# Math 281x - Arakelov Theory on Arithmetic Surfaces 

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This course was taught by Héctor Pastén, on Mondays, Wednesdays, and Fridays from 1 to 2 pm . There was a small midterm exam and a final paper, for students taking the course for credit. Lang's Introduction to Arakelov Theory was provided as a general reference. An official set of notes can by found on the instuctor's website http://math.harvard.edu/~hpasten/S2018math281X. html.

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## 1 January 22, 2018

### 1.1 Arithmetic surfaces

Let $A$ be a 1-dimensional Dedekind domain, and $S=\operatorname{Spec} A$.
Definition 1.1. An arithmetic surface is a pair $(\mathfrak{X}, \pi)$ where $\mathfrak{X}$ is an integral scheme and $\pi: \mathfrak{X} \rightarrow S$ is a flat proper relative curve of finite type.

In particular, the generic fiber $\mathfrak{X}_{\eta}$ is a projective integral curve over $K=$ $\operatorname{Frac}(A)$. Also, $\operatorname{dim} \mathfrak{X}=2$.

Definition 1.2. We say that $(\mathfrak{X}, \pi)$ is normal (regular) if $\mathfrak{X}$ is. (So it's not relative.)

If $\mathfrak{X}$ is regular (or normal) then $\mathfrak{X}_{\eta}$ is smooth. However, $\mathfrak{X}_{s}$ for $s \in S$ a closed point can be singular or non-reduced or reducible.

So given an arithmetic surface, we get a good curve over the fractional field of $A$. But we might want to do the other way round. Suppose we have a curve $X / K$ (projective smooth finite type and connected). Does there exist a "nice" algebraic surface $(\mathfrak{X}, \pi)$ with $\mathfrak{X}_{\eta} \cong X$ ? If "nice" means normal, we can do this by spreading out. Take the equations cutting out $X$ in projective space, and clear out the denominators. For "nice" equals regular, this is Lichtenbaum's theorem.

Theorem 1.3. If $(\mathfrak{X}, \pi)$ is regular, then $\pi$ is projective.
Here are the ideas. We have good intersection theory for $\operatorname{Div}_{s}(\mathfrak{X}) \times \operatorname{Div}(\mathfrak{X}) \rightarrow$ $\mathbb{Z}$. This is defined by first looking at $\operatorname{Div}(X)$ as a sheaf, considering the normalization of $\operatorname{Div}_{s}(X)$, and pull back the sheaf. This is going to be sheaf on a nice projective variety, so we can define degree well. Then construct an effective divisor $D$ on $X$ satisfying

- $\operatorname{supp}(D)$ contains no fiber component,
- $D$ meets every fiber component.

Now note that $\left.D\right|_{X_{0}}$ is ample, and deduce that $D$ is ample for $\pi$.
Theorem 1.4. Let $X / K$ be a "nice curve".
(1) There is a regular integral model $\mathfrak{X}$ for $X$.
(2) We can take $\mathfrak{X}$ to be minimal.
(3) If $g(X) \geq 1$, then the "minimal" model of (2) is unique.

Definition 1.5. If ( $\mathfrak{X}, \pi$ ) is "minimal" among the regular models, it is called relatively minimal. If it is unique, then it is called minimal.

## 2 January 24, 2018

Last time we discussed some motivations. Our setting is $A$ a Dedekind domain (of dimension 1), $S=\operatorname{Spec} A$, and $K=\operatorname{Frac} A$. An arithmetic surface is a pair ( $\mathfrak{X}, \pi$ ) where $\pi: \mathfrak{X} \rightarrow S$ is proper, integral, flat, finite-type, and a relative curve. We are going to assume some conditions on this: one is normality and another one is regularity. Last time we sketched the proof of the following:

Theorem 2.1 (Litchenbaum). Regular arithmetic surfaces are projective.
Definition 2.2. Let $X / K$ be a smooth projective integral curve. An integral model for $X$ is an arithmetic surface $(\mathfrak{X}, \pi)$ with an isomorphism $\phi: \mathfrak{X}_{\eta} \cong X$.

Under which conditions is there a "nice" model? First of all, models exist by a spreading-out argument. Secondly, normal model exist by taking normalization. Regular models also exist. The idea is blow up at the singular locus, normalize, blow up, and repeat. That this stabilizes is due to Lipman, but we won't prove it.

### 2.1 Minimal models

Definition 2.3. A regular model $(\mathfrak{X}, \pi)$ for $X / k$ is relatively minimal if for every proper birational $S$-morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of regular models for $X$, we have that $f$ is an isomorphism.

All tools like Castelnuovo are available, so a relatively minimal model is one obtained by blowing down if there is an exceptional divisor.

Definition 2.4. We say that $(\mathfrak{X}, \pi)$ is a minimal model of $X$, if it is relatively minimal and all relatively minimal models are isomorphic.
Example 2.5. Consider $X=\mathbb{P}_{K}^{1}$ and $\mathfrak{X}=\mathbb{P}_{S}^{1}$. Here, all the fibers have selfintersection number 0 , and it can be shown that $\mathfrak{X}$ is minimal. Take a point on the fiber $F$, and blow-up at a point on $F$. Then we get an exceptional divisor $E$ and the fiber $\tilde{F}$. Then $\tilde{F}$ has self-intersection -1 , and then we can contract it to get a different model.

Now let us define the intersection number. Let $(\mathfrak{X}, \pi)$ be any arithmetic surface an $s \in S$ a closed point. Let $C$ be a projective curve over a field $k$ with a closed immersion $i: C \rightarrow \mathfrak{X}$. Let $F=i(C)$. Then $F$ is considered in a fiber, and say $F \subseteq \mathfrak{X}_{s}$. Let $D$ be a divisor on $\mathfrak{X}$. Then we define

$$
i_{s, k}(F, D)=\operatorname{deg}_{C / k} i^{*} \mathscr{O}_{\mathfrak{X}}(D) \in \mathbb{Z} .
$$

This is well-defined because $C$ is projective.
We can then define a pairing. Let $F$ be the reduced fiber component at $s$, and let $\nu: C \rightarrow F$ be the normalization, and take $k=\mathfrak{x}_{s}$. Then we get

$$
i_{s}: \operatorname{Div}_{s}(\mathfrak{X}) \times \operatorname{Div}(\mathfrak{X}) \rightarrow \mathbb{Z}
$$

There is another interesting choice. Take $k=H^{0}\left(F, \mathscr{O}_{F}\right)=H^{0}\left(C, \mathscr{O}_{C}\right)$.

Theorem 2.6 (Castelnuovo-Lischtenbaum). Let $\mathfrak{X}$ be a regular arithmetic surface over $S$. A prime divisor $E$ on $\mathfrak{X}$ is contractible (i.e., there is a proper birational $S$-morphism $\mathfrak{X} \rightarrow \mathfrak{Y}$ with $\mathfrak{Y}$ regular, mapping $E$ to a point, and isomorphism away from $E$ ) if and only if the following holds:
(i) E is fibered, say, in $\mathfrak{X}_{s}$ for $s \in S$ closed.
(ii) $H^{1}\left(E, \mathcal{O}_{E}\right)=0$.
(iii) $i_{s, k}(E, E)=-1$ where $k=H^{0}\left(E, \mathscr{O}_{E}\right)$.

In this case, we get a $k$-isomorphism $E \cong \mathbb{P}_{k}^{1}$.
In (iii), the field $k$ has to be chosen carefully, because we want to say that something is a blow-down even when the residue fields decrease.

Theorem 2.7. Let $X / K$ be a smooth projective curve. Assume that $K$ is algebraically closed in $K(X)$. Then:
(1) If $\mathfrak{X} / S$ and $\mathfrak{X}^{\prime} / S$ are regular integral models for $X$, then there exists another regular integral model $\mathfrak{Y} / S$ for $X$ such that there exist $\phi, \phi^{\prime}: \mathfrak{Y} \rightarrow$ $\mathfrak{X}, \mathfrak{X}^{\prime}$ proper and birational $S$-morphisms of models for $X$.
(2) Given $\mathfrak{X} / S$ a regular model for $X$, make a sequence $\mathfrak{X}^{(0)} \rightarrow \mathfrak{X}^{(1)} \rightarrow \mathfrak{X}^{(2)} \rightarrow$ $\cdots$ of contractions of exceptional curves. Then the sequence stabilizes on some regular integral model for $X$.
(3) Relatively minimal models exist.
(4) If $\mathrm{g}(X) \geq 1$ then minimal models exist.

Proof. (1), (2), (3) are similar (up to technical steps) as in the theory of complex surfaces. For (4), we may as well assume that $A$ is a DVR, because we can do everything fiberwise. Now the point of the argument is to show that the order of blowing down doesn't matter. This is delicate because there can be a problem if two exceptional divisors meet, they can mess up the order. But the point is that if $E \neq C$ are exception on a regular model $\mathfrak{X}$, then $E$ and $C$ don't meet.

How do we show this? Suppose they meet, and contract at $E$. Then the blow-down $\mathfrak{X} \rightarrow \mathfrak{Y}$ sends $E$ to $p$ and $C$ to some $D$. If we write $f^{*} D=C+m E$, then

$$
i_{s, k_{s}}(D, D)=\left[k_{1}: k_{2}\right]\left(i_{s, k_{1}}(C, C)+m i_{s, k_{1}}(C, E)\right) \geq i_{s, k_{1}}(C, C)+1=0 .
$$

Now $D$ is in a fiber, so the intersection theory on the fiber shows that this is negative semi-definite. So the self-intersection number of $D$ is 0 , and $n D=\mathfrak{Y}_{s}$. If $\mathscr{I}$ is the ideal sheaf of $D$, with $s=(\pi)$, we have $\pi \mathscr{O}_{Y}=\mathscr{I}^{n}$.

Now

$$
H^{1}\left(\mathfrak{Y}, \mathscr{I}^{r} / \mathscr{I}^{r+1}\right)=H^{1}\left(D,\left(\nu^{*} \mathscr{I}\right)^{\otimes r}\right)=0
$$

because $\operatorname{deg}_{D / k_{2}} \nu^{*} \mathscr{I}$ is the self-intersection number, which is 0 . The exact sequence $0 \rightarrow \mathscr{I}^{r} / \mathscr{I}^{n+1} \rightarrow \mathscr{O}_{\mathfrak{Y}} / \mathscr{I}^{r+1} \rightarrow \mathscr{O}_{\mathfrak{Y}} / \mathscr{I}^{r} \rightarrow 0$ induces

$$
H^{1}\left(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}\right) \xrightarrow{\pi} H^{1}\left(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}\right) \rightarrow H^{1}\left(\mathfrak{Y}, \mathscr{I}^{r} / \mathscr{I}^{r+1}\right)=0
$$

and so $H^{1}\left(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}\right)$ is torsion. But this cannot happen, because then the fiber over the generic point is then zero. This is precisely the genus condition $g(X) \geq$ 1 prevents.

## 3 January 26, 2018

Let me do this computation, because this seems to be not clear.
Proposition 3.1 (Mumford?). Let $\mathfrak{X} / S$ be a regular arithmetic surface. Suppose that for $s \in S$, the special fiber $\mathfrak{X}_{s}$ is connected. Then $i_{s}: \operatorname{Div}_{s}(\mathfrak{X}) \times$ $\operatorname{Div}_{s}(\mathfrak{X}) \rightarrow \mathbb{Z}$ is negative semidefinite, and given $D \in \operatorname{Div}_{s}(\mathfrak{X}), D^{2}=0$ if and only if $D=\lambda \mathfrak{X}_{s}$ for some $\lambda \in \mathbb{Q}$.

