considering B as a module over itself, we can attempt to apply generic freeness. But we need A to be a domain to apply generic freeness.

We may assume that A is reduced, and this is because we are talking about topological properties. So replace A with $A/\mathfrak{N}(A)$, and also replace B by $B \otimes_A A/\mathfrak{N}(A)$. Also, without loss of generality, we can assume that A is irreducible. This is taking the quotient by minimal primes. (There are a finite number of them since A is Noetherian.)

Now we can actually apply generic freeness. Then B_f is free as an A_f module. Then the map $\operatorname{Spec} B_f \to \operatorname{Spec} A_f$ is surjective or $B_f = 0$, and you can check this. This means that $\operatorname{im} \pi$ either contains $\operatorname{Spec} A_f$ or is disjoint from it. Now go to $A \setminus \operatorname{Spec} A_f$. Then we can do the same thing again. By Noetherian induction, this has to stop after finitely many steps. □

We can prove Nullstellensatz.

Theorem 10.4. *Suppose K/k is finitely generated as an algebra. Then K/k is finite as a module.*

Proof. Pick generators $x_1, \ldots, x_n$ of K over k as an algebra. We want to show that any ring generated over k in K by each x_i is finitely generated as a k-module. Suppose $k[x_i] \to K$ has zero kernel. Then

$$\operatorname{Spec} K \to \operatorname{Spec} k[x_i]$$

is a morphism, that is of finite type. The image is the generic point. Then Chevalley implies that the generic point in $\operatorname{Spec} k[x]$ is constructible. Then the generic point is closed inside $\mathbb{A}^1_k$ is an open set. The complement is all the irreducible polynomials of $k[x]$. Then there is some $f \neq 0 \in k[x]$ that divides all irreducible polynomials. This is clearly impossible, by Euclid's theorem. $\square$

11 March 9, 2017

We showed Chevalley's theorem, that says that if $\pi : X \to Y$ is a finite type morphism of Noetherian scheme, then the image of a constructible set is constructible. To prove this, we first reduced to the case when X and Y are affine, and using generic freeness, showed that B_f is a free A_f-module.

We didn't show last time that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, if B is a free A-module. For a point $p \in \operatorname{Spec} A$, the fiber is given by

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} (\text{point } p) = \operatorname{Spec}(B \otimes_A k(p)).$$

Now the prime ideals of $B \otimes_A k(p)$ exist because the ring is nonzero (unless $B = 0$).

Corollary 11.1. *If $\pi : X \to Y$ is a morphism of finite type schemes over k, then π is surjective if and only if it is surjective on closed points.*

Proof. By Chevalley's theorem, $\operatorname{im} \pi$ is constructible and contains all closed points. You have to use the following lemma. If you have this lemma, use induction on the height of the prime. □

Lemma 11.2. *A constructible subset of a Noetherian scheme is closed if and only if it is closed under specialization.*

Proof. Let $C = \bigcup_{i=1}^n C_i$ with C_i locally closed. We want to show $\overline{C_i} \subseteq C$ so that $C = \bigcup \overline{C_i}$.

We have that C_i is dense open in $\overline{C_i}$. Because $\overline{C_i}$ is Noetherian, it breaks into finitely many irreducible components. So we may assume that $\overline{C_i}$ is irreducible. Then C_i contains the generic point of $\overline{C_i}$. Furthermore, every point of $\overline{C_i}$ is a specialization of the general point. So $\overline{C_i}$ is contained in C. □

11.1 Closed embeddings

A closed embedding is not just a closed subset. If the closed subset is $V(I) \subseteq \operatorname{Spec} A$, then there is an obvious scheme structure $\operatorname{Spec} A/I$. But this is not well-defined because $\operatorname{Spec} k[x]/(x)$ and $\operatorname{Spec} k[x]/(x^2)$ are the same sets.

Definition 11.3. A **closed embedding** $\iota : X \hookrightarrow Y$ is defined by (the following are equivalent)

(i) there exists an affine open cover $\{\operatorname{Spec} A\}$ of Y such that $\iota|_{\operatorname{Spec} A} \cong (\operatorname{Spec} A/I \to \operatorname{Spec} A)$.

(ii) for affine open $\operatorname{Spec} A \subseteq Y$, the same thing.

This is just a quasicoherent sheaf of ideals in $\mathscr{O}_Y$, just by definition.

There is another possible definition you can give. Recall that affine morphisms to Y is just the quasicoherent sheaf of algebras on Y. Closed embeddings are affine, and so this is a quasicoherent sheaf of algebras. Because $0 \to I \to A \to A/I \to 0$ is an exact sequence, we have

$$0 \to \mathscr{I}_{X/Y} \to \mathscr{O}_Y \to \pi_* \mathscr{O}_X \to 0.$$

Here $\mathscr{I}_{X/Y}$ denotes the ideal sheaf.

Definition 11.4. A **locally closed embedding** $Z \to X$ is defined by the composition of a closed $Z \hookrightarrow Y$ and an open $Y \hookrightarrow X$. (You shouldn't compose the other way round.)

For example, $\operatorname{Spec} k[y]_x$ into $\mathbb{A}^2$ is a locally closed embedding.

Given two closed subschemes, like $k[x,y]/(y-x^2)$ and $k[x,y]/(y)$, what should be the scheme-theoretic intersection? It has to be a fat point in the direction of the x-axis. That is, it should be $k[x,y]/(x^2,y)$.

Roughly, the intersection of $\operatorname{Spec} A/I_1$ and $\operatorname{Spec} A/I_2$ are going to correspond to the sum of the two ideals, and the union has to correspond to the intersection of ideals. Here, the intersection should be $I_1 \cap I_2$ instead of $I_1 I_2$, because we don't want the union of $k[x]/(x)$ and $k[x]/(x)$ to be $k[x]/(x^2)$.

Definition 11.5. If X_1 and X_2 are two closed subsets cut out by $\mathscr{I}_1$ and $\mathscr{I}_2$, then the **union** of given by $\mathscr{I}_1 \cap \mathscr{I}_2$ and the **intersection** is given by $\mathscr{I}_1 + \mathscr{I}_2$.

So closed subschemes are quasicoherent sheaves of ideals. Now closed subsets correspond to ideals. Here, $I_1, I_2 \subseteq A$ give the same closed subset if and only if $\sqrt{I_1} = \sqrt{I_2}$. So given a closed subset of a scheme, can you give it a closed subscheme structure? There is a canonical choice for each affine open chart, namely the radical ideal. This means that there is a **canonical reduced induced closed subscheme** structure.

Example 11.6. Apply this construction to the entirety of a scheme X. Then we get a **reduction of** X, called X^{red}. For example, the reduction of $\operatorname{Spec} k[x,y]/(xy,y^2)$ is $\operatorname{Spec} k[x]/(x)$.

12 March 21, 2017

12.1 Fiber products

I started talking about this last time. Forget about fiber for a moment and let's talk about products first. For schemes X and Y, does $X \times Y$ exists? We first try the case $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. The maps $S \to \operatorname{Spec} A$ and $S \to \operatorname{Spec} B$ is the same thing as maps $A \to \Gamma(S, \mathscr{O}_S)$ and $B \to \Gamma(S, \mathscr{O}_S)$. This is the same thing as a map $A \otimes B \to \Gamma(S, \mathscr{O}_S)$, which is the data of $S \to \operatorname{Spec} A \otimes B$. So we get

$$\operatorname{Spec} A \times \operatorname{Spec} B \cong \operatorname{Spec}(A \otimes B).$$

By the same reasoning, we see that the fiber product of $\operatorname{Spec} A \to \operatorname{Spec} C \leftarrow \operatorname{Spec} B$ is going to be

$$\begin{array}{ccc} \operatorname{Spec} A \otimes_C B & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} C. \end{array}$$

In general, we can ask if $X \times_Z Y$ exists? The answer is yes, and we can do this by gluing affines over affines. But what is the motivation for doing this? Suppose we really do care about the morphism $X \to Y$. If we know how to construct $X \times_Y R$ for some random R, then we can "reduce" studying the map $X \to Y$ to studying $X \times_Y R \to R$.

For example, consider a parameter space X of curves over $Y = \mathbb{A}^1$. We can look at something like

$$X = \operatorname{Spec} k[x, y, t]/(y^2 - x^3 + x^2 + t) \to \operatorname{Spec} k[t] = Y.$$

If we want to look at the curve over $p \in Y$, we can consider the fiber product

$$\begin{array}{ccc} X \times_Y \operatorname{Spec} \kappa(p) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \kappa(p) & \longrightarrow & Y. \end{array}$$

Definition 12.1. The **fiber** of a morphism of schemes $X \to Y$ at $p \ni Y$ is $X \times_Y \operatorname{Spec} \kappa(p)$.

Or maybe something hot is happening over around $[0] \in \mathbb{A}^1$. Then you might want to pull back along the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $t \mapsto t^2$. Also suppose that you have something like $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 + 1) \to \operatorname{Spec} \mathbb{R}$. But you don't know non-algebraically closed fields and want to work in $\mathbb{C}$. Then you can pull back along $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

One other nice fact of fiber products is that lots of properties of morphisms are preserved under under pullbacks. When I say that $X \to Y$ is finite, it roughly means that all the fibers are finite and they somehow glue nicely and

are uniform. So any reasonable property of a morphism should be preserved under pullback. Also, if I have a "good" cover of Y and app pullbacks satisfies some property P, then the original $f : X \to Y$ should also satisfy P.

Theorem 12.2. *Fiber products exist.*

Ravi has a proof in his notes, but I don't remember and I will give another proof.

"Proof". Alright, it works locally, and then patch. If we have $f : X \to Z$ and $g : Y \to Z$ then write $Z = \bigcup \operatorname{Spec} C_\alpha$ and write $f^{-1}(C_\alpha) = \bigcup \operatorname{Spec} A_{\alpha\beta}$ and $g^{-1}(C_\alpha) = \bigcup \operatorname{Spec} B_{\alpha\gamma}$. Then you should be able to glue them with the ridiculous amount of gluing data. But this is going to be horrible. This makes us sad. □

Here is the way I would do it.