Proof. Assume that $A$ is a DVR and $\varpi$ is the uniformizer. Let us write $\mathfrak{X}_{s}=$ $\sum_{j} m_{j} C_{j}$ where $C_{j}$ are components with multiplicities $m_{j}$. For all $j, C_{j} \cdot \mathfrak{X}_{s}=0$, and so

$$
m_{j} C_{j}^{2}=-\sum_{i \neq j} m_{j} C_{i} \cdot C_{j}
$$

for all $j$.
Now consider a general divisor $D \in \operatorname{Div}_{s}(\mathfrak{X})$, and write $D=\sum_{j} a_{j} m_{j} C_{j}$ for $a_{j} \in \mathbb{Q}$. We have

$$
\begin{aligned}
D^{2} & =\sum_{j} a_{j}^{2} m_{j}^{2} C_{j}^{2}+\sum_{i \neq j} a_{i} a_{j} m_{i} m_{j} C_{i} \cdot C_{j} \\
& =-\frac{1}{2} \sum_{i \neq j}\left(a_{i}-a_{j}\right)^{2} m_{i} m_{j} C_{i} \cdot C_{j} \leq 0
\end{aligned}
$$

So this is semi-definite, and it is an equality if and only if $a_{i}$ are all equal. (Here, we're using that the fiber is connected.)

We have focused on good properties in an absolute sense, not in a relative sense. This is one issue, and another issue is base changing. For instance, consider $A[x, y] /(x y-\varpi)$ where $A$ is a DVR and $\pi$ is a uniformizer. This is a regular ring, but the special fiber is not nice. Also, suppose we're going to base change to $B=A\left[\varpi^{1 / r}\right]$. Then we get $B[x, y] /\left(x y-\lambda^{r}\right)$, which is singular for $r \geq 2$. This will motivate the discussion on semi-stable models.

### 3.1 Semi-stable models

From now on, all residue fields are perfect.
Definition 3.2. Let $k=k^{\text {alg }}$ and $C / k$ be a 1 -dimensional connected projective $k$-scheme. We say that
(1) $C$ has normal crossings if if is reduced and the only singularities or nodes. Completely locally, these look like $k[[x, y]] /(x y)$.
(2) $C$ is semi-stable if it has normal crossings and every component isomorphic to $\mathbb{P}^{1}$ has at least 2 intersection points with other components.

Definition 3.3. Let $\mathfrak{X} / S$ be an arithmetic surface, and $s \in S$.
(1) $\mathfrak{X}_{s}$ has normal crossings if $\mathfrak{X}_{s} \otimes_{k_{s}} k_{s}^{\text {alg }}$ is.
(2) $\mathfrak{X}_{s}$ is semi-stable if $\mathfrak{X}_{s} \otimes_{k_{s}} k_{s}^{\text {alg }}$ is.
(3) $\mathfrak{X}$ has normal crossings/is semi-stable if $\mathfrak{X}_{s}$ is for all $s \in S$.

Lemma 3.4. Take $\mathfrak{X} / S$ an arithmetic surface with smooth geometric fiber $X / K$, and $K=H^{0}\left(X, \mathscr{O}_{X}\right)$. (This implies that all fibers are geometrically connected.) why? Assume that $\mathfrak{X}$ has normal crossings. Then
(1) $\mathfrak{X}$ is normal.
(2) Let $L / K$ be finite and $B \subseteq L$ be the integral closure of $A$. Let $T=\operatorname{Spec} B$. Then $\mathfrak{X}_{T} / T$ is an arithmetic surface with normal crossings.
(3) The same holds for "semi-stable" instead of "normal crossings".

Proof. (1) is some commutative algebra verification. (2) is trivial because we are base changing to the algebraic closure to verify the conditions.

Theorem 3.5. Let $\mathfrak{X} / S$ be an arithmetic surface with smooth generic fiber $X / K$, and assume that $K=H^{0}\left(X, \mathscr{O}_{X}\right)$. Also assume that $\mathfrak{X}$ has normal crossings. There is a finite extension $K_{0} / K$ such that for all $L / K_{0}$ finite, if $B \subseteq L$ is the integral closure of $A$ and $T=$ Spec $B$, then the following holds for the normal crossings surface $\mathfrak{Y}=\mathfrak{X}_{T}$ : if $s \in T$ with uniformizer $\varpi$, then the only singularities of $\mathfrak{X}_{T}$ above $s$ are ordinary double points of the form

$$
\widehat{\mathfrak{O}_{\mathfrak{Y}, y}} \cong \hat{B}[[X, Y]] /\left(X Y-\varpi^{n}\right)
$$

for some $n \geq 2$. The same holds for "semi-stable" instead.
To resolve these singularities, we can just blow up repeatedly. This replaces to singular point $y$ by a chain of $n-1(-2)$-curves isomorphic to $\mathbb{P}_{k_{s}}^{1}$. In the normal crossing case, this is all written up in detail in Liu. For the semi-stable, you can do this by just getting you hands dirty and tracking all the $\mathbb{P}^{1}$ s.

## 4 January 29, 2018

Last time we introduced the notion of stability. The basic claim here is that normal crossing singularities on the fiber lifts to a completed local ring of the form $\widehat{B}[[x, y]] /\left(x y-\varpi^{n}\right)$. If we pass to a field where we can see the singularities and the components passing through the singularity, we get a local description. Then it is just an exercise to resolve the singularity.

Let $k=\kappa_{s}$ and $\bar{k}=\kappa_{s}^{\text {alg }}$. We then base change to the Witt vectors $W(\bar{k})$, and consider $\mathfrak{X} \otimes_{A} W(\bar{k}) \rightarrow \mathfrak{X}$. (Technically $A$ can be ramified, so we should base change to $R=\left(\mathcal{O}_{\mathfrak{X}, \bar{x}}\right)^{\wedge} \cong\left(\mathcal{O}_{\mathfrak{X}, x}^{\text {s.h. }}\right)^{\wedge}$. $)$ Then we now all the flat $\left.\widehat{A}[[x, y, z]]\right] /(x y-$ $z) \rightarrow R$.
???
Theorem 4.1. Let $X / K$ be a smooth projective curve, with $K=H^{0}\left(X, \mathscr{O}_{X}\right)$, with $g_{X} \geq 1$. Assume there exists a semi-stable model $\mathfrak{X} / A$. Then there is a regular semi-stable model for $X / A$, call it $\mathfrak{X}^{\prime}$, obtained by blow-up of singular points with a proper birational map $p: \mathfrak{X}^{\prime} \rightarrow \mathfrak{X}$ whose connected fibers are chains of $(-2)$-curves, and $\mathfrak{X}^{\prime}$ is minimal regular.

We can resolve the singularity by blowing up and normalizing. Now here, we're stating that it can be down only with blow-ups.

Proof. In the result last time, the extension $K_{0} / K$ just needs "long enough residue extension". This can be achieved by an unramified extension $K_{0} / K$. (We're working locally over a DVR.) Now we can take the integral closure $B=$ $\tilde{A} \subseteq K_{0}$, and then $\mathfrak{X}_{B} \rightarrow \mathfrak{X}$ is étale.

Now suppose $\mathfrak{X}^{\prime}$ is not minimal regular. Then there exists some $C \subseteq \mathfrak{X}_{s}^{\prime}$ with $C \cong \mathbb{P}_{k}^{1}$ where $k=H^{0}\left(C, \mathcal{O}_{C}\right)$, and $i_{s, k}(C, C)=-1$. Base change so that $k=\kappa_{s}$. (This is étale.) Then

$$
0=i_{s}\left(C, \mathfrak{X}_{s}^{\prime}\right)=i_{s}(C, C)+i_{s}\left(C, \mathfrak{X}_{s}^{\prime}-C\right)
$$

But then $i_{s}\left(C, \mathfrak{S}^{\prime}-C\right)=1$, which is impossible because $\mathfrak{X}^{\prime}$ is semi-stable.

### 4.1 Duality

Assume that everything is Noetherian.
Definition 4.2. Let $n \geq 0$ be an integer, and let $X \rightarrow Y$ be a morphism. Assume $f$ is proper of relative dimension $n$. An $n$-relative dualizing sheaf is a pair $(\omega, t)$ where $\omega$ is a quasi-coherent $\mathscr{O}_{X}$-module and

$$
t: R^{n} f_{*} \omega \rightarrow \mathscr{O}_{Y}
$$

is a map such that for every quasi-coherent $\mathscr{O}_{X}$-module $\mathscr{F}, t$ induces an isomorphism

$$
f_{*} \mathscr{H} \operatorname{om}_{\mathscr{O}_{X}}(\mathscr{F}, \omega) \cong \mathscr{H} \operatorname{om}_{\mathscr{O}_{Y}}\left(R^{n} f_{*} \mathscr{F}, \mathscr{O}_{Y}\right)
$$

In our case, we have $Y=S=\operatorname{Spec} A$, and $n=1$. So the trace is

$$
t: H^{1}(X, \omega) \rightarrow A
$$

and the condition translate to

$$
\operatorname{Hom}_{\mathscr{O}_{X}}(\mathscr{F}, \omega) \cong \operatorname{Hom}_{A}\left(H^{1}(X, \mathscr{F}), A\right)
$$

But the left hand side is $H^{0}\left(X, \omega \otimes_{\mathscr{O}_{Y}} \mathscr{F}^{\vee}\right)$.
The question is existence of the dualizing sheaf $\omega$.
Definition 4.3. Let $f: X \rightarrow Y$ be of finite type. We say that $f$ is l.c.i. if for all $x \in X$, there exists a neighborhood $x \in U$ such that there exist a smooth $p: Z \rightarrow Y$ and $i: U \rightarrow Z$ a regular immersion such that $\left.f\right|_{U}=p \circ i$.

In this setting, one has the duality theory developed by Kleiman. Here are some ways of checking lci:

- If $f: X \rightarrow Y$ is of finite type and $Y^{\prime} \rightarrow Y$ is faithfully flat, then $f$ is lci if and only if $X \times_{Y} Y^{\prime} \rightarrow Y^{\prime}$ is.
- If $f: X \rightarrow Y$ is flat and of finite type, $f$ is lci if and only if all fibers are.

It will turn out that all semi-stable arithmetic surfaces are lci, after playing around with these properties.

## 5 January 31, 2018

Last time, the two key points were the definition of a relative dualizing sheaf, and the notion of a lci morphism. We don't yet have existence of a relative dualizing sheaf. Also, we want our dualizing sheaf to be invertible.

Example 5.1. Consider $f: \mathbb{P}_{Y}^{n} \rightarrow Y$. In this case, $\omega_{f}$ exists and is invertible. Explicitly, it is the determinant of $\omega_{f}=\operatorname{det} \Omega_{f}^{1}=\Omega_{\mathbb{P}_{Y}^{n} / Y}^{n}$.

Here is one of the possible statements for existence.
Theorem 5.2 (Kleiman p.52,55,58). Let $f: X \rightarrow Y$ be flat, projective, of pure relative dimension $n$.
(1) If the fibers of $f$ are Cohen-Macauly, then $\omega_{f}$ exists.
(2) If the fibers are Gorenstein then $\omega_{f}$ exists and is invertible.

In particular, if $f$ is lci then (2) holds.

### 5.1 Adjunction formula

But this is not what we are going to use.
Theorem 5.3 (Kleiman, Corollary 19, Theorem 21). Let $f: X \rightarrow Y$ be flat, projective, lci, of pure relative dimension $n$. (So in particular, $\omega_{f}$ exists and is invertible.) Now let $h: Z \rightarrow X$ be a regular closed immersion with ideal $\mathscr{I}$ on $X$. (Locally it means that it is a complete intersection.) Suppose $g=f \circ h: Z \rightarrow Y$ is flat of pure relative dimension $m \leq n$. Then $g$ admits a relative dualizing sheaf $\omega_{g}$, invertible, and with a canonial isomorphism

$$
\omega_{g} \cong \operatorname{deg}\left(h^{*} \mathscr{I}\right)^{\vee} \otimes_{\mathscr{O}_{Z}} h^{*} \omega_{f}
$$

What you can do is to check that this really is a relative dualizing sheaf. Then if you have a dualizing sheaf on projective space, you can somehow restrict it to closed subscheme.

Corollary 5.4. Let $(\mathfrak{X}, \pi)$ be a regular arithmetic surface over $S$. Then $\omega_{\pi}$ exists and is invertible. Explicitly, $\pi$ is projective, so we can write $i: \mathfrak{X} \rightarrow \mathbb{P}_{S}^{n}$ with ideal $\mathscr{I}$.


Here $i$ is a regular closed immersion because $\mathfrak{X}$ is itself regular. Then

$$
\omega_{\pi} \cong \operatorname{det}\left(i^{*} \mathscr{I}\right)^{\vee} \otimes i^{*} \Omega_{\mathbb{P}_{S}^{n} / S}^{n}
$$

Corollary 5.5. Let $(\mathfrak{X}, \pi)$ be a regular, normal crossings arithmetic surface withe regular dualizing sheaf $\omega_{\pi}$. Let $s \in S$ be a closed point, and let $k=\kappa_{s}$. Let $i: C \hookrightarrow \mathfrak{X}$ be a fiber component at $s$. Then $\omega_{C / k}$ exists, is invertible, and

$$
\omega_{C / k} \cong i^{*}\left(\omega_{\pi} \otimes \mathscr{O}_{\mathfrak{X}}(C)\right)
$$

Proof. We have $\mathscr{I} \cong \mathscr{O}_{\mathfrak{X}}(-C)$. We pull first pull back to $\mathfrak{X}_{s}$ over $s$, and then use the adjunction formula.

Here is another corollary.
Lemma 5.6. Let $f: X \rightarrow Y$ and $h: Z \rightarrow X$ where $h$ is a closed immersion with ideal $\mathscr{I}$. Consider $g=f \circ h: Z \rightarrow Y$. Then

$$
h^{*} \mathscr{I} \rightarrow h^{*} \Omega_{X / Y}^{1} \rightarrow \Omega_{Z / Y}^{1} \rightarrow 0
$$

is exact. Moreover, if $f$ is smooth then the first map is injective on the smooth locus of $g$.

Corollary 5.7. Let $f: X \rightarrow Y$ be flat, projective, lci of pure relative dimension $n$ (so that $\omega_{f}$ exists and is invertible). Suppose $X$ and $Y$ are regular. Let $U \subseteq X$ be the smooth locus. Then $\left.\omega_{f}\right|_{U} \cong \Omega_{U / Y}$.

Proof. Let us factor $f: X \rightarrow Y$ as $X \xrightarrow{v} \mathbb{P}_{Y}^{\ell} \xrightarrow{u} Y$ where $v$ is a regular closed immersion with ideal $\mathscr{I}$. We're in a situation where we can apply adjunction. Then

$$
\omega_{f} \cong \operatorname{deg}\left(v^{*} \mathscr{I}\right)^{\vee} \otimes_{\mathscr{O}_{X}} v^{*} \Omega_{u}^{\ell}
$$

The lemma is giving me that on the smooth locus, I have the exact sequence

$$
\left.\left.\left.0 \rightarrow v^{*} \mathscr{I}\right|_{U} \rightarrow v^{*} \Omega_{u}^{1}\right|_{U} \rightarrow \Omega_{f}^{1}\right|_{U} \rightarrow 0
$$

So if we take determinant, we get

$$
\left.\left.\left.\Omega_{f}^{n}\right|_{U} \otimes \operatorname{det}\left(v^{*} \mathscr{I}\right)\right|_{U} \cong v^{*} \Omega_{u}^{\ell}\right|_{U}
$$

They will then agree.

## 6 February 2, 2018

### 6.1 Surfaces with rational singularities

Let $S$ be a (Noetherian) scheme with $\operatorname{dim} S=2$, and assume $S$ is normal. Let $p \in S$ is a singular point. By normality, the singular point is going to be isolated.

Definition 6.1 (Artin). The point $p$ is a rational singularity if there is a strong desingularization (that is, blow-up only at $p$ ) $\pi: S^{\prime} \rightarrow S$ such that $R^{1} \pi_{*} \mathscr{O}_{S^{\prime}}=0$.

In this case, we can desingularize by blowing up, and it will be a chain of $(-2)$-curves. In our case, semi-stable arithmetic surfaces are of this type. So we can use the theory of Artin.

Lemma 6.2. Let $(\mathfrak{X}, \pi)$ be a semi-stable arithmetic surface over $S=\operatorname{Spec} A$ with smooth $X=\mathfrak{X}_{\eta}$. Let $\mathscr{L}$ be a line sheaf on $\mathfrak{X}$ and $i \geq 0$. Then the natural map

$$
H^{i}(\mathfrak{X}, \mathscr{L}) \rightarrow H^{i}\left(\mathfrak{X}^{\prime}, p^{*}, p^{*} \mathscr{L}\right)
$$

is an isomorphism. Here $p: \mathfrak{X}^{\prime} \rightarrow \mathfrak{X}$ is the minimal desingularization.
Proof. We have the map $\mathscr{O}_{\mathfrak{X}} \rightarrow p_{*} p^{*} \mathscr{O}_{\mathfrak{X}}$. This is an isomorphism because $\mathfrak{X}$ is normal and use Hartog's lemma. So for all $\mathscr{L}$, the natural $\mathscr{L} \rightarrow p_{*} p^{*} \mathscr{L}$ is an isomorphism because everything is local. Now it is enough to show that

$$
H^{i}\left(\mathfrak{X}, p_{*} \mathscr{F}\right) \cong H^{i}\left(\mathfrak{X}^{\prime}, \mathscr{F}\right)
$$

for $\mathscr{F}$ a line sheaf on $\mathfrak{X}^{\prime}$. This can be seen from the Leray spectral sequence.

### 6.2 Base-change trick

Let $L / K$ be a finite extension. (All residue fields are perfect and everything is Noetherian.) Let $B \subseteq L$ be the integral closure of $A \subseteq K$. Let $S=\operatorname{Spec} A$ and $T=\operatorname{Spec} B$. Let $\mathfrak{X} / S$ be a regular semi-stable arithmetic surface, and write $X=\mathfrak{X}_{\eta}$. Let $\mathfrak{Y}=\mathfrak{X} / T$ be the base-change. This is semi-stable, but there is no reason for it to be regular. Let $\mathfrak{Y}^{\prime} \rightarrow \mathfrak{Y}$ be the minimal desingularization.


We would like to relate the cohomology on $\mathfrak{X}$ to cohomology on $\mathfrak{Y}^{\prime}$.
Proposition 6.3. Let $\mathscr{L}$ be a line sheaf on $\mathfrak{X}$ and $i \geq 0$. Then there exists a canonical isomorphism

$$
H^{i}\left(\mathfrak{Y}^{\prime}, f^{*} \mathscr{L}\right) \cong H^{i}(\mathfrak{X}, \mathscr{L}) \otimes_{A} B
$$

Proof. First, $B$ is flat over $A$, so by flat base change,

$$
H^{i}(\mathfrak{X}, \mathscr{L}) \otimes_{A} B \cong H^{i}\left(\mathfrak{Y}, f^{*} \mathscr{L}\right)
$$

Then use the previous discussion.
Consider the relative dualizing sheaves $\omega_{\mathfrak{X} / S}$ and $\omega_{\mathfrak{Y}}{ }^{\prime} / S$. We don't know from last time that $\omega_{\mathfrak{Y} / T}$ exists, but pullbacks of relative dualizing sheaves are relative dualizing, and so we can define

$$
\omega_{\mathfrak{Y} / T}=f^{*} \omega_{\mathfrak{X} / S} .
$$

Now we want to compare between $\omega_{\mathfrak{Y}}{ }^{\prime} / T$ and $p^{*} \omega_{\mathfrak{Y} / T}=f^{*} \omega_{\mathfrak{X} / S}$. They differ only where the surfaces differ.

Proposition 6.4. We have a canonical isomorphism $\omega_{\mathfrak{Y}^{\prime} / T} \cong f^{*} \omega_{\mathfrak{X} / S}$.
Proof. Let $U \subseteq \mathfrak{X}$ be the smooth locus of $\mathfrak{X} / S$. Then $\operatorname{codim} U^{c}=2$. Let $V \subseteq \mathfrak{Y}_{T}$ be the smooth locus and $V^{\prime} \subseteq \mathfrak{Y}^{\prime} / T$ be the smooth locus. By projectiveness, we have canonical isomorphisms

$$
\left.\omega_{\mathfrak{X} / S}\right|_{U} \cong \Omega_{U / S}^{1}, \quad \omega_{\mathfrak{Y}} /\left.T\right|_{V^{\prime}} \cong \omega_{V^{\prime} / T}
$$

Because $q^{*} \Omega_{\mathfrak{X} / S}^{1} \cong \Omega_{\mathfrak{Y} / T}^{1}$, we can look at their smooth locus and get $q^{-1}(U)=$ $V$. Then we have $\omega_{\mathfrak{Y}} /\left.T\right|_{V} \cong \Omega_{V / T}^{1}$ by restricted to $V$. Then generally, we get

$$
\omega_{\mathfrak{Y}} / T T \cong p^{*} \omega_{\mathfrak{Y} / T} \otimes \mathscr{O}_{\mathfrak{Y}^{\prime} / T} \otimes \mathscr{O}_{\mathfrak{Y}^{\prime}}(D)
$$

where $D$ is a divisor supported on the $p$-contracted ( -2 )-curves. Now I claim that $D=0$, and the reason this holds is adjunction. At this point, it is okay to assume that $B$ is a DVR. It is enough to show that $D^{2}=0$, because then $D$ is a rational multiple of the fiber, but $D$ does not surject onto $\mathfrak{Y}_{s}$.

It is enough to show that for all $C$ irreducible components of the support of $D$, we have $C \cdot D=0$. But we can compute

$$
\begin{aligned}
C \cdot D & =\left.\operatorname{deg}_{C / k}\left(p^{*} \omega_{\mathfrak{Y} / T} \otimes \mathscr{O}_{\mathfrak{Y}^{\prime}}(D)\right)\right|_{C} \\
& =\left.\operatorname{deg}_{C / k}\left(\omega_{\mathfrak{Y}} / T\right)\right|_{C}=-C^{2}+\left(2 g_{C}-2\right)=2-2=0
\end{aligned}
$$

because $C$ is a $(-2)$-curve, which is a $\mathbb{P}^{1}$.