12.2 Functor of points

There is a fully faithful embedding $\mathsf{Sch} \to \operatorname{Funct}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$, given by $X \mapsto (h_X : Y \mapsto \operatorname{Mor}(Y, X))$. Given a functor, we are going to call it **representable** if it is in the essential image. We are first going to show that $X \times_Z Y$ exists in the functor category and after that it is representable. In other words, we are going to show that $h_X \times_{h_Z} h_Y$ is representable.

But how do we show that a functor is representable? The idea is that local representability should suffice. Suppose I have a functor h, and we secretly want $h = h_X$. If we have an open cover h_α of h, (which secretly correspond to the open cover U_α of X) with each h_α representable, then we can glue the schemes representing h_α to get X.

Definition 12.3. A functor $F : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is an **open subfunctor** of $G : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ (with a map $F \to G$) if for any $h_X \to G$, the pullback $F \times_G h_X \to h_X$ is isomorphic to $h_U \to h_X$ for some open subscheme U.

$$\begin{array}{ccc} F \times_G h_X & \longrightarrow & F \\ \downarrow & & \downarrow \\ h_X & \longrightarrow & G. \end{array}$$

We can likewise define an open cover in a similar way: a collection is an open cover if any pullback along a representable functor form an open cover.

Theorem 12.4. *If $h \in \operatorname{Funct}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$ is a Zariski sheaf and has an open cover by representable subfunctors $\{h_\alpha\}$, then h is representable.*

Definition 12.5. A functor $h : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ is a **Zariski sheaf** if for nay Zariski open cover S_α of S there is the exact sequence of sets:

$$\bullet \to h(S) \to \prod h(S_\alpha) \rightrightarrows \prod h(S_{\alpha\beta}).$$

Proof. Each h_α is representable by U_α. To patch, I need for each $\alpha, \beta \in I$, I need (1) an open subschemes $U_{\alpha\beta} \subseteq U_\alpha$, (2) isomorphisms $\varphi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ that satisfies the cocycle condition.

We want $U_{\alpha\beta} = U_\alpha \times_X U_\beta$. So we define $h_{\alpha\beta} = h_\alpha \times_h h_\beta$. This is representable since h_β are open embeddings.

By the Zariski sheaf condition, the glued one should be the scheme representing h. □

This gives us an easy way to prove Theorem 12.2. It is easy to show that $h_X \times_{h_Z} h_Y$ is a Zariski sheaf and we have an open cover by $h_{A_{\alpha\beta}} \otimes_{h_{C_\alpha}} h_{B_{\alpha\gamma}}$. So it must be representable.

These ideas are used in moduli space problems. For instance, there is a Hilbert scheme $\mathrm{Hilb}(X)$, which is the moduli space of closed subschemes of X. You first define a functor S mapping to the set of closed subschemes of $S \times X$. You then somehow show that this is representable.

13 March 23, 2017

13.1 Properties of fiber products

We defied fiber products. The maxim is that properties should be preserved.

Proposition 13.1. *Pullback of an open embedding is an open embedding.*

Proof. The inverse image of an open subscheme should be an open subscheme. I claim that this is the fiber product. This is because there is a pullback square

$$\begin{array}{ccc} \operatorname{Spec} B_{g(f)} & \longrightarrow & \operatorname{Spec} A_f \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A. \end{array}$$

So you can glue them. Alternatively, you can directly check this categorically. □

Proposition 13.2. *Pullbacks of affine morphisms are affine.*

Proposition 13.3. *Pullbacks of closed embeddings are closed embeddings.*

Proof. You can pullback the sheaf of ideals and that should cut out the fiber product. □

If some $X \to Y$ has connected fibers, then is it true that any pullback have connected fibers? This is false in general, because $\operatorname{Spec} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is two $\operatorname{Spec} \mathbb{C}$. This is because $\operatorname{Spec} \mathbb{C}$ is like a double cover of $\operatorname{Spec} \mathbb{R}$.

If $X \to Y$ has irreducible fibers, then does any pullback $X' \to Y'$ have irreducible fibers? This is clearly false by the previous example.

Definition 13.4. A scheme X over k is **geometrically connected** if $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is connected.

Proposition 13.5. *If all fibers are geometrically connected, then any pullback have geometric connected.*

Definition 13.6. A scheme X over k is **geometrically irreducible** if $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is connected.

The same thing is false for irreducibility. Consider the variety

$$X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2).$$

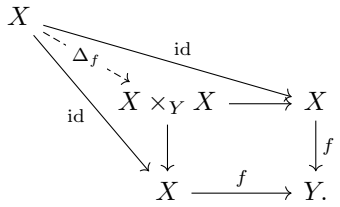
Then X is irreducible but $X \times_{\mathbb{R}} \mathbb{C}$ is connected but not irreducible. The intuition is that X is like the quotient by $\operatorname{Gal}(\bar{k}/k)$ of $X \times_k \bar{k}$.

13.2 Separated morphisms

We want the notion of Hausdorff and compactness. But we have a retarded topology, and so notions that most geometers are interested in are going to be very hard to define. In the future we would want to define stuff like π_1, H^*, H_*.

Let us first look at Hausdorff. We don't want sequences to have two different limits, like $\mathbb{A}^1_{00}$ with the double origin. Here is a weird way of saying it. If $(x_1, x_1), (x_2, x_2), \dots$ converges to (x, x'), then $x = x'$. So this is the same as saying that the diagonal $\Delta \subseteq X \times X$ is closed.

Definition 13.7. A morphism $f : X \to Y$ is **separated** if $\Delta_f : X \to X \times_Y X$ is a closed embedding.



Being separated over $\mathbb{Z}$ does not imply that X is Hausdorff as a set. This is because $X \times X$ as a topological space is not the same as the square of X as a topological space. They are not even the same as sets.

Example 13.8. The affine line with double origin $\mathbb{A}^1_{00}$ over k is not separated. We that $\mathbb{A}^1_{00} \times_k \mathbb{A}^1_{00}$ is going to be the affine plane $\mathbb{A}^2$ with double x-axis, double y-axis, and four origins. The diagonal has two of those origins. You can check topologically that the closure of any neighborhood of the origins on the diagonal contains all four origins.

Proposition 13.9. *(1) Affine morphisms are separated, and separatedness can be checked target-space locally.*
(2) Open embeddings are separated.
(3) Pullbacks and compositions of separated are separated.

Proof. (1) Let us first check target locality. You can check this categorically by pulling back the whole diagram along U_α.

Now to show that affine morphisms are separated, you now need to show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is separated. This is because $B \otimes_A B \to B$ is surjective.

(2) The map $U \to U \times_X U$ is just an isomorphism.

(3) We are going to check this next time. □

Definition 13.10. A **variety** is a finite type reduced separated scheme over a field k.

14 March 28, 2017

Proposition 14.1. *If K/k is a finite Galois extension, then $K \otimes_k K \cong K^{[K:k]}$.*

Proof. In particular, K/k separable and so $K = k(\alpha)$. So if $f(\alpha)$ is the minimal polynomial then $K \cong k[\alpha]/f(\alpha)$. Then

$$K \otimes_k K \cong K[\alpha]/(f(\alpha)).$$

But $f(\alpha)$ splits completely in K with distinct roots. By the Chinese remainder theorem, we get the result. □

A better way of writing this is $\operatorname{Spec}(K \otimes_k K) \cong \operatorname{Gal}(K/k) \times \operatorname{Spec} K$ with an Galois-equivariant isomorphism. These kind of stuff is a big deal in algebraic geometry. $\operatorname{Spec} K \to \operatorname{Spec} k$ is like a topological cover, and there should be a analogue of "Galois extension" for rings and schemes. This is called an étale cover or and étale extension. For instance, take like $\mathbb{G}_m \to \mathbb{G}_m$ given by $z \mapsto z^2$. If you have a sheaf over $\mathbb{G}_m$ with a $\mathbb{Z}/2$ action, then you should be able to descend to a sheaf over $\mathbb{G}_m$.

Let us finish up separatedness.

Proposition 14.2. *Pullbacks of separated morphisms are separated, and compositions of separated morphisms are separated.*

Proof. We claim that

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X' \times_{Y'} X' & \longrightarrow & X \times_Y X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a pullback. You can check this categorically. For composition, we have a pullback diagram

$$\begin{array}{ccc} X \times_Y X & \longrightarrow & X \times_Z X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y. \end{array}$$

So we get a composition of closed embeddings. □

If X is separated (i.e., $X \to \operatorname{Spec} k$ is separated), and if U, V are affine open in X, then $U \cap V$ is affine. This is because we have a pullback diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \times V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

Then $U \cap V \to U \times V$ is an closed embedding, which is affine.

14.1 Proper morphisms

This is the algebraic geometer's way of saying compact Hausdorff. We are going to assume separatedness and finite type. In topology, if $f : X \to Y$ is proper, it should be a closed map, assuming some stuff. But this is not enough because $g : \mathbb{A}^1 \to k$ is a closed map since the topology on k is stupid. But this is not universally closed, i.e., pullbacks are not always closed. We have

$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & k \end{array}$$

and $\mathbb{A}^2 \to \mathbb{A}^1$ is just the projection. The projection of $xy = 1$ is going to be $\mathbb{A}^1 \setminus 0$, which is not closed.

Definition 14.3. A morphism $f : X \to Y$ is **proper** if it is separated, finite type, and universally closed.

Theorem 14.4. *$\mathbb{P}^n$ is proper.*

This is going to be hard. You need to actually go into the algebra. We are probably going to spend most of next class doing this.

Proposition 14.5. *Closed embeddings are proper.*

Proof. We know that is is separated. Finite type is also easy because A/I is finitely generated over A. We now need to show that if $f : X \to Y$ is a closed embedding, then any pullback $f' : X' \to Y'$ is a closed map. We have checked that $f' : X' \to Y'$ is a closed embedding, and closed embeddings are closed maps. □

Definition 14.6. A morphism $f : X \to Y$ is **finite** if the following equivalent conditions hold:

(1) For any affine open $\operatorname{Spec} A \subseteq Y$ its inverse image is $f^{-1}(\operatorname{Spec} A) \cong \operatorname{Spec} B$ with B finite as an A-module.

(2) The above holds for some affine cover.