## 7 February 5, 2018

Today, we are going to do some analytic preliminaries. There are some analogies with Riemann surfaces.

| Arithmetic surfaces | Curves over $\mathbb{C}$ |
| :---: | :---: |
| "good" integral models | suitable metrics on line sheaves |
| geometric intersection numbers | Green function $g(P, Q)$ |

Table 1: Analogies between arithmetic surfaces and Riemann surfaces

### 7.1 Metrized line sheaves

Let $X / \mathbb{C}$ be a complex projective manifold of dimension $n$. Let $\mathscr{L}$ be an invertible sheaf on $X$.

Definition 7.1. A smooth metric on $\mathscr{L}$ is a morphism of sheaves of sets $\mathscr{L} \rightarrow \operatorname{Cont}\left(X, \mathbb{R}_{\geq 0}\right)$ such that
(0) For every local generating section $s \in \mathscr{L}(U)$, the map $\|s\|: U \rightarrow \mathbb{R}_{\geq 0}$ is smooth,
(1) given $f \in \mathscr{O}_{X}(U)$ and $\phi \in \mathscr{L}(U)$, we have $\|f \cdot \phi\|=|f|\|\phi\|$,
(2) $\|\phi\|(p)=0$ if and only if $\phi(p)=0$ in $\left.\mathscr{L}\right|_{p}=\mathscr{L} \otimes k_{p}$.

Example 7.2. If $X=\mathbb{P}^{n}$ and $\mathscr{L}=\mathscr{O}_{X}(1)$, there is a global section. The global section

$$
s=\alpha_{0} x_{0}+\cdots+\alpha_{n} x_{n}
$$

generates this away from the zero section. Then

$$
\|s\|_{\mathrm{FS}}=\frac{\left|\alpha_{0} x_{0}+\cdots+\alpha_{n} x_{n}\right|}{\sqrt{\left|x_{0}\right|^{2}+\cdots+\left|x_{n}\right|^{2}}}
$$

gives a norm. This generates the line sheaf as an $\mathscr{O}_{X}$-module, so it determines the whole metric.

Let us write $\overline{\mathscr{L}}=(\mathscr{L},\|-\|)$.

- We can tensor them and get define $\overline{\mathscr{L}}_{1} \otimes \overline{\mathscr{L}}_{2}$. If $s_{1}$ and $s_{2}$ are locally generating sections, we can define

$$
\left\|s_{1} \otimes s_{2}\right\|=\left\|s_{1}\right\|\left\|s_{2}\right\|
$$

- We can dualize them by $\overline{\mathscr{L}}^{\vee}=\mathscr{H} \operatorname{om}\left(\overline{\mathscr{L}}, \overline{\mathscr{O}}_{X}\right)$ and for $p \in X$ and $\phi$ a section of $\mathscr{L}^{\vee}$ near $p$, define

$$
\|\phi\|(p)=\sup _{\|s\|(p)=1}|\phi(s)|(p)
$$

Then if we defined things correctly, we should be able to check that $\overline{\mathscr{L}} \otimes \overline{\mathscr{L}} \cong$ $\overline{\mathscr{O}}_{X}$. We can check this by using the fact that sup is not really useful.

Lemma 7.3. Every line sheaf on $X$ admits a smooth metric.
Proof. If $X$ is projective, we can put metrics on very ample line bundles by pulling back the Fubini-Study metric. But any line bundle is a very ample line bundle tensored with a dual of a very ample line bundle.

### 7.2 Differential operators

For $U \subseteq \mathbb{C}$ a domain, there are operators

$$
\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i \frac{\partial}{\partial y}\right)
$$

on $C^{\infty}(\Omega, \mathbb{C})$. These are defined so that $\frac{\partial}{\partial z}=1, \frac{\partial}{\partial \bar{z}} z=0$, and so on. We then define

$$
\begin{aligned}
\partial f & =\frac{\partial f}{\partial z} d z, \quad \bar{\partial} f=\frac{\partial f}{\partial \bar{z}} d \bar{z} \\
4 \partial \bar{\partial} f & =2 \partial\left(\left(f_{x}+i f_{y}\right) d \bar{z}\right) \\
& =\left(\left(f_{x x}+i f_{y x}\right)-i\left(f_{x y}+i f_{y x}\right)\right) d z \wedge d \bar{z}=\left(f_{x x}+f_{y y}\right) d z \wedge d \bar{z}
\end{aligned}
$$

Let $X$ be a compact Riemann surface. This means that $X / \mathbb{C}$ is a projective curve. Take $\overline{\mathscr{L}}$ on $X$.

Definition 7.4. The curvature is

$$
\operatorname{curv}(\overline{\mathscr{L}})=\partial \bar{\partial} \log \|s\|^{2}
$$

on $U \subseteq X$ with $s$ a locally generating on $U$.
Explicitly, say $s$ is a local generator on $U$, and $z$ be a local chart on $U$. Then

$$
\left.\operatorname{curv}(\overline{\mathscr{L}})\right|_{U}=\frac{\partial^{2} \log \|s\|^{2}}{\partial z \partial \bar{z}} d z \wedge d \bar{z}
$$

If we choose another section $t=f s$ with $f \in \mathscr{O}_{X}^{\times}(U)$, we need to check that

$$
\frac{\partial^{2}}{\partial z \partial \bar{z}} \log |f|^{2}=0
$$

on $U$. But the thing is that $f=f \cdot \bar{f}$, and so $\log f$ is just a sum of some holomorphic and anti-holomorphic. But they are both killed by $\partial$ and $\bar{\partial}$. So $\operatorname{curv}(\overline{\mathscr{L}})$ is a $C^{\infty}(1,1)$-form on $X$.
Lemma 7.5. $\int_{X} \operatorname{curv}(\overline{\mathscr{L}})=2 \pi i \operatorname{deg}(\mathscr{L})$.
We will check this next time.

## 8 February 7, 2018

Today we are going to continue computing curvature. Recall:
Theorem 8.1 (Stokes's theorem). Let $U \subseteq \mathbb{C}$ be a domain (with nice boundary) and $\Omega \supseteq \bar{U}$ a domain. Let $f \in C^{\infty}(\Omega, \mathbb{C})$. Then

$$
\int_{U} \partial(f d \bar{z})=\int_{\partial U} f d \bar{z}, \quad \int_{U} \bar{\partial}(f d z)=\int_{\partial U} f d z
$$

Lemma 8.2. Let $X$ be a compact Riemann surface, and let $\overline{\mathscr{L}}=(\mathscr{L},\|-\|)$ be a metrized line bundle. Then

$$
\int_{X}(2 \pi i)^{-1} \operatorname{curv}(\overline{\mathscr{L}})=\operatorname{deg} \mathscr{L}
$$

Proof. Let $s$ be a meromorphic section. Let $D$ be the divisor, and write $D=$ $\sum_{p} v_{p, \mathscr{L}}(s) \cdot p$. Let $\epsilon>0$ be small and consider closed balls $B(p, \epsilon)$ that do not overlap for $p \in \operatorname{supp} D$. Let $U=X-\bigcup_{p} B(p, \epsilon)$ be an analytic open set. Then

$$
\left.\operatorname{curv}(\mathscr{L})\right|_{U}=\partial \bar{\partial} \log \|s\|^{2}
$$

and so

$$
\lim _{\epsilon \rightarrow 0} \int_{U(s, \epsilon)} \operatorname{curv}(\overline{\mathscr{L}})=\int_{X} \operatorname{curv}(\overline{\mathscr{L}})
$$

because the curvature is actually smooth. By Stokes's theorem,

$$
\int_{U(s, \epsilon)} \operatorname{curv}(\overline{\mathscr{L}})=-\sum_{p \in \operatorname{supp}(D)} \int_{\partial B(p, \epsilon)} \bar{\partial}\|s\|^{2}
$$

Now the claim is that

$$
\lim _{\epsilon \rightarrow 0} \int_{\partial B(p, \epsilon)} \bar{\partial} \log \|s\|^{2}=-2 \pi i v_{p, \mathscr{L}}(s) .
$$

But locally on $p$, we can write $s=f t$ for $f \in \mathscr{O}_{X}^{\times}$on $B(p, \epsilon) \backslash\{p\}$ and $t$ a locally generating section. Then

$$
\lim _{\partial B(p, \epsilon)} \bar{\partial} \log \|f t\|^{2}=\int \bar{\partial} \log (f \bar{f})
$$

because $\bar{\partial} \log \|t\|^{2}$ is $C^{\infty}$. Then $\log f$ vanishes after $\bar{\partial}$, and then use Cauchy or something.

For the purpose of Arakelov theory, you really need to get the normalization right.

Example 8.3. Take $X=\mathbb{P}^{1}$ and $\mathscr{L}=\mathscr{O}(1)$ with the Fubini-Study metric. We use the chart $z=\frac{x_{1}}{x_{0}}$ and the section $s=x_{0}$. Then

$$
\|s\|^{2}=\frac{\left|x_{0}\right|^{2}}{\left|x_{0}\right|^{2}+\left|x_{1}\right|^{2}}=\frac{1}{1+|z|^{2}}=\frac{1}{1+z \bar{z}}
$$

Curvature is then

$$
\operatorname{curv}(\overline{\mathscr{L}})=\partial \bar{\partial} \log \|s\|^{2}=\frac{\partial^{2}}{\partial z \partial \bar{z}} \log \frac{1}{1+z \bar{z}} d z \wedge d \bar{z}=-\frac{1}{\left(1+|z|^{2}\right)^{2}} d z \wedge d \bar{z}
$$

( $d z \wedge d \bar{z}=-2 i d x \wedge d y$ has the negative orientation for some traditional reason.)
Then

$$
\int_{X} \operatorname{curv}(\overline{\mathscr{L}})=\int_{U} \frac{-1}{\left(1+|z|^{2}\right)^{2}} d z \wedge d \bar{z}
$$

If we switch to $z=r e^{2 \pi i \theta}$, we get $2 \pi i$.
In Arakelov theory, we want only to work with metrics with curvature a constant times a certain fixed $(1,1)$-form. But there is the question of existence, and we can first construct a nonvanishing curvature metric.

### 8.1 Curvature on projective manifolds

Let $X$ be a projective manifold, and $\overline{\mathscr{L}}=(\mathscr{L},\|-\|)$ be a metrized line bundle. We similarly define curvature locally. Let $z=\left(z_{1}, \ldots, z_{n}\right)$ be a chart, and let us use a generating section $s \in \mathscr{L}(U)$. Then we can properly define

$$
\left.\operatorname{curv}(\overline{\mathscr{L}})\right|_{U}=\sum_{j=1}^{n} \sum_{k=1}^{n} \partial_{j} \bar{\partial}_{k} \log \|s\|^{2}
$$

This is well-defined and $C^{\infty}$ by the same argument.
Lemma 8.4. We have $\operatorname{curv}\left(\overline{\mathscr{L}}_{1} \otimes \overline{\mathscr{L}}_{2}\right)=\operatorname{curv}\left(\overline{\mathscr{L}}_{1}\right)+\operatorname{curv}\left(\overline{\mathscr{L}}_{2}\right)$ and same for duals.

Definition 8.5. A smooth (1, 1)-form is self-adjoint if it locally can be written as

$$
\omega=\frac{i}{2} \sum_{j, k} f_{j k} d z_{j} \wedge d \bar{z}_{k}
$$

with the matrix $\left[f_{j k}\right]$ self-adjoint pointwise. We say that $\omega$ is positive if is positive-definite.