This property is target-space local and preserved under pullback and composition.

Proposition 14.7. *Finite morphisms have finite fibers and finite geometric fibers. (Having finite fibers is also called **quasifinite**.)*

Open embeddings are quasifinite but not finite. For instance, $k[t, t^{-1}]$ is not finite as a $k[t]$-module. We are going prove that finite morphisms are proper.

Theorem 14.8. *Finite is equivalent to proper and quasifinite.*

This is a very deep theorem and we won't be able to prove this in this course.

Proposition 14.9. *Finite morphisms are proper.*

Proof. Finite morphisms are affine and thus separated. Finite morphisms are also of finite type. Pullbacks of finite morphisms are finite morphisms, so we only need to show that finite morphisms are closed maps.

Let $Z \subseteq X$ be a closed set. Recall that there is a reduced induced closed subscheme structure. So $Z \to X$ can be thought of as a closed embedding. Because closed embeddings are finite, we've now reduced to showing that finite morphisms have closed image.

This is true in general, but we are going to do it under the Noetherian hypotheses. Apply Chevalley. Then the image is a constructible set. We want to show that $f(x)$ is closed under specialization. Let us assume $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ using target space locality. Now being closed under specialization is just a restatement of the going-up theorem. □

15 March 30, 2017

We defined proper.

Definition 15.1. A morphism $f : X \to Y$ is **proper** if it is separated, of finite type, and universally closed.

15.1 Projective space is proper

Theorem 15.2. *$\mathbb{P}^n$ is proper.*

Recall that we have built $\mathbb{P}^n$ by patching together a bunch of $\mathbb{A}^n_A$. This comes with a structure morphism $\mathbb{P}^n_A \to A$. In fact, we have a pullback

$$\begin{array}{ccc} \mathbb{P}^n_A & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} \mathbb{Z}. \end{array}$$

So what we are claiming is $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.

Finite type is obvious. We want to show that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is universally closed. We need to show that $\mathbb{P}^n \times X \to X$ is a closed map. This can be checked locally on the base, and so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map.

Let $Z \subseteq \mathbb{P}^n_A$ be a closed set. We want to show that $\pi(Z)$ is closed, where $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$. Closed sets in $\mathbb{P}^n_A$ are cut out by homogeneous ideals. Suppose $(f_1, f_2, \ldots) \subseteq A[x_0, \ldots, x_n]$ is a homogeneous ideal that cuts out $Z \subseteq \mathbb{P}^n_A$.

As an aside, $\mathbb{P}^n$ can be viewed as coming with a fibration $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. The charts are given as

$$A[x_0/x_i, \ldots, x_n/x_i] \to A[x_0, \ldots, x_n, x_i^{-1}] \ (= \text{open subset of } \mathbb{A}^{n+1}),$$

and they can be glued together. So we can think of $\mathbb{P}^n \cong (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m)$. This is probably something you already know from algebraic topology of geometry.

Anyways, we need to see when $p \in \operatorname{Spec} A$ is in the image of Z. One trick we can use is to take the fiber $\mathbb{P}^n_{\kappa(p)}$, and the ideal $(f_1, f_2, \ldots)$ gives and homogeneous ideal $(\bar{f}_1, \bar{f}_2, \ldots)$ over $\kappa(p)$. We are asking if the subset of $\mathbb{P}^n_{\kappa(p)}$ cut out by $(\bar{f}_1, \bar{f}_2, \ldots)$ empty or nonempty. In projective space, this is tricky. Essentially this question is whether $(\bar{f}_1, \ldots)$ is just the ideal that cuts out the origin in $\mathbb{A}^{n+1}$ or not.

To make this rigorous, recall that

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i, \quad \text{where } U_i = \operatorname{Spec} \kappa(p)[x_0/x_i, \ldots, x_n/x_i].$$

When does $\bar{I} = (\bar{f}_1, \ldots, \bar{f}_n)$ have empty intersection with all U_i? Empty intersection with U_i means that $\sqrt{\bar{I}_{(x_i)}}$ is the unit ideal, and this means that $x_i \in \sqrt{\bar{I}}$. So empty intersection with all U_i means that

$$(x_0, x_1, \ldots, x_n) = \sqrt{\bar{I}}.$$

So for $p \in \operatorname{Spec} A$, we have that $p \in \pi(Z)$ if and only if $\sqrt{\bar{I}} \subsetneq (x_0, \ldots, x_n)$. This means that

$$(x_0, x_1, \ldots, x_n)^N \subsetneq \bar{I}$$

for every N. Let us denote $S_N = (x_0, x_1, \ldots, x_n)^N$ the finite dimensional $\kappa(p)$-vector space. Continuing on, $p \in \pi(Z)$ is then equivalent to

$$\bigoplus_\alpha S_{N-\deg f_\alpha} \to S_N; \quad (s_i) \mapsto \sum s_i f_i$$

not being surjective. Now there is a matrix M_N representing this map, with S_N many rows and a lot of columns. Our equivalence continues as the rank being less than $\dim S_N$. This is then equivalent to all $\dim S_N \times \dim S_N$ minors of M_N having zero determinant, for all N.

All of these are happening over $\kappa(p)$. But note I can form these matrices M_N over A for all N. Then the determinants of S_N minors of M_N yields a bunch of elements of A. Then $p \in \pi(Z)$ is equivalent to p containing all these bunch of elements in A. If we call J the ideal they generate in A, we can say that $p \in \pi(Z)$ if and only if $J \subseteq p$. This locus is going to be closed.

Okay, we showed universally closed. Now let us show that it is separated. We want to show that $\mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$ is a closed embedding. This can be checked locally. We have a charge $\mathbb{P}^n = \bigcup_{i=0}^n U_i$. We need to check that $U_i \cap U_j \to U_i \times U_j$ is closed embeddings. For $i = j$ this is clear because affines are separated.

Without loss of generality let $i = 0$ and $j = 1$. We have

$$U_0 \cap U_1 = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}].$$

This maps to

$$U_0 \times U_1 = \operatorname{Spec} \mathbb{Z}[x_1^{-1}, x_2, \ldots, x_n] \times \operatorname{Spec} \mathbb{Z}[x_1, x_2, \ldots, x_n].$$

This is clearly surjective.

But this is a boring proof. How else can we check that $\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ is a closed embedding?

Definition 15.3. A **projective scheme** over k is a closed subscheme of a projective space, $Z \hookrightarrow \mathbb{P}^N_k$. More generally, $f : X \to Y$ is **projective** if it factors as $X \hookrightarrow Y \times \mathbb{P}^n \xrightarrow{\pi} Y$.

There is something called a **Segre embedding**

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{(n_1+1)(n_2+1)-1}$$

given by the map

$$k[x_0, \ldots, x_{n_1}] \otimes k[y_0, \ldots, y_{n_2}] \leftarrow k[z_{0,0}, \ldots, z_{n_1,n_2}]; \quad z_{i,j} \mapsto x_i \otimes y_j.$$

This is going to be a closed embedding. Also there is an almost second Veronese embedding $\mathbb{P}^n \to \mathbb{P}^{n^2+2n}$ that is closed. Then we get a commutative diagram.

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\Delta} & \mathbb{P}^n \times \mathbb{P}^n \\ & \searrow^{\text{Ver.}} & \downarrow{\scriptstyle \text{Seg.}} \\ & & \mathbb{P}^{n^2+2n} \end{array}$$

Proposition 15.4. *Let $f : Y \to Z$ and $g : X \to Y$.*

(1) If $f \circ g$ and f are closed embeddings, then g is a closed embedding.

(2) If $f \circ g$ is proper and f is separated, then g is separated.

(3) ...

There is a general framework for these properties.

Proposition 15.5. *If P is a property stable under composition and pullback, and $f \circ g \in P$ and $\Delta_f \in P$, then $g \in P$.*

Proof. We have a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Delta_f} \\ X \times_Z Y & \longrightarrow & Y \times_Z Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

This Γ is called the graph of f. Now g factors into Γ and the projection $X \times_Z Y \to Y$. Both are in P and so their composition g is also in P. □

Definition 15.6. A **quasiprojective morphism** is an open embedding in a projective morphism.

Quasiprojective implies separated.

16 April 4, 2017

Definition 16.1. A morphism is it is **proper** if it is separated, finite type, and universally closed.

Theorem 16.2. *The projective space $\mathbb{P}^n_A$ is closed over* $\operatorname{Spec} A$.

To show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated, we used the following lemma:

Lemma 16.3 (Cancellation)**.** *If P is a property stable under pullback and composition, and $f \circ g, \Delta_f \in P$, then $g \in P$.*

For any $f : X \to Y$, the diagonal $\Delta_f : X \to X \times_Y X$ is a locally closed embedding. $f : X \to Y$ being separated is Δ_f being a closed embedding, and f being a categorical monomorphism is Δ_f being an isomorphism. In particular, locally closed embeddings are monomorphisms. This is because open embeddings, and closed embeddings are monomorphisms since

$$Z(S) = \{f \in X(S) : f^* \mathscr{I}_Z = 0\}.$$

As an aside, if $f : X \to Y$ is a map of schemes, there are

$$f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y), \quad f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X),$$

locally modeled on f_* forgetful and f^* tensoring up. These are adjoint functors.

16.1 Properties of proper morphisms

Why is this the right definition of proper? I won't prove this, but here is a remark. Consider the finite type schemes over $\mathbb{C}$. These are actually cut out by polynomials in some $\mathbb{C}^n$. There is a forgetful functor

$$F : \mathsf{Var}/\mathbb{C} \to \mathsf{CplxAnalSpaces} \to \mathsf{Top}$$

giving the analytic topology. Then

(1) X is separated if and only if $F(X)$ is Hausdorff,

(2) X is proper if and only if $F(X)$ is compact Hausdorff.

Let us now actually prove stuff. Let X be proper over k. We would expect $\Gamma(X, \mathscr{O}_X) = k$, as an analogue of the maximum principle. But there are some things that can happen. First of all if X is not connected, then we can have many copies of k. So we will assume that X is connected. Note that $\operatorname{Spec} k[\epsilon]/\epsilon^2$ is proper over $\operatorname{Spec} k$, because nothing much changes. So we want X to be reduced. Finally, we don't want something like $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ and so we assume k is algebraically closed.

Theorem 16.4 (Maximum principle)**.** *Let X be a connected reduced proper scheme over k, where k is algebraically closed. Then $\Gamma(X, \mathscr{O}_X) = k$.*

Proof. A global function $f \in \Gamma(X, \mathscr{O}_X)$ is a map $f : X \to \mathbb{A}^1_k$. Since $X \to k$ is proper and $\mathbb{A}^1_k \to k$ is separated, we conclude by the cancellation lemma that $X \to \mathbb{A}^1$ is proper.

Now we can compose $X \to \mathbb{A}^1 \to \mathbb{P}^1$, and this is also going to be proper by the same argument over k.

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{\bar{f}} & \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

Then $\bar{f}$ is closed, but its image lies in $\mathbb{A}^1$. Thus it has finite many points and so the image of f also is a finite number of points. By connectedness, it has to be a single point. Let it be $k[t]/(t-a)$.

We want to check that $f : X \to \mathbb{A}^1$ factors through $* \hookrightarrow \mathbb{A}^1$. We can check this on open affines $\{\operatorname{Spec} A\}$ of X. We want to show that $(t-a)$ is in the kernel of $k[t] \to A$. By the condition, for any prime $\mathfrak{p} \subseteq A$, its inverse is $(t-a)$. Then $f^{-1}(\bigcap \mathfrak{p}) = (t-a)$ but $\bigcap \mathfrak{p}$ is the nilradical, which is 0. □

What sort of schemes over k are both proper and affine? In complex analytic spaces, proper corresponds to compact and affine corresponds to Stein spaces. Suppose $\operatorname{Spec} R \to \operatorname{Spec} k$ is proper. Forget about the maximal principle for now. How did we show that $\mathbb{A}^1$ is not proper? We made a base change to $\mathbb{A}_1$ and the hyperbola in $\mathbb{A}^2$ did not project down to a closed set.

Let us do a similar thing. Make a base change to $\operatorname{Spec} k[t]$ and consider the image of the closed subscheme $Z = \operatorname{Spec} R[t]/(rt-1) = \operatorname{Spec} R_r$ in $\operatorname{Spec} R[t]$. Recall that $\mathfrak{p} \in \mathbb{A}^1$ in the image $\pi(Z)$ if and only if $R[t]/(rt-1) \otimes_{k[t]} \kappa(\mathfrak{p})$ is not the zero ring. If $\mathfrak{p} = (t-a)$, then this ring is $R/(ar-1)$. In particular, $[0] \in \mathbb{A}^1$ is always not in the image.

Let us think about when the generic point $\eta \in \mathbb{A}^1$ is in the image. This is asking if $R[t]/(rt-1) \otimes_{k[t]} k(t) = 0$, which is the same thing as asking if $rt-1$ is a unit in $R \otimes_k k(t)$. The ring $R \otimes_k k(t)$ consists of something in $R[t]$ divided by something in $k[t]$. Suppose

$$\sum r_i t^i (rt-1) = \sum k_i t^i,$$

where we can assume $k_0 = 1$. If you work though, this turns out to be the condition that r is integral over k. This shows that in order for $\operatorname{Spec} R$ to be proper, every element need to be integral, i.e., R is integral over k.

Theorem 16.5. *Universally closed and affine is equivalent to integral.*

17 April 6, 2017

17.1 Proper, finite, integral

The main theorem we want to prove is

Theorem 17.1. *Proper and affine is equivalent to finite.*

Mostly proper stuff are going to be projective. Here is a way of constructive a proper but non-projective scheme. Take two curves $C, C' \in \mathbb{P}^3$ that intersect at two points. We can look at the blowup $\mathrm{Bl}_{C'} \mathrm{Bl}_C \mathbb{P}^3$, and look at the other blowup $\mathrm{Bl}_C \mathrm{Bl}_{C'} \mathbb{P}^3$. These two are going to look the same away from the intersection. So take a scheme that looks like one at one intersection point, and the other at the other intersection.

Last time we had this discussion that affine over a field and universally closed implies integral.

Definition 17.2. An **integral morphism of rings** is a morphism $\phi : A \to B$ such that B is integral over $\phi(A)$.

Definition 17.3. An **integral morphism of scheme** is an affine morphism such that locally for $\operatorname{Spec} B \to \operatorname{Spec} A$, B is integral over A.

Theorem 17.4. *Universally closed and affine is equivalent to integral.*

The reason this is relevant is that integral is almost finite.

Proposition 17.5. *If $A \hookrightarrow B$ is an injective morphism of rings, $b \in B$ is integral over A if there exists a subalgebra $b \in B' \subseteq B$ such that B' is finite as an A-module.*

Proof. One direction is easy. For the other direction, consider generators $m_1, \dots, m_n$ of B' as an A-module. Let us write $bm_i = \sum a_{ij} m_j$ for some a_{ij}. Then we get a matrix equation

$$(bI - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

Now recall that there is something called $\operatorname{adj}(M)$, that satisfies $\operatorname{adj}(M)M = \det(M)I$. Let $M = bI - A$. Multiplying $\operatorname{adj}(M)$ on the left gives

$$(\det M) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But $m_1, \dots, m_n$ has $1 \in B$ as a linear A-combination. So $\det M = 0$. □

Proposition 17.6. *Finite is the same as integral and finite type. (This is both for rings and schemes.)*

Proof. We showed that finite implies integral. So the forward direction is proved.

For the other direction, let $b_1, \dots, b_n$ be generators as an algebra. Take N such that $b_i^N \in A\langle 1, b_i, \dots, b_i^{N-1}\rangle$. Then $\prod_i b_i^{l_i}$ for $l_i < N$ generates B as an A-module. □

Proof of affine + univ. closed ⇒ integral. We can immediately reduce everything to rings. So we want to prove that if $A \to B$ is a morphism of rings such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is universally closed, then B is integral over A.

We are again going to use the hyperbola trick. Base change and look at the hyperbola $\operatorname{Spec} B[t]/(bt-1)$.

$$\begin{array}{ccccc} \operatorname{Spec} B[t]/(bt-1) & \hookrightarrow & \operatorname{Spec} B[t] & \longrightarrow & \operatorname{Spec} B \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} A[t] & \longrightarrow & \operatorname{Spec} A \end{array}$$

Before doing this, we can simply factor $\phi : A \to B$ as $A \to A/\ker\phi \to B$. If $\operatorname{Spec} B \to \operatorname{Spec} A$ is universally closed, then $\operatorname{Spec} B \to \operatorname{Spec} A/\ker\phi$ is also universally closed. In other words, we can restrict our attention to this map and assume that ϕ is an injection of rings.

Here is a remark. We know what it means for $A \to B$ to be surjective in terms of $\operatorname{Spec} B \to \operatorname{Spec} A$. It is just a closed embedding. Now what does it mean for $A \to B$ to be injective. Roughly it would mean something like being surjective. Assume that A and B are both domains. Then ϕ being injective is $\phi^{-1}(0) = 0$. This means that the generic point in $\operatorname{Spec} B$ maps to the generic point in $\operatorname{Spec} A$. This is equivalent to the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ being dense.

Let's move on. We have $A \hookrightarrow B$ is universally closed.

Lemma 17.7. *If $A \hookrightarrow B$ is closed, then $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective on sets.*

Proof. Suppose otherwise, and suppose $\operatorname{im} f = \operatorname{Spec} A/I$ as a set. Recall that, in general, $f^{-1}(V(I)) = V(I \cdot B)$. Then $f^{-1}(V(I))$ is the whole thing, i.e., $IB \subseteq \mathfrak{N}(B)$. Then $I \subseteq \mathfrak{N}(A)$ and so $V(I) = \operatorname{Spec} A$. □

Corollary 17.8. $\phi^{-1}(B^\times) = A^\times$

Proof. Suppose $a \in A$ such that $(a)B = B$. If we take $I = (a)$ then $f^{-1}(V(I)) = \emptyset$ and $V(I) = \emptyset$. Then $(a) \subseteq A$ is the unit ideal and so $a \in A^\times$. □

Let us now actually look at the hyperbola. We can factor

$$\begin{array}{ccc} \operatorname{Spec} B[t]/(ft-1) & \hookrightarrow & \operatorname{Spec} B[t] \\ \downarrow & & \downarrow \\ Z = \operatorname{Spec} A[t]/I & \hookrightarrow & \operatorname{Spec} A[t]. \end{array}$$

Here I is the kernel of $\xi : A[t] \to B[t]/(ft-1)$. Then $A[t]/I \hookrightarrow B[t]/(ft-1) = B_f$ satisfies the hypotheses of the lemma and the corollary. By the corollary, $t \in (A[t]/I)$ is invertible.

Then we can write

$$t\sum_{i=0}^{n} a_i t^i = 1$$

in $A[t]/I$, and so for sufficiently large n,

$$(tf)\sum_{i=0}^{n} a_i t^i f^n = f^{n+1}.$$

Because $tf = 1$, we have

$$f^{n+1} = \sum_{i=0}^{n} a_i f^{n-i}$$

in B. This shows that f is integral over A. □

Proof of integral ⇒ universally closed. Integrality is closed under base change. So we just have to worry about closed. Next we only have to consider integral extensions, because closed embeddings are closed.

Lemma 17.9 (Lying over)**.** *Let $A \hookrightarrow B$ is an integral extension. Then* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective.*

Proof. Consider the map $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$. This is still an integral extension. Take any maximal $\mathfrak{m}$ in $B_{\mathfrak{p}}$. We want to show that it restricts to $\mathfrak{p}A_{\mathfrak{p}}$. That is, we want to show that $\mathfrak{m} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Note that $A_{\mathfrak{p}}/(\mathfrak{m} \cap A_{\mathfrak{p}}) \to B_{\mathfrak{p}}/\mathfrak{m}$ is again an integral extension.