For $\operatorname{dim} X=1$, we have $\frac{i}{2} d z \wedge d \bar{z}=f d x \wedge d y$, so this really the right thing. Also, note that the choice of a coordinate does not matter.

Definition 8.6. For $\overline{\mathscr{L}}$, we say that it is positive if $(2 \pi i)^{-1} \operatorname{curv}(\bar{L})$ is positive.
Proposition 8.7. On $\mathbb{P}^{n},\left(\mathscr{O}(1),\|-\|_{F S}\right)$ is positive.

Proof. We can write down the metric and the curvature. This is going to be

$$
(2 \pi i)^{-1} \operatorname{curv}(\overline{\mathscr{L}})=\frac{i}{2}\left(\sum_{j=1}^{n} \frac{\pi^{-1}}{1+|z|^{2}} d z_{j} \wedge d \bar{z}_{j}-\sum_{j=1}^{n} \sum_{j=1}^{n} \frac{\pi^{-1} \bar{z}_{j} z_{k}}{\left(1+|z|^{2}\right)^{2}} d z_{j} \wedge d \bar{z}_{k}\right)
$$

and you can check that it is positive.

## 9 February 9, 2018

Last time I tried to convince you that the Fubini-Study metric is positive. We computed it on the open set $U=\left\{x_{0} \neq 0\right\}$ and the coordinate $z=\left(z_{j}\right)$ with $z_{j}=x_{j} / x_{0}$. The claim is that $(2 \pi i)^{-1} \operatorname{curv}\left(\mathscr{O}(1),\|-\|_{F S}\right)$ is positive. The metric is

$$
\|s\|^{2}=\frac{1}{1+\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}}=\frac{1}{1+|z|^{2}}
$$

Then we can compute the curvature

$$
\begin{aligned}
\partial \bar{\partial}\|s\|^{2} & =\sum_{j, k} \frac{\partial^{2}}{\partial z_{j} \partial \bar{z}_{k}}\|s\|^{2} d z_{j} \wedge d \bar{z}_{k} \\
& =\sum_{j, k}\left(\frac{-\delta_{j k}}{1+|x|^{2}}+\frac{\bar{z}_{j} z_{k}}{\left(1+|z|^{2}\right)^{2}}\right) d z_{j} \wedge d \bar{z}_{k}
\end{aligned}
$$

We need to check that

$$
\sum_{j=1}^{n} \frac{\pi^{-1}}{1+|z|^{2}} d z_{j} \wedge d \bar{z}_{j}-\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\pi^{-1} \bar{z}_{j} z_{k}}{\left(1+|z|^{2}\right)^{2}} d z_{j} \wedge d \bar{z}_{k}
$$

is positive definite. We can check this using Cauchy-Schwartz.

### 9.1 Constructing positive metrics

Lemma 9.1. Let $f: X \rightarrow Y$ be a regular closed immersion of projective complex manifolds. Let $\overline{\mathscr{L}}$ be a positive metrized line sheaf. Then $f^{*} \overline{\mathscr{L}}$ is also positive.

Proof. Take $z=\left(z_{1}, \ldots, z_{n}\right)$ be a local holomorphic chart at $p \in X$. Then the claim is that there exists a local holomorphic chart $w=\left(w_{1}, \ldots, w_{l}\right)$ at $q=f(p)$ such that $w$ extends $z$. That is, for $j \leq m, z_{j}=f^{*} w$. Now the claim follows the fact that principal minors of positive definite matrices are positive definite.

Theorem 9.2. Let $X$ be a projective complex manifold, and let $\mathscr{L}$ be an ample line sheaf. Then there exists a metric $\|-\|$ on $\mathscr{L}$ which is positive.

Proof. If it is very ample, you can embed into projective space and then you can pull it back. For ample $\mathscr{L}$, take $r \geq 1$ such that $\mathscr{L}^{\otimes r}$ such that $\mathscr{L}$ is very ample. Take $\|-\|$ positive on $\mathscr{L}^{\otimes r}$ and define $\|-\|_{0}$ on $\mathscr{L}$ locally as

$$
\|s\|_{0}=\|s \otimes \cdots \otimes s\|^{1 / r}
$$

We need to check that this is a metric. It is positive because $\operatorname{curv}\left(\mathscr{L},\|-\|_{0}\right)=$ $\frac{1}{r} \operatorname{curv}\left(\mathscr{L}^{\otimes r},|-|\right)$.

Actually, the converse is true as well.
Theorem 9.3 (Kodaira). Let $\mathscr{L}$ be a line sheaf on a projective complex manifold. Suppose $\mathscr{L}$ admits a positive metric. Then $\mathscr{L}$ is ample.

### 9.2 Poisson equation

This is in general the name of the Laplace equation $\Delta f=\phi$. We can take $\phi$ to be a distribution and ask $f$ to be a distribution. We can take $\phi$ to be a current, which is like a $(1,1)$-form with distributional coefficients.

Lemma 9.4 (Weyl). Let $X$ be a compact Riemann surface. Let $\omega$ be a smooth $(1,1)$-form on $X$. Suppose $\partial \bar{\partial} f=\omega$ has a weak solution. Then it $f$ is in fact represented by a smooth function.

This is an example of elliptic regularity. In our context, $X$ is a compact Riemann surface and $\omega$ is a smooth $(1,1)$-form. We want to solve $\partial \bar{\partial} f=\omega$ where $f \in C^{\infty}(X, \mathbb{C})$ because it is going to be smooth if it has any meaning.

If $f$ is real-valued, then $\partial \bar{\partial} f$ is $i$ times real-valued. Also, $\partial \bar{\partial} f=0$, then $f$ is constant because locally $f$ should be harmonic.

Also, if $\partial \bar{\partial} f=\omega$ has a solution, then $\int_{X} \omega=0$. This is because

$$
\int_{X} \omega=\int_{X} \partial \bar{\partial} f=\int_{\partial X} \bar{\partial} f=0
$$

## 10 February 12, 2018

Last time we were looking at the Poisson equation

$$
\partial \bar{\partial} f=\phi
$$

on a compact Riemann surface. For us $\phi$ is a smooth $(1,1)$-form, because Weyl's lemma tells us that if there is a solution $f$ in the weak sense then it is represented by $C^{\infty}(X, \mathbb{C})$.

If there exists a solution $f$, then

- $f$ is unique up to adding a constant, because the difference is going to be locally the solution of the Laplacian.
- if there exists a $C^{\infty}$ solution $f$, then

$$
\int_{X} \phi=0
$$

by Stokes's theorem.
Theorem 10.1. Let $\omega$ be a smooth $(1,1)$-form on a compact Riemann surface $X$. Then there is a smooth $f \in C^{\infty}(X, \mathbb{C})$ such that it solves the equation $\partial \bar{\partial} f=\omega$, provided that $\int_{X} \omega=0$.

Locally, you can solving it by convoluting with the fundamental solution. But the question is whether you can glue them.

Proof. We solve this for $f \in L^{2}$. The key step is to decompose 1-forms by "orthogonal projection" into $f d z$ and $g d \bar{z}$ and a harmonic part. Then elliptic regularity will tell you that it is smooth.

What is interesting is that this shows that it can be expressed by integration against a kernel.

### 10.1 Admissible metrics

Let $X$ be a compact Riemann surface.
Definition 10.2. A smooth $(1,1)$-form $\omega$ is a volume form if it is positive.
To each volume form we can associate a measure

$$
\mu(A)=\int_{A} \omega
$$

Since $\omega$ is positive, this is strictly positive for $m(A)>0$. This means that locally $\left.\omega\right|_{U}=f d x \wedge d y=\frac{i}{2} f d z \wedge d \bar{z}$ where $f$ is positive valued. For us, we are going to normalize this. We say that $\omega$ is a probability volume form if $\mu(X)=\int_{X} \omega=1$.

Definition 10.3. Fix a probability volume form $\omega$ on $X$. Let $\overline{\mathscr{L}}=(\mathscr{L},\|-\|)$ be a metrized line sheaf. We say that $\|-\|$ on $\overline{\mathscr{L}}$ is $\omega$-admissible if there exists a $\lambda \in \mathbb{C}$ such that

$$
(2 \pi i)^{-1} \operatorname{curv}(\overline{\mathscr{L}})=\lambda \omega .
$$

Note that necessarily $\lambda=\operatorname{deg} \mathscr{L}$.
Question. Given a probability volume form $\omega$ and $\mathscr{L}$, does there exists a $\|-\|$ on $\mathscr{L}$ that is $\omega$-admissible? Is it unique?

Proposition 10.4. Given a probability volume form $\omega$ on $X$, let $\overline{\mathscr{L}}=(\mathscr{L},\|-\|)$ be $\omega$-admissible. Let $\|-\|_{0}$ be another $\omega$-admissible metric on $\mathscr{L}$. Then there exists an $c>0$ such that $\|-\|_{0}=c\|-\|$.

Proof. Let $U, z, s$ be local data for $\mathscr{L}$. Then

$$
\left.\partial \bar{\partial} \log \|s\|^{2}\right|_{U}=\left.(2 \pi i \operatorname{deg} \mathscr{L}) \omega\right|_{U}=\left.\partial \bar{\partial} \log \|s\|_{0}^{2}\right|_{U}
$$

and so

$$
\left.\partial \bar{\partial} \log \frac{\|s\|^{2}}{\|s\|_{0}^{2}}\right|_{U}=0
$$

But $\varphi=\|s\|^{2} /\|s\|_{0}^{2}$ extends to a global $C^{\infty}$-function, which is everywhere harmonic, and hence constant.

So for instance, on the structure sheaf, the only $\omega$-admissible metrics are constant times the absolute value.

Theorem 10.5. Given $\omega$ a probability volume form on $X$, and $\mathscr{L}$ a line sheaf on $X$, there is a metric on $\mathscr{L}$ which is $\omega$-admissible.

Proof. It suffices to assume that $\mathscr{L}$ is very ample. This is because curvature is additive in tensor products and duals. So let us deal with the very ample case.

Let $\mathscr{L}$ be very ample. Let $\|-\|_{0}$ be a psotive metric. Consider

$$
\xi=(2 \pi i \operatorname{deg} \mathscr{L})^{-1} \operatorname{curv}\left(\mathscr{L},\|-\|_{0}\right)
$$

This is a probability volume form. Now $\omega-\xi$ is a smooth $(1,1)$-form, and satisfies

$$
\int_{X}(\omega-\xi)=1-1=0
$$

So we can solve the Poisson equation

$$
\partial \bar{\partial} \phi=(2 \pi i \operatorname{deg} \mathscr{L})^{-1}(\omega-\xi)
$$

This solution $\phi$ is smooth, and unique up to adding a constant.
If $f$ is some function $f \in C^{\infty}(X, \mathbb{R})$, then $\partial \bar{\partial} f=i \cdot$ (real). This shows that $\Im(\phi)$ is a constant, so that we can throw it away. Then we get a real-valued solution $f \in C^{\infty}(X, \mathbb{R})$ with

$$
\partial \bar{\partial} f=(2 \pi i \operatorname{deg} \mathscr{L})(\omega-\xi)
$$

$$
\begin{aligned}
& \text { Let } \gamma=e^{f / 2} \text { and let } \\
& \qquad\|-\|=\gamma \cdot\|-\|_{0} .
\end{aligned}
$$

This is a metric, and its curvature is, locally,

$$
\partial \bar{\partial} \log \left(\|s\|^{2}\right)=\partial \bar{\partial} \log \left(\gamma^{2}\|s\|_{0}^{2}\right)=\partial \bar{\partial} f+(2 \pi i \operatorname{deg} \mathscr{L}) \xi=(2 \pi i \operatorname{deg} \mathscr{L}) \omega
$$

So one can construct admissible metrics.
Let $f \in C^{\infty}(\Omega, \mathbb{C})$, and consider $\overline{B(0, \epsilon)} \subseteq \Omega$. Then

$$
\int_{B(0, \epsilon)} f(z) \log |z|^{2} d z \wedge d \bar{z}
$$

converges absolutely.
With this, the following makes sense. Let $D$ be a divisor on $X$ and $\omega$ be a probability volume form.