So we have reduced things to showing that if $A \hookrightarrow B$ is integral and B is a field then A is a field. Consider $a \in A \setminus \{0\}$. There exists a $b \in B$ such that $ab = 1$ but $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$. Multiply by a^{n-1} and we get that $b \in A$. □

This finishes the proof. □

Corollary 17.10. *Finite is equivalent to affine and proper.*

Proof. Integral is equal to affine and universally closed, and finite is equal to integral and finite type. Also affine implies separated anyways. □

18 April 13, 2017

Last time we finished showing that proper plus affine is equal to finite.

Theorem 18.1. *Proper + affine = finite.*

What are the finite schemes over k? In other words, which k-algebras have finite dimension as k-vector spaces? There can be finitely many points, with finite fatness, which are finite field extensions of k.

Theorem 18.2. *Suppose A is a reduced k-algebra which is finite over k. Then A is a finite direct sum of finite field extensions of k.*

Proof. We apply the Chinese remainder theorem, which says that if I_α is a finite set of ideals of A such that $I_\alpha + I_\beta = (1)$ then

$$A\Big/\bigcap_{\alpha\in S} I_\alpha \to \bigoplus_\alpha A/I_\alpha$$

is an isomorphism.

We apply this to the set of maximal ideals. The coprimality condition is automatically satisfied. Also $A/\mathfrak{m}_\alpha$ are finite field extensions of k.

In this situation, all primes are maximal. Suppose $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is a integral domain that is finite over k. By the following lemma, it has to be a field. Then the intersection of the maximal ideals is the nilradical, which is (0). □

Lemma 18.3. *A integral domain A that is integral over k is a field.*

Proof. Suppose $a \in A \setminus \{0\}$. There is a relation

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0.$$

Then we can assume that $c_0 \neq 0$ and then assume that $c_0 = -1$ since it is in the ground field. Then $a^{n-1} + c_{n-2}a^{n-2} + \cdots + c_1$ is the obvious inverse of a. □

18.1 Nakayama's lemma

Proposition 18.4 (Nakayama's lemma)**.** *Let $I \subseteq A$ be an ideal and M be finite over A. If $M = IM$ then there exists a $a \equiv 1 \pmod{I}$ such that $aM = 0$.*

Proof. Pick generators $m_1, \ldots, m_n$ for M. For each $i = 1, \ldots, n$, write

$$m_i = \sum i_{ij} m_j$$

for some $i_{ij} \in I$. Now look at the matrix $\mathscr{I} = (i_{ij})$. Then $\mathrm{Id} - \mathscr{I}$ acts on (m_i) trivially, i.e., (m_i) is in the kernel. The determinant of $\mathrm{Id} - \mathscr{I}$ is 1 mod I. □

Corollary 18.5. *Suppose $M = IM$ and M is finite over A, where $I \subseteq \bigcap_{\mathfrak{m}\in \mathrm{mSpec}\, A} \mathfrak{m}$. Then $M = 0$.*

Proof. There exists a $a \equiv 1 \pmod{I}$ such that $aM = 0$. Then a is not in any maximal ideal and so a is a unit. Then $M = 0$. □

Corollary 18.6. *Let M be finite over A and let A be a local ring. If $M = \mathfrak{m}M$ then $M = 0$.*

Corollary 18.7. *Suppose M is finite over A and $N \subseteq M$, with A a local ring. If $M = \mathfrak{m}M + N$ then $M = N$.*

Proof. Apply the previous statement to M/N. □

Corollary 18.8. *Let M be finite over $(A, \mathfrak{m})$ a local ring. If $\bar{m}_1, \dots, \bar{m}_n$ span $M/\mathfrak{m}M$, then any lifts $m_1, \dots, m_n$ will generate M.*

Proof. Apply the previous statement to $N = A\langle m_1, \dots, m_n \rangle \subseteq M$. □

Corollary 18.9. *Let $\mathscr{F} \in \mathsf{Coh}(X)$, and p is a closed point of X. Then generators of the fiber $\mathscr{F}|_p$ lifts to generators of $\mathscr{F}$ in a neighborhood of p.*

Here is a quick recap. If $\mathscr{F} \in \mathsf{Shv}(X)$ and $p \in X$, then the stalk is roughly $\mathscr{F}_p = \mathscr{F} \otimes \mathscr{O}_{X,p}$ and the fiber is roughly $\mathscr{F}|_p = \mathscr{F} \otimes \kappa(p)$.

If $\mathscr{F} \in \mathsf{Coh}(X)$, locally given by a module M on $\operatorname{Spec} A \subseteq X$, then the stalk is M_p and the fiber is M_p/pM_p.

18.2 Dimension

In other parts of geometry, dimension is intrinsic in the definition. Well it is not obvious that a ball in $\mathbb{R}^n$ and a ball in $\mathbb{R}^m$ is not homeomorphic if $n \neq m$, but this can be shown.

Definition 18.10. For A a commutative ring, we define its **Krull dimension** as

$$\dim A = \max\{n : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}.$$

Example 18.11. For $A = k$ a field, $\dim A = 0$. We have $\dim \mathbb{Z} = 1$ and $\dim k[x] = 1$. Localization and completion doesn't change dimension that much. For example $\dim \mathbb{Z}_{(5)} = 1$ and $\dim k[[x]] = 1$.

Example 18.12. If the scheme is not irreducible, it might not work nicely. For example, the dimension of $k[x, y, z]/(xz, yz)$, which is a union of a plane and a line, is going to be 2.

It is really hard to compute the dimension. For instance, we can show that $\dim \mathbb{Z}[x] \geq 2$, by exhibiting $(0) \subseteq (x) \subseteq (2, x)$. But how do we know that the dimension is exactly 2? Likewise we want $\dim \mathbb{A}^n = n$, and we have $(0) \subseteq (x_1) \subseteq (x_1, x_2) \subseteq \cdots$. But it is hard to actually show this.

Definition 18.13. Let X be a scheme. We define its **dimension** as

$$\dim X = \max\{n : Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n \text{ irreducible closed}\}.$$

Definition 18.14. Let X be a scheme and $p \in X$ be a point. We define the **local dimension** as

$$\operatorname{loc\,dim}_p X = \dim \mathscr{O}_{X,p} = \max\{\text{dim. of irr. comp. containing } p\}.$$

Proposition 18.15. *If* $X = \bigcup \operatorname{Spec} A_\alpha$ *then* $\dim X = \max \dim A_\alpha$.

Proof. We claim that $n = \dim A_\alpha \leq \dim X$. Then there is a chain of irreducible closed sets $Z_0 \subsetneq \cdots \subsetneq Z_n \subseteq \operatorname{Spec} A_\alpha$. You can check that $\overline{Z}_0 \subsetneq \cdots \subsetneq \overline{Z}_n \subseteq \overline{\operatorname{Spec} A_\alpha}$ are irreducible closed sets.

To show the other direction, take $p \in Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$. Consider $p \in \operatorname{Spec} A_\alpha$. Then taking the intersection with $\operatorname{Spec} A_\alpha$ gives the chain of closed sets in $\operatorname{Spec} A_\alpha$. □

Proposition 18.16. *If* $A \subseteq B$ *is an integral extension, then* $\dim B = \dim A$.

Proof. We know that the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, by the lying over theorem. If $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$, we can lift each prime $\mathfrak{p}_i$ to $\mathfrak{q}_i \in \operatorname{Spec} B$ starting from $\mathfrak{q}_0$, one by one, using surjectivity. (This is the going-up theorem.) So $\dim B \geq \dim A$.

To show that $\dim A \geq \dim B$, it suffices to show that if $\mathfrak{q} \subseteq \mathfrak{q}' \in \operatorname{Spec} B$ maps to $\mathfrak{p} \in \operatorname{Spec} A$ then $\mathfrak{q} = \mathfrak{q}'$. We localize and quotient as $B_{\mathfrak{q}'}/\mathfrak{q}$, over the field $A_{\mathfrak{p}}/\mathfrak{p}$. This is integral domain which is integral over a field, and so it is a field. That is, $\mathfrak{q} = \mathfrak{q}'$. □

Here is why this dimension makes sense.

Theorem 18.17. *If* A *is a finitely generated domain over* k*, then* $\dim A$ *is the transcendence degree of* $\kappa(A)$ *over* k.

Theorem 18.18 (Noether normalization)**.** *Let* A *be a finitely generated domain over* k*. Then* $\dim A = n$ *if and only if there is a finite map* $A \to \mathbb{A}^n_k$.

19 April 18, 2017

We had the notion of a Krull dimension of a ring/scheme as the length of the maximal chain of primes/closed irreducible sets. We proved some random fact.

Proposition 19.1. *If $A \subseteq B$ is an integral extension, then* $\dim A = \dim B$.

For A a domain, consider the **normalization** $\mathrm{Norm}(A)$, the integral closure of A in $\mathrm{Frac}(A)$. What does this look like geometrically? The map $\operatorname{Spec}\mathrm{Norm}(A) \to \operatorname{Spec} A$ has to be a finite surjection, and it has to have degree 1 because the field of fraction does not change. That is, it is birational. Also there cannot be something like this over it.

Example 19.2. Since $k[t]$ consider $A = k[t^2, t^3] \cong k[x,y]/(y^2 - x^3)$. Then $\operatorname{Spec}\mathrm{Norm}(A) \to \operatorname{Spec} A$ is $\mathbb{A}^1$ mapping down to the cuspidal cubic.

Definition 19.3. Let $Z \subseteq X$ be an irreducible closed subset in an irreducible scheme X. The **codimension** is defined as

$$\operatorname{codim}(Z/X) = \max\{n : Z = Z_0 \subsetneq \cdots \subsetneq Z_n = X\}.$$

Of course, the same definition works for commutative rings. For a domain A and a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$,

$$\begin{aligned}\operatorname{codim}_{V(\mathfrak{p})/\operatorname{Spec} A} &= (\text{length of maximal chain between } (0) \text{ and } \mathfrak{p}) \\ &= \dim A_{\mathfrak{p}} = \operatorname{ht}_{\mathfrak{p}}.\end{aligned}$$

Proposition 19.4. *A prime ideal $\mathfrak{p} \subseteq A$ is principal if and only if it is height 1 (if A is a UFD).*

Definition 19.5. A **hypersurface** is a codimension 1 space of $\mathbb{A}^n$ or $\mathbb{P}^n$.

The proposition shows that these are in one-to-one correspondence with irreducible (homogeneous) polynomials.

Proof. Let's first show that if (f) is prime, i.e., f is an irreducible element, then it is of height 1. Assume that A is a UFD. Suppose $(0) \subseteq \mathfrak{p} \subseteq (f)$. What is $\mathfrak{p}$? If it is nonzero, it contains some $f^n g$, where $f \nmid g$ and $n \geq 1$. If $f \notin \mathfrak{p}$ then $g \in \mathfrak{p}$ and we get a contradiction. So $f \in \mathfrak{p}$.

Now suppose that $\mathfrak{p}$ is of height 1. We want to show that it is principal. Because $\mathfrak{p} \neq (0)$, consider some $p \neq 0 \in \mathfrak{p}$. Write $p = \prod f_i^{e_i}$, and suppose $\mathfrak{p} \supseteq (f_i) \supsetneq (0)$. Then $\mathfrak{p} = (f_i)$. □

19.1 Noether normalization

Theorem 19.6 (transcendnce = Krull)**.** *Let A be a finitely generated domain over k,*

$$\dim A = \operatorname{trdeg}_k \mathrm{Frac}(A).$$

Theorem 19.7 (Noether normalization)**.** *Let A be a finitely generated domain over k, and* $\operatorname{trdeg}_k \operatorname{Frac}(A) = n$*. Then there exist $a_1, \dots, a_n \in A$ such that A is integral over $k[a_1, \dots, a_n]$, and then* $\operatorname{Spec} A \to \mathbb{A}^n$ *is finite surjective.*

Corollary 19.8. *For A a finitely generated domain over k, the following are equivalent:*

(i) $\operatorname{trdeg}_k \operatorname{Frac}(A) = n$,

(ii) $\dim A = n$,

(iii) there exists a finite surjection $\operatorname{Spec} A \to \mathbb{A}^n$.

Note that A finitely generated is crucial. This is because you can do stupid things like removing all closed points of $\mathbb{A}^1$ and get $k(t)$. Nullstellensatz is an immediate corollary of all this.

Before we start, let's talk about transcendence degree of field extensions. Let us put an equivalence relation on field extension of k: $F \sim F'$ if and only if FF' is integral over F and over F'.

Definition 19.9. A field extension F/k has **transcendence degree** n if there exist algebraically independent elements $f_1, \dots, f_n \in F$ such that $F \sim \langle f_1, \dots, f_n \rangle$.

Well-definedness of the degree can be shown as in linear algebra. If you have two "bases" you can substitute one with the other by switching one elements at a time.

Proof of Noether normalization. Because A is a finitely generated domain over k, there exists a closed embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}^N$. Let's induct down on N. We will show that given a finite map $\operatorname{Spec} A \to \mathbb{A}^N$, if $N > n$ then we can find a finite map $\operatorname{Spec} A \to \mathbb{A}^{N-1}$. Our hope is that we can achieve this by taking a generic projection.

The map $\operatorname{Spec} A \to \mathbb{A}^N$ is the same thing as a map $k[x_1, \dots, x_N] \to A$, and A is a finite module over the image. This is the same as elements $a_1, \dots, a_N \in A$. Because the transcendence degree is n, which is smaller than N, there is a polynomial $f \in k[x_1, \dots, x_N]$ such that

$$f(a_1, \dots, a_N) = 0.$$

Our hope is to pick $N - 1$ new elements of A such that A is finite over the algebra they generate. The suggestion by the projecting process is to use

$$a_1' = a_1 - c_1 a_N, \dots, a_{N-1}' = a_{N-1} - c_{N-1} a_N,$$

where $c_i \in k$. Because A is a finite module over $k\langle a_1, \dots, a_N \rangle$, it suffices to make a_N integral over $k\langle a_1, \dots, a_{N-1} \rangle$.

Now we have our n-variable polynomial f. We have

$$0 = f(a_1' + c_1 a_N, a_2' + c_2 a_N, \dots, a_{N-1}' + c_{N-1} a_N, a_N).$$

If we expand this, we get

$$(\text{some nonzero polynomial in } c_1, \dots, c_{N-1}) a_N^{\square} + (\cdots) a_N^{\square - 1} + \cdots = 0.$$

We are after some choice $c_1, \dots, c_{N-1}$, this is going to be nonzero, if k is infinite. If k is finite, you can use $a_1' = a_1 - a_N^{r_1}$ and stuff to make the leading coefficient nonzero.

So we get some finite map $\operatorname{Spec} A \to \mathbb{A}^n$. There cannot be any algebraic relations between $a_1, \dots, a_n$ because if there is, we can again do the same thing and get a finite $\operatorname{Spec} A \to \mathbb{A}^{n-1}$. This contradicts the transcendence degree. $\square$

Proof of trdeg = dim. If $\operatorname{trdeg} A = n$, then there exists a finite surjection $A \to \mathbb{A}^n$. But from the proposition we proved last time, we know $\dim A = \dim \mathbb{A}^n$. So we only need to show that $\dim \mathbb{A}^n$.

Let's do this by induction on n. We have certainly done the base case. Suppose we have some chain of prime ideals. Consider the maximal chain

$$(0) \subseteq (f) \subseteq \cdots \subseteq \mathfrak{m}.$$

This first prime is a principal ideal because $k[x_1, \dots, x_n]$ is a UFD. Now we want to show that $V(f) = k[x_1, \dots, x_n]/(f)$ has dimension $n - 1$. But this is what we have been doing so far. So the transcendence degree of $k[x_1, \dots, x_n]/(f)$ is $n-1$ by the random projection thing. This implies that $\dim k[x_1, \dots, x_n]/(f) = n - 1$. $\square$

20 April 20, 2017

We define dimension and proved these big theorems.

Theorem 20.1. *For A a finitely generated domain over k, the following are equivalent:*

(i) $\operatorname{trdeg}_k \operatorname{Frac}(A) = n$,

(ii) $\dim A = n$,

(iii) there exists a finite surjective $\operatorname{Spec} A \to \mathbb{A}^n$.

Why do we care about dimension? We define curves, surfaces, threefolds, ... Then we can try to classifies curves. The other things is that, it is a good way to tell if stuff exists. If $\dim X \geq 0$ then $X \neq \emptyset$. On a K3-surface, we want to look at rational curves on the surface. Do they exist? One thing we can try to do is set up a moduli space of $\mathbb{P}^1$ inside K3, $\mathcal{M}(\mathbb{P}^1, \mathrm{K3})$. But this is hard. So we consider the K3 as a degree 4 hypersurface of $\mathbb{P}^3$, and then consider

$$\mathcal{M}(\mathbb{P}^1, \mathrm{K3}) \subseteq \mathcal{M}(\mathbb{P}^1, \mathbb{P}^3) = \operatorname{Gr}(2,4)/\operatorname{PGL}(4).$$

Then we hope that if we analyze the dimension, then you can show that this is nonempty. This is what people like Joe Harris does all the time. For some $X \to Y$, a generic fiber F satisfies

$$\dim Y = \dim X - \dim F.$$

Theorem 20.2. *Let X, Y be irreducible and finite type over k. Then* $\dim(X \times Y) = \dim X + \dim Y$.

For non-finite type, there is a counterexample $X = k(x)$ and $Y = k(y)$. You can check that $\dim k(x) \otimes_k k(y) = 1$.

Proof. We showed

$$\dim X = \max_{\operatorname{Spec} A \subseteq X} \dim \operatorname{Spec} A = \max_{\operatorname{Spec} A \subseteq X} \operatorname{trdeg}_k \operatorname{Frac}(A) = \operatorname{trdeg}_k \mathscr{O}_{X,\eta}.$$

One remark from this is that we always have

$$\dim X = \operatorname{trdeg}_k \mathscr{O}_{X,\eta} = \operatorname{trdeg}_k \mathscr{O}_{U,\eta} = \dim U$$

for open $U \subseteq X$, if X is irreducible and finite type. So dimension can be computed on any open.

Let us denote $\mathscr{O}_{X,\eta} = \kappa(X)$. Then I claim that

$$\mathscr{O}_{X \times Y, \mathrm{gen}} = \kappa(X) \otimes_k \kappa(Y).$$

Let's just assume that $k = \overline{k}$. Then you can show that if X and Y are irreducible then $X \times Y$ is irreducible. Then for affine open $\operatorname{Spec} A \subseteq X$ and $\operatorname{Spec} B \subseteq Y$, then $\operatorname{Spec} A \otimes_k B \subseteq X \times_k Y$. Then $A \otimes_k B$ is still a domain (by $k = \overline{k}$) and so

$$\operatorname{Frac}(A \otimes_k B) = \kappa(A) \otimes_k \kappa(B).$$

Then $\operatorname{trdeg}_k(\kappa(X) \otimes_k \kappa(Y)) = \operatorname{trdeg}_k \kappa(X) + \operatorname{trdeg}_k \kappa(Y)$. □

What if k is not algebraically closed? If X is irreducible and finite type over k and K/k is a finite extension, we can consider the extension of scalars $X_K = X \times_k \operatorname{Spec} K$. Then I claim that the dimension of any irreducible component of X_K is equal to $\dim_k X$. I don't remember the best way to prove this, but you can prove this for purely transcendental extensions (this shouldn't be too hard) and for algebraic extension, in which case the Galois group has to act on the irreducible components.

20.1 Codimension

I also defined codimension. They are useful for the same reason.

Theorem 20.3. *Let $Y \subseteq X$ be a closed subscheme, where both are irreducible and finite type over k. Then* $\dim Y + \operatorname{codim} Y/X = \dim X$.

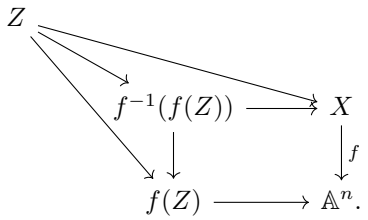
Proof. Suppose $\operatorname{codim} Y/X = n$. Then there exists a chain of closed sets

$$Z_0 = Z \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{n-1} \subsetneq Z_n = X.$$

Then it suffices to show that if $Z \subsetneq X$ is a maximal irreducible closed subset, then $\dim Z = \dim X - 1$.