Proposition 10.6. There is a unique "canonical" metric on $\mathscr{O}_{X}(D)$ which is $\omega$-admissible.

First, there is a canonical rational section for $\mathscr{O}_{X}(D)$. This is because $\mathscr{O}(D) \subseteq \mathscr{K}$ and there is the constant section 1 . Let $\|-\|$ be an admissible metric on $\mathscr{O}(D)$ satisfying

$$
\int_{X} \log \|\mathbf{1}\| d \mu=0
$$

This $\|-\|$ exists and is unique. This is going to be used to construct the Green's function.

## 11 February 14, 2018

Admissible metrics are important because that is how we define intersection numbers.

### 11.1 Global equations for points

Fix $X$ a compact Riemann surface and $\omega$ a probability volume form on $X$. Recall that there is a unique admissible metric $\|-\|_{D}$ on $\mathscr{O}_{X}(D)$ for any divisor $D$ on $X$ such that

$$
\int_{X} \log \|1\|_{D}(P) d \mu(P)=0
$$

This 1 is going to be the section with $\operatorname{div}(1)=D$.
Let $P \in X$ be a point, and define

$$
\varphi_{P}: X \rightarrow \mathbb{R} ; \quad \varphi_{P}(Q)=\|1\|_{P}(Q)
$$

Because 1 is locally generating away from $P$, it is smooth away from $P$.
Proposition 11.1. Let $P \in X$ and let $f: X \rightarrow \mathbb{C}$ be a function. We have $f=\varphi_{P}$ if and only if the following holds:
(1) $f$ is smooth and positive away form $P$,
(2) $f f$ has a simple zero at $P$ (this means that if $z$ is a local holomorphic chart near $P$ with $z(P)=0$ then there exists a $u$ smooth near $P$ such that locally $f=|z| u$ and $u(p) \neq 0)$,
(3) on $U=X \backslash p$, we have $\left.(2 \pi i)^{-1} \partial \bar{\partial} \log f^{2}\right|_{U}=\left.\omega\right|_{U}$,
(4) $\int_{X}(\log f) d \mu=0$.

Proof. First it is clear that $\varphi_{P}$ satisfies (1)-(4). For the other direction, take any other $f$ and divide by $\varphi_{P}$. Then this is smooth nonzero harmonic and so can be shown to be 1 .

Definition 11.2. $\varphi_{P}$ is called the global equation for $P \in X$.

### 11.2 Green function

Definition 11.3. We define the Green function as

$$
G: X \times X \rightarrow \mathbb{R} ; \quad(P, Q) \mapsto \varphi_{P}(Q)
$$

Since $\varphi_{P}$ has an axiomatic characterization, so does $G$.
Lemma 11.4. Let $\Omega \subseteq \mathbb{C}$ be a domain with $0 \in \Omega$. Let $\epsilon_{0}>0$ be such that $B\left(0, \epsilon_{0}\right) \subseteq \Omega$. Let $f \in C^{0}(\Omega, \mathbb{C})$. As $\epsilon \rightarrow 0$ with $0<\epsilon<\epsilon_{0}$, we have

$$
\text { (1) } \lim _{\epsilon \rightarrow 0} \int_{\partial B(0, \epsilon)}(\log |z|) f(z) d z=0 \text { and }
$$

(2) $\lim _{\epsilon \rightarrow 0} \int_{\partial B(0, \epsilon)} f(z) \partial \log |z|^{2}=2 \pi i f(0)$.

Proof. Parametrize the circle and just compute.
Theorem 11.5. $G$ is symmetric.
Proof. Let $P \neq Q$ in $X$. Let $\epsilon>0$ be small so that both $B(P, \epsilon)$ and $B(Q, \epsilon)$ do not meet. Consider the open set $U_{\epsilon}=X-(B(P, \epsilon) \cup B(Q, \epsilon))$, and compute

$$
\begin{aligned}
I_{\epsilon} & =\int_{U_{\epsilon}}\left(\log \varphi_{P} \partial \bar{\partial} \log \varphi_{Q}-\log \varphi_{Q} \partial \bar{\partial} \log \varphi_{P}\right) \\
& =\pi i \int_{U_{\epsilon}}\left(\log \varphi_{P} \cdot \omega-\log \varphi_{Q} \cdot \omega\right) \rightarrow 0
\end{aligned}
$$

as $\epsilon \rightarrow 0$, because logarithmic singularities don't contribute much.
Now note that

$$
\begin{aligned}
\partial\left(\log \varphi_{P} \bar{\partial} \log \varphi_{Q}\right) & =\partial \log \varphi_{P} \wedge \bar{\partial}+\log \varphi_{P} \partial \bar{\partial} \log \varphi_{Q} \\
\bar{\partial}\left(\log \varphi_{Q} \partial \log \varphi_{P}\right) & =\bar{\partial} \log \varphi_{Q} \wedge \partial \log \varphi_{P}+\log \varphi_{Q} \bar{\partial} \partial \log \varphi_{P} \\
& =-\partial \log \varphi_{P} \wedge \bar{\partial} \log \varphi_{Q}-\log \varphi_{Q} \partial \bar{\partial} \log \varphi_{P}
\end{aligned}
$$

So we can use Stokes to write

$$
\begin{aligned}
I_{\epsilon} & =-\int_{\partial B(P, \epsilon)+\partial B(Q, \epsilon)}\left(\log \varphi_{P} \bar{\partial} \log \varphi_{Q}+\log \varphi_{Q} \partial \log \varphi_{P}\right) \\
& =(\rightarrow 0)-\frac{1}{2} \int_{\partial B(P, \epsilon 0} \log \varphi_{Q} \partial \log \varphi_{P}^{2}-\frac{1}{2} \int_{\partial B(Q, \epsilon)} \log \varphi_{P} \bar{\partial} \log \varphi_{Q}^{2} \\
& \rightarrow-\frac{2 \pi i}{2}\left(\log \varphi_{Q}(P)-\log \varphi_{P}(Q)\right)
\end{aligned}
$$

This finishes the proof.
Since $\omega$ is a volume form, it is positive. This means that locally $\omega=\frac{i}{2} f d z \wedge d \bar{z}$ where $f>0$. Therefore the next construction works. Let $\phi \in C^{\infty}(X, \mathbb{C})$ be a smooth function, and write

$$
(\pi i)^{-1} \partial \bar{\partial}=(\Delta \phi) \omega
$$

This is the definition of

$$
\Delta: C^{\infty}(X, \mathbb{C}) \rightarrow C^{\infty}(X, \mathbb{C})
$$

which is called the Laplacian.

## 12 February 16, 2018

Today we invert the Laplacian. We have the Laplacian

$$
\Delta: C^{\infty}(X, \mathbb{C}) \rightarrow C^{\infty}(X, \mathbb{C})
$$

with respect to $\omega$, given by the equation

$$
(\pi i)^{-1} \partial \bar{\partial} f=(\Delta f) \omega
$$

Theorem 12.1. Let $\psi \in C^{\infty}(X, \mathbb{C})$. The equation $\Delta \phi=\psi$ has a smooth solution if and only if

$$
\int_{X} \psi \omega=0
$$

Furthermore, the solution is unique up to additive constant in this case. In particular, $\phi$ is unique if we additionally require $\int_{X} \psi \omega=0$.

Proof. We already know this.
Corollary 12.2. Let $C^{\infty}(X, \mathbb{C})^{0}=C^{\infty}(X, \mathbb{C})_{\omega}^{0}$ be the space of all $f \in C^{\infty}(X, \mathbb{C})$ with $\int_{X} f \omega=0$. Then $\Delta$ restricts to a linear operator $\Delta: C^{\infty}(X, \mathbb{C})^{0} \rightarrow$ $C^{\infty}(X, \mathbb{C})^{0}$, which is a linear bijection.

### 12.1 Inverting the Laplacian

We define $g=\log G: X \times X \backslash \Delta \rightarrow \mathbb{R}$. Now I can define the following operator $\Gamma$ pointwise. For $\psi \in C^{\infty}(X, \mathbb{C})$ and a point $P \in X$, we define

$$
(\Gamma \psi)(P)=\int_{X}(-g(P,-)) \psi \omega \in \mathbb{C}
$$

So we have $\Gamma \psi: X \rightarrow \mathbb{C}$.
Theorem 12.3. $\Gamma$ restricts to a linear bijection $\Gamma: C^{\infty}(X, \mathbb{C})^{0} \rightarrow C^{\infty}(X, \mathbb{C})^{0}$ and it is the inverse of $\Delta$ on this space.

Proof. Because $\Delta$ is bijective, it suffices to check that $\Gamma(\Delta f)=f$ for all $f \in$ $C^{\infty}(X, \mathbb{C})^{0}$. Let $P \in X$, and let us check this pointwise. First note that

$$
\partial(g(P,-) \bar{\partial} f)=\partial g(P,-) \wedge \bar{\partial} f+g(P,-) \partial \bar{\partial} f
$$

Then

$$
\begin{aligned}
(\Gamma \Delta f)(P) & =\int_{X}-g(P,-)(\Delta f) \omega=\int_{X}-g(P,-)(\pi i)^{-1} \partial \bar{\partial} f=-\frac{1}{\pi i} \int_{X} g(P,-) \partial \bar{\partial} f \\
& =\frac{1}{\pi i} \int_{X} \partial g(P,-) \wedge \bar{\partial} f
\end{aligned}
$$

by Stokes and the lemma we had last time.

Also, we have

$$
\begin{aligned}
\bar{\partial}(f \partial g(P,-)) & =\bar{\partial} f \wedge \partial g(P,-)+f \bar{\partial} \partial g(P,-) \\
& =-\partial g(P,-) \wedge \bar{\partial} f-f \partial \bar{\partial} g(P,-)=-\partial g(P,-) \wedge \bar{\partial} f-f \frac{1}{2} 2 \pi i \omega
\end{aligned}
$$

So if we let $U_{\epsilon}=X \backslash B(p, \epsilon)$ then

$$
-\int_{\partial B(p, \epsilon)} f \partial g(P,-)=-\int_{U_{\epsilon}} \partial g(P,-) \wedge \bar{\partial}-\frac{1}{2} \int_{U_{\epsilon}} f \omega .
$$

So as $\epsilon \rightarrow 0$, we get

$$
-\frac{1}{2} 2 \pi i f(P)=-\pi i(\Gamma \Delta f)(P)
$$

The spectral theory of the Laplacian will play a role later.

### 12.2 Arakelov's theory

Let $K$ be a number field and $\mathcal{O}_{K}$ be the ring of integers, and $S=\operatorname{Spec} \mathcal{O}_{K}$. Take $\mathfrak{X}$ a regular semi-stable arithmetic surface. Let $X=\mathfrak{X}_{\eta}$ the fiber, and assume that it is geometrically irreducible. For all $\sigma: K \rightarrow \mathbb{C}$, let $X_{\sigma}=X \otimes \mathbb{C}$ and look at it as a compact Riemann surface. For each $\sigma$, let $\omega_{\sigma}$ be a fixed choice of a probability volume form on $X_{\sigma}$, and $\mu_{\sigma}$ be the measure associated to $\omega_{\sigma}$, so that integration against $d \mu_{\sigma}$ is integration against $\omega_{\sigma}$. Let us package this data into

$$
\hat{\mathfrak{X}}=\left(\mathfrak{X}, \pi,\left\{d \mu_{\sigma}\right\}_{\sigma}\right) .
$$

Definition 12.4. An Arakelov divisor is a divisor $D=D_{\text {fin }}+D_{\infty}$ where $D_{\text {fin }} \in \operatorname{Div}(\mathfrak{X})$ and $D_{\infty}=\sum_{\sigma: K \rightarrow \mathbb{C}} \alpha_{\sigma} F_{\sigma}$ (where $F_{\sigma}$ are formal symbols), and the set of divisors is denoted $\operatorname{Div}(\mathfrak{X})$.

Let $f \in K(\mathfrak{X})^{\times}$and we can define

$$
\widehat{\operatorname{div}}(f)=\operatorname{div}(f)+\sum_{\sigma: K \rightarrow \mathbb{C}} v_{F_{\sigma}}(f) F_{\sigma}
$$

where

$$
v_{F_{\sigma}}(f)=-\int_{X_{\sigma}} \log |f| \cdot d \mu_{\sigma}
$$

These are the principal divisors $\operatorname{PDiv}(\mathfrak{X})$. Then we can also define

$$
\mathrm{Cl}(\hat{\mathfrak{X}})=\frac{\operatorname{Div}(\hat{\mathfrak{X}})}{\operatorname{PDiv}(\hat{\mathfrak{X}})}
$$

A admissible line sheaf on $\mathfrak{X}$ is $\left(\mathscr{L},\left\{\|-\|_{\sigma}\right\}\right)$ where
(1) $\mathscr{L}$ is a line sheaf on $\mathfrak{X}$,
(2) $\|-\|_{\sigma}$ is a $\omega_{\sigma}$-admissible metric on $\mathscr{L}_{\sigma}$.

This forms a group, and $\operatorname{Pic}(\hat{\mathfrak{X}})$ is the group of such objects up to isometric isomorphism.

## 13 February 21, 2018

I missed 20 minutes of class.

## fill in

### 13.1 Comparison of class group and Picard group

Lemma 13.1. If $\overline{\mathscr{L}}_{1} \cong \overline{\mathscr{L}}_{2}$ and $s_{i} \neq 0$ are nonzero sections on $\mathscr{L}_{i}$, then $\operatorname{div} \mathscr{L}_{1}\left(s_{1}\right) \sim \operatorname{div}_{\mathscr{L}_{2}}\left(s_{2}\right)$.
Proposition 13.2. The previous construction gives a group morphism

$$
\xi: \operatorname{Pic}(\widehat{\mathfrak{X}}) \rightarrow \operatorname{Cl}(\widehat{\mathfrak{X}}) ; \quad \overline{\mathscr{L}} \mapsto \operatorname{div}_{\overline{\mathscr{L}}}(s)
$$

where $s$ is a rational section of $\overline{\mathscr{L}}$.
Proof. This is well-defined. If we take tensors $\mathscr{L}_{1} \otimes \mathscr{L}_{2}$, then we can compute this on $s_{1} \otimes s_{2}$. On $\overline{\mathscr{O}}$, we take $s=1$.

Lemma 13.3. $\xi$ is injective.
Proof. Say $\overline{\mathscr{L}}$ is such that for $s \neq 0$ a rational section with $\operatorname{div}_{\overline{\mathscr{L}}}(s) \sim 0$. Then Take $f \neq 0$ such that $\operatorname{div}(f)=\operatorname{div}_{\overline{\mathscr{L}}}(s)$, and let $t=f^{-1} s$ which is a rational section of $\overline{\mathscr{L}}$ and $\operatorname{div}_{\overline{\mathscr{L}}}(t)=0$. Now define the map

$$
\varphi: \mathscr{O} \rightarrow \mathscr{L} ; \quad 1 \mapsto t
$$

This is an isomorphism, and let us check what happens on the metrics. Both


$$
0=-\int_{X_{\sigma}} \log |1|_{\sigma} d \mu_{\sigma}
$$

and

$$
-\int_{X_{\sigma}} \log \left(\varphi^{*}\|-\|_{\sigma}\right)(1)(P) d \mu_{\sigma}(P)=-\int_{X_{\sigma}} \log \|\varphi(1)\|_{\sigma} d \mu_{\sigma}=v_{F_{\sigma}, \overline{\mathscr{L}}}(t)=0
$$

So they have the same normalization, and thus equal.
Let $D=D_{\text {fin }}+D_{\infty}$ be an Arakelov divisor, with $D_{\infty}=\sum_{\sigma} \alpha_{\sigma} F_{\sigma}$. We would like to define an admissible $\overline{\mathscr{O}}(D)$, given by

$$
\overline{\mathscr{O}}(D)=\left(\mathscr{O}(D)_{\mathrm{fin}},\left\{e^{-\alpha_{\sigma}}\|-\|_{\sigma}\right\}_{\sigma}\right),
$$

where $|-|_{\sigma}$ is the unique normalized admissible $d \mu_{\sigma}$.
Lemma 13.4. $\overline{\mathscr{O}}(D) \otimes \overline{\mathscr{O}}(E) \cong \overline{\mathscr{O}}(D+E)$.
Proof. Clear.
Lemma 13.5. $D \sim 0$ implies $\overline{\mathscr{O}}(D) \cong \overline{\mathscr{O}}$.
Proof. Say $D=\operatorname{div}(f)$, with $f \in K(\mathfrak{X})^{\times}$. Then define $\varphi ; \mathscr{O} \rightarrow \mathscr{O}\left(D_{\text {fin }}\right)$ that maps $1 \mapsto f \in H^{0}\left(\mathfrak{X}, \mathscr{O}\left(D_{\text {fin }}\right)\right)$. This is well-defined and an isomorphism. For the metric parts, it is going to be admissible. So we check normalization.

## 14 February 23, 2018

After the computation we had before, we have an injective group homomorphism

$$
\xi: \operatorname{Pic}(\hat{\mathfrak{X}}) \rightarrow \operatorname{Cl}(\hat{\mathfrak{X}}) ; \quad \overline{\mathscr{L}} \mapsto \operatorname{div}_{\overline{\mathscr{L}}}(s)
$$

Also we have a map

$$
\zeta: \operatorname{Div}(\hat{\mathfrak{X}}) \rightarrow \text { (admiss. line sheaves); } \quad D \mapsto \overline{\mathscr{O}}(D)
$$

This induces a group morphism $\bar{\zeta}: \operatorname{Cl}(\hat{\mathfrak{X}}) \rightarrow \operatorname{Pic}(\hat{\mathfrak{X}})$.
Proposition 14.1. $\bar{\zeta} \circ \xi=\operatorname{id}_{\operatorname{Pic}(\hat{\mathcal{X}})}$ so that $\xi$ is an isomorphism.
Proof. Let $\overline{\mathscr{L}}$ be an admissible line sheaf. Choose $0 \neq s$ a rational section. It is enough to show that $\overline{\mathscr{L}} \cong \overline{\mathscr{O}}(\operatorname{div} \overline{\mathscr{L}}(s)$. Since $\xi$ is injective, it suffices to show that

$$
\operatorname{div}_{\overline{\mathscr{O}}\left(\operatorname{div}_{\overline{\mathscr{L}}}(s)\right)}(1)=\operatorname{div}_{\overline{\mathscr{L}}}(s)
$$

This can be checked by direct computation.

### 14.1 Arakelov point of view on height

Height measures the complexity of a point with respect to a divisor. If you look at the theory, there is always a $O(1)$ hanging around. But this has a meaning.

Let me first look at the 1-dimensional case. Forget about the admissibility condition now. In general, let $K$ be a number field, and $\mathcal{O}_{K}$ its ring of integers, and $S=\operatorname{Spec} \mathcal{O}_{K}$. Let $\mathfrak{Y} / S$ be regular flat projective. Assume $Y=\mathfrak{Y}_{\eta}$ be irreducible over $K$. In this setting, we have metrized line sheaves over $Y$.

Assume $\mathfrak{Y}$ has dimension 1 (relative dimension 0 ), and in particular take $\mathfrak{Y}=S$. Let $\widehat{S}=(S,\{\sigma: K \rightarrow \mathbb{C}\})$. A metrized line sheaf $\overline{\mathscr{L}}=\left(\mathscr{L},\left\{|-|_{\sigma}\right\}\right)$ is
(1) $\mathscr{L}$ a line sheaf on $S$,
(2) $\|-\|_{\sigma}$ is a metric on $\left.\mathscr{L}\right|_{\eta} \otimes_{\sigma} \mathbb{C}$.

Note that $S$ is affine. So $\mathscr{M}=\left(\mathscr{M},\left\{\|-\|_{\sigma}\right\}\right)$ has exactly the same data of $\bar{M}=\left(M,\left\{\|-\|_{\sigma}\right\}\right)$ where $M$ is a projective module over $\mathcal{O}_{K}$ of rank 1 and $\|-\|_{\sigma}$ is a hermitian metric on the $\mathbb{C}$-vector space $M \otimes_{\sigma} \mathbb{C}$. The interesting thing is that we can define degree.

Definition 14.2. Let $\bar{M}$ be a metrized line sheaf on $S$. Let $\tilde{n} \in M$ be a nonzero element. We define the degree as

$$
\widehat{\operatorname{deg}}_{K}(\bar{M}, \eta)=\log \# M / \mathcal{O}_{K} \cdot \eta-\sum_{\sigma: K \rightarrow \mathbb{C}} \log \|\eta\|_{\sigma}
$$

Proposition 14.3. The degree $\operatorname{deg}_{K}(\bar{M}, \eta)$ does not depend on the choice of $\eta$.

Proof. Note that projective modules of rank 1 are just fractional ideals. So it is enough to compare $\eta$ and $\beta \eta$ for $\beta \in \mathcal{O}_{K}-\{0\}$. By the Chinese remainder theorem, we have

$$
M / \mathcal{O}_{K} \eta \cong \prod_{\mathfrak{p}}\left(M / \mathcal{O}_{K} \eta\right)_{\mathfrak{p}}
$$

Now we can compute

$$
\#\left(M / \beta \mathcal{O}_{K} \eta\right)_{\mathfrak{p}}=\#\left(\mathcal{O}_{k} / \mathfrak{p}\right)^{v_{\mathfrak{p}}(\beta)} \#\left(M / \mathcal{O}_{K} \eta\right)_{\mathfrak{p}}
$$

All together, we get

$$
\widehat{\operatorname{deg}}_{K}(\bar{M}, \beta \eta)=\widehat{\operatorname{deg}}_{K}(\bar{M}, \eta)+\sum_{\mathfrak{p}} v_{\mathfrak{p}}(\beta) \log \#\left(\mathcal{O}_{K} / \mathfrak{p}\right)-\sum_{\sigma} \log |\sigma(\beta)|
$$

The last two terms cancel by the product formula in algebraic number theory.