Take $\operatorname{Spec} A \subseteq X$ is a dense open set. Consider the intersection $\operatorname{Spec} A/I \subseteq Z$. Then $\operatorname{Spec} A/I \subseteq \operatorname{Spec} A$ is again a maximal closed irreducible. So we can suppose that everyone is affine.

Let $n = \dim X$ and we want to show that $\dim Z = n - 1$. There exists a finite surjective map $f : X \to \mathbb{A}^n$.



Then we know that $f^{-1}(f(Z)) \to f(Z)$ is a finite surjection because this is a base change of a finite surjection, and $f^{-1}(f(Z) \to X$ is separated and $Z \to X$ is a closed embedding and so $Z \to f^{-1}(f(Z))$ is a closed embedding by factorization. Then $Z \to f(Z)$ is a finite surjection. So $\dim Z = \dim f(Z)$.

Now we want to show that $\dim f(Z) = n - 1$. But we see that $f(Z)$ is irreducible, and it is a maximal irreducible closed set. Then Z corresponds to some height 1 prime ideal of $k[x_1, \ldots, x_n]$. This is has to be a principal ideal $\mathfrak{p} = (f)$, and then we have shown last time that $\dim V(f) = n - 1$. □

So in particular, if A is a finitely generated domain over k, then all maximal chains of ideals have the same length.

Theorem 20.4 (Hauptidealsatz)**.** *Let X be locally Noetherian over k and $f \in \Gamma(X, \mathscr{O}_X)$. Then* $\operatorname{codim} V(f)/X$ *is zero or one.*

This is actually quite hard to prove, because we don't have access to the transcendence degree. Dimension theory should be true in greater generality.

In the beginning of the semester, we have shown that for $U = \mathbb{A}^2 \setminus (x, y)$, the sections is $\Gamma(U, \mathscr{O}_U) = k[x, y]$, which is the global sections of $\mathbb{A}^2$.

Theorem 20.5 (Hartog's lemma)**.** *Let A be an integrally closed Noetherian domain. Then*

$$A = \bigcap_{\text{ht } \mathfrak{p}=1} A_{\mathfrak{p}}$$

inside $\text{Frac}(A)$.

Why is this Hartog's lemma? $\kappa(A)$ are the meromorphic functions on $\text{Spec } A = X$. Then $A_{\mathfrak{p}}$ are the rational functions who are well-defined on $\overline{\mathfrak{p}} = V(\mathfrak{p})$. So if you're defined on except for a codimension 2 set, then it is well-defined on any codimension 1 set.

21 April 25, 2017

Theorem 21.1. *For X irreducible and finite type over k, and $Y \hookrightarrow X$ an irreducible closed subscheme,* $\dim Y + \operatorname{codim} Y/X = \dim X$.

Proof. We reduced to the affine case, and then were were claiming that if Z is maximal then $f(Z)$ is an hypersurface. To do this, we had to show that $f(Z)$ is also maximal in $\mathbb{A}^n$, which we skipped last time. Suppose that $f(Z) \subsetneq C \subsetneq \mathbb{A}^n$, and we want to show that it pulls back to $f^{-1}(f(Z)) \subsetneq f^{-1}(C) \subsetneq X$. This needs some going-up/down. □

Theorem 21.2 (Going-up/down)**.** *Let $A \subseteq B$ be an integral extension of rings with A normal. If $\mathfrak{p} \subsetneq \mathfrak{p}' \subseteq A$ and $\mathfrak{q}' \subseteq B$ sitting inside $\mathfrak{p}'$, then there exist $\mathfrak{q} \subsetneq \mathfrak{q}' \subseteq B$ sitting over $\mathfrak{p} \subsetneq \mathfrak{p}'$.*

Corollary 21.3. *All maximal chains of prime ideals in domains finitely generated over k have the same length.*

Theorem 21.4 (Hauptidealsatz)**.** *If X is finite type over k (more generally, locally Noetherian), $f_1, \dots, f_r \in \Gamma(X, \mathscr{O}_X)$, and $Y = V(f_1) \cap \cdots \cap V(f_r)$ is a closed subscheme of X, then* $\operatorname{codim} Y/X \leq r$.

Proof. Immediate from previous methods. □

Theorem 21.5 (algebraic Hartog's lemma)**.** *If A is integrally closed and Noetherian. Then*

$$A = \bigcap_{\operatorname{ht} \mathfrak{p}=1} A_{\mathfrak{p}} \subseteq \operatorname{Frac}(A).$$

Definition 21.6. If $f \in \operatorname{Frac}(A)$ and A is a domain, we call f a **rational function** or a **meromorphic function** on $\operatorname{Spec} A$. We define the **polar locus** of f as

$$P(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin A_{\mathfrak{p}}\}.$$

I just made up this definition, but $P(f)$ is probably closed for reasonable A. An analogous definition should make sense for quasicompact quasiseparated schemes. Now if $P(f) = \emptyset$, we probably want $f \in A$.

Lemma 21.7. *For A a domain, $A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$.*

Proof. We further claim that A is the intersection over $A_{\mathfrak{m}}$ for maximal $\mathfrak{m}$. Suppose $f \in \operatorname{Frac}(A)$. Define the **ideal of denominators** of f,

$$D_f = \{a \in A : af \in A\}.$$

If $f \in A_{\mathfrak{m}}$, then there exists some $a \notin \mathfrak{m}$ such that $af \in A$. That is, $D_f \not\subseteq \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$. This shows that $D_f = (1)$. □

Proof of Hartog's lemma. Fix f and let I be the ideal of denominators. Suppose $\mathfrak{q}$ is a minimal that contains I. We localize at q by replacing A with $A_{\mathfrak{q}}$. (Here, $(D_f)_{\mathfrak{q}} = D_{(f_{\mathfrak{q}})}$.) Now $\mathfrak{q}$ is the unique maximal ideal in the local ring A, and so is the unique prime that contains I.

Note that $\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p} = \mathfrak{q}$. Because $\sqrt{I}$ is finitely generated as ideals, say by $q_1, \dots, q_d$, there exists some $n_1, \dots, n_d$ such that $q_i^{n_i} \in I$. Then picking $n = \sum n_i$, if $q \in \mathfrak{q}$ then $q^n \in I$. Take n minimal so that $\mathfrak{q}^n \subseteq I$. We have

$$\mathfrak{q} \supseteq \mathfrak{q}^2 \supseteq \cdots \supseteq \mathfrak{q}^{n-1} \supseteq \mathfrak{q}^n.$$

Choose $a \in \mathfrak{q}^{n-1} \setminus I$, which exists because n is minimal. Consider D_{af}. This is not the unit ideal because $a \notin I$. But $D_{af} \supseteq \mathfrak{q}$ because $a\mathfrak{q} \subseteq \mathfrak{q}^n \subseteq I$. Now if we write $x = af$, then $D_x = \mathfrak{q}$.

The next question is what is $x\mathfrak{q} \subseteq \mathfrak{q}$. Because $\mathfrak{q}$ is the maximal ideal, either $x\mathfrak{q} = A$ or $x\mathfrak{q} \subseteq \mathfrak{q}$. Suppose $x\mathfrak{q} \subseteq \mathfrak{q}$. Then we can again use the determinant trick and then

$$x^n - \operatorname{tr}(a_{ij})x^{n-1} + \cdots = 0.$$

Then $x \in A$ by integrally closed and so $afin D_f$ and so $a \in D_f$. This is a contradiction.

So $x\mathfrak{q} = A$ and so $x^{-1} \in \mathfrak{q}$ and $\mathfrak{q} = (x^{-1})$. This is principal, and it should probably be of height 1. So I is in a prime ideal of height 1. □

Proposition 21.8. *Principal prime ideals in domains are of height* 1.

We will prove this later.

21.1 Regularity

Smoothness and regularity are not quite the same. Now we are starting differential geometry. Because we are doing algebra, we are going to need some notion of a differential algebra(?).

Definition 21.9. Let X is a scheme and $p \in X$. The **Zariski cotangent space** is defined as

$$T_{X,p}^{\vee} = \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$$

where $\mathfrak{m}_{X,p} \subseteq \mathscr{O}_{X,p}$ is the local ring. This is a vector space over $\mathscr{O}_{X,p}/\mathfrak{m}_{X,p} = \kappa(p)$. The **tangent space** is its dual

$$T_{X,p} = (\mathfrak{m}/\mathfrak{m}^2)^{\vee}.$$

Example 21.10. Let $X = \mathbb{A}^n$, $k[x_1, \dots, x_n]$. Let us look at the cotangent space at the origin, $T_{X,0}^{\vee}$. The local ring and the maximal ideal are

$$\mathscr{O}_{X,p} = k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}, \quad \mathfrak{m}_{X,p} = (x_1, \dots, x_n)k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}.$$

Then

$$\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 = k\langle x_1, \dots, x_n \rangle.$$

What have I done here? The geometric interpretation is that $\mathscr{O}_{X,p}$ is the germs of functions near the origin. $\mathfrak{m}_{X,p}$ is the germs of functions vanishing at p to order 1, and $\mathfrak{m}_{X,p}$ is the germs of functions vanishing at p to order 2.

Proposition 21.11. *If X be irreducible and finite type over k and $p \in X$, then*

$$\dim_{\kappa(p)} T_{X,p}^{\vee} \geq \dim X.$$

Proof. We can only look at the affine case, because the cotangent space clearly cares about the local behavior, and also $\dim X$ can be computed on an open affine chart. Let $n = \dim_{A_p/\mathfrak{m}_p} \mathfrak{m}_p/\mathfrak{m}_p^2$. Pick a basis $m_1, \dots, m_n$ a basis for $\mathfrak{m}/\mathfrak{m}^2$. If we pick any lifts $\tilde{m}_1, \dots, \tilde{m}_n \in \mathfrak{m}$, by Nakayama, they still generate the ideal $\mathfrak{m}$.

Now consider

$$\kappa(p) = A/(\tilde{m}_1, \dots, \tilde{m}_n).$$

This has dimension 0 because it is a field, but it is cut out by n equations and so must have dimension at least $\dim A - n$. (This is the Hauptidealsatz business.) So $0 \geq \dim A - n$ and $\dim A \leq n$. □

Example 21.12. Consider $\operatorname{Spec} k[x,y]/(y^2 - x^3) = X$ and $T_{X,0}$. The maximal ideal is going to be $\mathfrak{m} = (x,y)k[x,y]_{(x,y)}/(y^2 - x^3)$ and then $\mathfrak{m}^2 = (x^2, xy, y^2)k[x,y]_{(x,y)}/(y^2 - x^3)$. This implies that

$$\mathfrak{m}/\mathfrak{m}^2 = (x,y)k[x,y]_{(x,y)}/(x^2, xy, y^2),$$

which is just the cotangent space of $\mathbb{A}^2$. So $\dim T_{X,0} = 2 > \dim X = 1$.

Definition 21.13. A scheme X is **regular** if $\dim T_{X,p} = \dim X$ for all $p \in X$.

Proposition 21.14. *If $\mathfrak{p} \subseteq A$ is a principal prime ideal inside a finitely type domain A over k, then* $\operatorname{ht} \mathfrak{p} = 1$.

Proof. We want to show that $\dim A_{\mathfrak{p}} = 1$. We have

$$\dim A_{\mathfrak{p}} \leq \dim_{A/\mathfrak{p}} \mathfrak{p}/\mathfrak{p}^2.$$

But I can pick the generator f as a spanning set of $\mathfrak{p}/\mathfrak{p}^2$. So $\dim T_{X,p}^{\vee} \leq 1$ and so $\operatorname{ht} \mathfrak{p} \leq 1$. □

Example 21.15. Consider the map $k[t^2, t^3] \hookrightarrow k[t]$. This is the map of $\mathbb{A}^1$ to the cuspidal curve. This is actually a homeomorphism of topological spaces. But if we write $x = t^2$ and $y = t^3$, then y/x is a rational function that has no singularities. This is a counterexample to Hartog's lemma.

22 May 1, 2017

I have discussed regularity last time. I defined it as $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim X$. But whatever smoothness is, we should be able to construct tangent/cotangent bundles and we should be able to differentiate functions.

Here is a regular scheme that is not smooth. Consider $\operatorname{Spec} \mathbb{F}_p[t^{1/p}]$ over $\operatorname{Spec} \mathbb{F}_p[t]$. Differentiation is then a bit awkward. Let us write $\mathbb{F}_p[t] = R$ and $\mathbb{F}_p[t^{1/p}] = R[u]/(u^p - t)$. Given a $f \in R[u]$, we would expect there to be some df. Now we have $dt = d(u^p) = pu^{p-1}du = 0$.

22.1 Cotangent bundle

We have defined the cotangent space $T^\vee_{X,p}$. We would need the cotangent bundle to be a quasicoherent sheaf. We would also want a notion of relative cotangent bundles.

Let us go to the affine case. Consider $\operatorname{Spec} B \to \operatorname{Spec} A$. Let us define the **sheaf of differentials** $\Omega_{B/A}$. This is going to be a B-module, with elements like db for $b \in B$. We define it as

$$\Omega_{B/A} = \left\langle db, b \in B : \begin{array}{c} d(b_1 + b_2) = db_1 + db_2 \\ d(b_1 b_2) = b_1 db_2 + b_2 db_1 \\ da = 0 \end{array} \right\rangle.$$

Here is another idea. For any M a B-module, define the **derivations** as

$$\operatorname{Der}_A(B, M) = \left\{ (f : B \to M) : \begin{array}{c} f \text{ is } A\text{-linear} \\ f(b_1 b_2) = b_1 f(b_2) + b_2 f(b_1) \end{array} \right\}.$$

You can show that the map $M \mapsto \operatorname{Der}_A(B, M)$ is corepresentable by $\Omega_{B/A}$, i.e., for any B-module M,

$$\operatorname{Hom}_B(\Omega_{B/A}, M) \cong \operatorname{Der}_A(B, M).$$

Example 22.1. Consider $B = k[x_1, \ldots, x_n]$ and $A = k$. What is the sheaf of differentials? For some intuition, we can compute

$$d(x_1^2 x_2 + x_3^3 + 1) = 2x_1 x_2 dx_1 + x_1^2 dx_2 + 3x_3^2 dx_3.$$

So we are going to have

$$\Omega_{k[x_1,\ldots,x_n]} = \langle dx_1, \ldots, dx_n \rangle.$$

So $\Omega_{B/A}$ is going to be the free module of rank n over B generated by $dx_1, \ldots, dx_n$.

Example 22.2. Take $B = k[x, y, z]/(x^2 + y^2 + 2z)$ and $A = k$. What is $\Omega_{B/A}$? This is still going to be generated by dx, dy, dz, but there will be more relations, like

$$0 = d(x^2 + y^2 + 2z) = 2xdx + 2ydy + 2dz.$$

I claim that

$$\Omega_{B/A} = B\langle dx, dy, dz \rangle / (2xdx + 2ydy + 2dz) = 0.$$

In fact, you can show that if $B = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ then $\Omega_{B/A}$ is the cokernel of the Jacobian matrix.

Proposition 22.3. *Let $X \to k$ be a finite type scheme, and let $p \in X$ be a closed k-rational point (the residue field is k). Then $\Omega_{X/k}|_p \cong \mathfrak{m}_p/\mathfrak{m}_p^2$.*

I haven't actually defined $\Omega_{X/Y}$ in general. You can show that $\Omega_{B/A}$ localizes well, and then show that they can be glued well by some cocycle condition.

Proof. We can reduce this to the affine case. Let $\operatorname{Spec} B \to \operatorname{Spec} k$. The point will correspond to some maximal ideal $\mathfrak{m} \subseteq B$ so that $B/\mathfrak{m} = k$. We then want to show that

$$\Omega_{B/k} \otimes_B B/\mathfrak{m} \cong \mathfrak{m}/\mathfrak{m}^2.$$

Instead, we check that we have a canonical isomorphism

$$\operatorname{Hom}_{B/\mathfrak{m}}(\Omega_{B/k} \otimes_B B/\mathfrak{m}, k) \cong \operatorname{Hom}_{B/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, k).$$

the left hand side is $\operatorname{Hom}_B(\Omega_{B/k}, k)$. This is the same as $\operatorname{Der}_k(B, k)$ by the universal property. So the problem is to give an isomorphism $\operatorname{Der}_k(B, k) \cong \operatorname{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$.

Given a derivation $f : B \to k$, we can restrict it to $f|_{\mathfrak{m}}^2$. Then $f(m_1 m_2) = m_1 f(m_2) + m_2 f(m_1) = 0$ because $m_1, m_2 \in \mathfrak{m}$ send anything to 0. So we get a map $\mathfrak{m}/\mathfrak{m}^2 \to k$. The other direction is also easy. Now I have a $f : \mathfrak{m} \to k$ with $\mathfrak{m}^2 \to 0$. We want to lift this to $\tilde{f} : B \to k$. We have a short exact sequence

$$0 \to \mathfrak{m} \to B \to B/\mathfrak{m} = k \to 0,$$

but we have a section $k \to B$ because B was already a k-algebra. So we get a projection $B \to \mathfrak{m}$ and we can compose this with $\mathfrak{m} \to k$. □

In the general case, you can deal it with a trick. Suppose you have the map $X \to \operatorname{Spec} E$ corresponding to the point, for some extension E/k. Then we can make a base change to $X_E \to \operatorname{Spec} E$, and this is going to satisfy the condition we needed.

Definition 22.4. For C a proper regular curve over k, we can define its **genus** as

$$g = \dim \Gamma(C, \Omega_{C/k}).$$

22.2 Smoothness

The **ramification locus** of $X \to Y$ (finite or quasifinite morphism) is like the places we have branching, as $\operatorname{supp} \Omega_{X/Y}$. Consider the map $z \mapsto z^2$. We are going to expect something to happen at $z = 0$. We can actually compute $\Omega_{k[\epsilon]/\epsilon^2} = B\langle d\epsilon \rangle/(2\epsilon d\epsilon)$, which is nonzero. In fact, you can define étale maps as maps with vanishing $\Omega_{X/Y}$.

Let K be a number field, a finite extension of $\mathbb{Q}$. Then there is a finite map $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} \mathbb{Z}$, and then $\Omega_{\mathcal{O}_K/\mathbb{Z}}$ is called the **different ideal**. Then the

ramification locus becomes the discriminant ideal. In number theory, there is a theorem that says that the discriminant is always greater than 1. This can be thought of as $\pi_1^{\text{ét}} \operatorname{Spec} \mathbb{Z} = *$. I also think $\pi_1^{\text{ét}} \mathbb{Z}[i] = *$.

We have a B-module morphism $B \to \Omega_{B/A}$ given by $b \mapsto db$. This patches together to give a morphism $d : \mathscr{O}_X \to \Omega_{X/Y}$.

Definition 22.5. A morphism $X \to Y$ is **smooth** if $\Omega_{X/Y}$ is a vector bundle.

In this case, we can define $\Omega^i_{X/Y} = \wedge^i \Omega_{X/Y}$, which is the sheaf of i-forms. We can extend the d map to

$$0 \to k \to \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X/k} \to 0$$

and get a chain complex.

Proposition 22.6 (algberaic de Rham)**.** *If X is smooth and proper over k, then this complex is exact.*

This is a resolution of the constant sheaf $\underline{\mathbb{C}}$. The cohomology of X then is going to be

$$H^*(X; \underline{\mathbb{C}}) \cong H^*(X, \{\Omega^i_{X/k}\}).$$

On the other hand, $H^*(X; \underline{\mathbb{C}})$ algebraically is $H^*(X; \mathbb{C})$ topologically. (I think you need to go through GAGA to see this.) Now then there is a spectral sequence $E_2^{p,q} = H^q(X, \Omega^p_{X/k})$. The claim is that this collapses on the E_2 page. Then we get a decomposition

$$H^n(X; \mathbb{C}) = \bigoplus_{p+q=n} H^q(X, \Omega^p_{X/k}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

This is now Hodge theory. Here, these two match because $\Omega^p_{X/k}$ is like the holomorphic p-forms and some complex geometry nonsense suggests that H^q can be computed by anti-holomorphic stuff.

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