Now let $\mathfrak{Y} / S$ have relative dimension $n$. For $P \in Y(\bar{k})$, let $D_{P} \subseteq \mathfrak{Y}$ be the subscheme that is the closure of $P$. It is easy to check that $D_{P}$ is finite and flat over $S$. Then $\operatorname{deg} D_{P} / S=\left[k_{P}: k\right]$, where $K\left(D_{P}\right)=k_{P}$. Now take the normalization


This is a curve, normal and finite over $S$. So we get $B_{p}=\operatorname{Spec} \mathcal{O}_{L}$ where $L=k_{P}$. Now let $\overline{\mathscr{L}}$ be a metrized line sheaf on $\mathfrak{Y}$. It makes sense to pull back to get a metrized line sheaf $v^{*} \overline{\mathscr{L}}$ on $B_{p}$. This is

$$
\left(v^{*} \overline{\mathscr{L}}\right)=\left(v^{*} \mathscr{L},\left\{\|-\|_{\tau}\right\}\right)
$$

where $\|-\|_{\tau}$ agrees with $\|-\|_{\sigma}$ for $\sigma=\left.\tau\right|_{k}$. Finally, we can define height.
Definition 14.4. The height of $Y$ with respect to $\overline{\mathscr{L}}$ is defined as

$$
h_{\overline{\mathscr{L}}}: Y(\bar{K}) \rightarrow \mathbb{R} ; \quad P \mapsto \frac{1}{\left[k_{P}: k\right]} \widehat{\operatorname{deg}}_{k_{P}} v^{*} \overline{\mathscr{L}}
$$

## 15 February 26, 2018

Last time we had $\mathfrak{Y} / S$ an arithmetic scheme (flat, projective, $Y / K$ geometrically irreducible, normal, regular). Then for a line sheaf $\overline{\mathscr{L}}$ we defined height as

$$
h_{\overline{\mathscr{L}}, K}: Y(\bar{K}) \rightarrow \mathbb{R} ; \quad P \mapsto \frac{1}{\left[K_{P}: K\right]} \widehat{\operatorname{deg}}_{K_{P}} \nu_{p}^{*} \overline{\mathscr{L}} .
$$

Here are some properties:
(1) Linearity on $\overline{\mathscr{L}}: \widehat{\operatorname{deg}}$ is compatible with $\otimes$.
(2) Functoriality: if $f: \mathfrak{Y} \rightarrow \mathfrak{Z}$ is an $S$-morphism, and $\overline{\mathscr{L}}$ is on $\mathfrak{Z}$, then

$$
f^{*} h_{\overline{\mathscr{L}}, K}=h_{f^{*} \overline{\mathscr{L}}, K} .
$$

To prove this, just use that we can enlarge $K_{P}$ and the number is the same. Then use the fact that $\nu^{*} \circ f^{*}=(f \circ \nu)^{*}$.

The compatibility of Neron functions is implicitly hidden in that there is a morphism between integral models.
(3) Let $\overline{\mathscr{L}}$ on $\mathfrak{Y} / S$ and let $\mathscr{L}$ be ample. Then for all $A, B>0$ the set

$$
\left\{P \in Y(\bar{K}): h_{\overline{\mathscr{L}}, K}(P)<A \text { and }\left[K_{P}: K\right]<B\right\}
$$ is finite.

Proof. For $\mathfrak{Y}=\mathbb{P}_{S}^{n}$ and $\mathcal{L}=\mathcal{O}(1)$ with the Fubini-Study metric, this is classical. You can check for $\mathbb{P}^{1}$, and then embed $\mathbb{P}^{n}$ into some product of copies of $\mathbb{P}^{1}$. The general case follows from this, (1), (2), and norm comparison.

### 15.1 Intersection pairing

Let $\mathfrak{X} / S$ be a regular semi-stable arithmetic surface, with $d \mu_{\sigma}$ a probability volume form, so that we have $\hat{\mathfrak{X}}$. Let us construct an intersection pairing

$$
(-,-): \operatorname{Div}(\hat{\mathfrak{X}}) \times \operatorname{Pic}(\hat{\mathfrak{X}}) \rightarrow \mathbb{R}
$$

(0) We make this linear on the first component. So we want to define on $F_{\sigma}$, irreducible fiber components, and horizontal curves $D_{p}$ for $P \in X(\bar{K})$.
(i) $F_{\sigma}$ : We define $\left(F_{\sigma}, \overline{\mathscr{L}}\right)=\left.\operatorname{deg}_{\mathbb{C}} \mathscr{L}\right|_{X_{\sigma}}=\left.\operatorname{deg}_{K} \mathscr{L}\right|_{X}$.
(ii) irreducible fiber component: say $\nu: \tilde{C} \rightarrow \mathfrak{X}_{s} \subseteq \mathfrak{X}$ so that $C=\nu(\tilde{C})$. Then we define

$$
(C, \overline{\mathscr{L}})=i_{s}(C, \mathscr{L}) \cdot \log \# k_{s}
$$

Recall that $i_{s}(C, \mathscr{L})=\left.\operatorname{deg}_{k_{s}} \mathscr{L}\right|_{C}=\operatorname{deg}_{\tilde{C} / k_{s}} \nu^{*} \mathscr{L}$.
(iii) horizontal curves: say $\nu_{P}: B_{P} \rightarrow D_{P} \subseteq \mathfrak{X}$, for $P \in X(\bar{K})$. Then

$$
\left(D_{P}, \overline{\mathscr{L}}\right)=\widehat{\operatorname{deg}}_{k_{P}} \nu_{p}^{*} \overline{\mathscr{L}}=\left[K_{P}: K\right] h_{\overline{\mathscr{L}, K}}(P)
$$

Actually, we have a bilinear map

$$
\operatorname{Div}(\hat{\mathfrak{X}}) \times \operatorname{Pic}(\hat{\mathfrak{X}}) \rightarrow \mathbb{R}
$$

This is because we are only taking degree. Also note that we have a map $\zeta: \operatorname{Div}(\hat{\mathcal{X}}) \times \operatorname{Pic}(\hat{\mathfrak{X}})$. So we get map

$$
(-,-): \operatorname{Div}(\hat{\mathfrak{X}}) \times \operatorname{Div}(\hat{\mathfrak{X}}) \rightarrow \mathbb{R}
$$

This new pairing is well-defined, bilinear, and respects linear equivalence on the second component. Also, we have an explicit description. The way we are going to check linear equivalence on the first component is by proving symmetry.

Proposition 15.1. $(-,-)$ is symmetric on $\operatorname{Div}(\hat{\mathfrak{X}}) \times \operatorname{Div}(\hat{\mathfrak{X}})$.
Proof. It is enough to check it for $\left(C_{1}, C_{2}\right)$ for $C_{1} \neq C_{2}$ "irreducible", meaning $F_{\sigma}$, irreducible fiber component, and horizontal $D_{p}$. Here are a bunch of cases.

1. For $C_{1}, C_{2}$ being either $F_{\sigma}$ or irreducible fiber components, we should get 0.
2. For $C_{1}=F_{\sigma}$ and $C_{2}$ horizontal, we need to show that

$$
\widehat{\operatorname{deg}} \nu_{P}^{*} \overline{\mathscr{O}}\left(\sum_{\sigma} \alpha_{\sigma} F_{\sigma}\right)=\left[K_{P}: K\right] \sum_{\sigma} \alpha_{\sigma}
$$

because $\left(F_{\sigma}, \overline{\mathscr{O}}\left(D_{P}\right)\right)=\left[K_{P}: K\right]$. Using $s=1$, we can evaluate the left hand side as

$$
-\sum_{\gamma: K_{P} \rightarrow \mathbb{C}} \log \left\|\nu^{*} 1\right\|_{\tau}=-\sum_{\sigma: K \rightarrow \mathbb{C}} \sum_{\tau \mid \sigma} \log \left(e^{-\alpha_{\sigma}} 1\right)=\left[K_{P}: K\right] \sum_{\sigma} \alpha_{\sigma}
$$

We will finish next time.

## 16 February 28, 2018

So far we have this pairing

where the map from $\operatorname{Div} \times \mathrm{Pic}$ is given by taking degree. We know that the upper map is

- bilinear,
- respects linear equivalence on the second component,
- well-defined and explicit.

We are trying to show that it is symmetric. Let's do only the hardest case now. Assume that $P, Q \in X(\bar{K})$ and this gives horizontal divisors $D_{P}, D_{Q}$. We are trying to show that $\left(D_{P}, D_{Q}\right)=\left(D_{Q}, D_{P}\right)$. By definition,

$$
\left(D_{P}, D_{Q}\right)=\widehat{\operatorname{deg}}_{K_{P}} \nu_{P}^{*} \overline{\mathscr{O}}_{\mathfrak{X}}\left(D_{Q}\right)=(\text { finite contribution for } \eta)+\sum_{\tau: K_{P} \rightarrow \mathbb{C}} \log \left\|\eta^{\tau}\right\|_{\left.\tau\right|_{K}}
$$

for some $\eta \in H^{0}\left(B_{P}, \nu_{P}^{*} \mathscr{O}_{\mathfrak{X}}\left(D_{Q}\right)\right)$. We are going to choose $\eta=1=\nu_{P}^{*} 1$. Here, because $P \neq Q$ we have $\eta \neq 0$. We can write the finite part as the sum of terms of the form

$$
\text { length }\left(\frac{\mathscr{O}_{\mathfrak{X}, x}}{(s, t)}\right) \log \# k_{x}
$$

where $x$ is a closed point of $\mathfrak{X}$ and $s, t \in \mathscr{O}_{\mathfrak{X}, x}$ are local equations for $D_{P}, D_{Q}$ respectively. The infinite part is the sum of the terms of the form

$$
\log \left(\nu^{*}\|-\|_{\sigma}\right)\left(\nu^{*} 1\right)=\log \left\|1_{Q}\right\|_{\sigma}(P)=\log \varphi_{Q}(P)
$$

We spent a week showing that this is symmetric.
So this bilinear paring is symmetric, and we can descend to

$$
\operatorname{Pic}(\hat{\mathfrak{X}}) \times \operatorname{Pic}(\hat{\mathfrak{X}}) \rightarrow \mathbb{R}
$$

Suppose $E_{1}, E_{2} \geq 0$ are effective Arakelov divisors. Say even that $E_{1}, E_{2}$ are irreducible "scheme-theoretic", and assume $E_{1} \neq E_{2}$. We still can have $\operatorname{supp}\left(E_{1}\right) \cap \operatorname{supp}\left(E_{2}\right) \neq \emptyset$ and $\left(E_{1}, E_{2}\right)<0$. There is some infinite contribution if we intersect horizontal divisors, and the Green function is not bounded below by 1 or anything.

### 16.1 Canonical class

Question. What should be a canonical class?
Our conditions on $\mathfrak{X}$ implies that there exists a relative dualizing sheaf $\omega=$ $\omega_{\pi}$, and it is invertible. Note that $g_{X} \geq 1$ by semi-stability. We are going to take $\omega$ not to be $\mathscr{O}(-)$. On the infinite part, we will be able to put "good" canonical metrics on $\omega$ when we choose

$$
d \mu_{\sigma}=d \mu_{\sigma}^{\mathrm{Ar}}
$$

Here, given a compact Riemann surface $Y$ with $Y \geq 1$, we define $d \mu^{\mathrm{Ar}}$ as

$$
d \mu^{\mathrm{AR}}=\frac{i}{2 g} \sum_{j=1}^{g} \alpha_{j} \wedge \bar{\alpha}_{j}
$$

where $\left\{\alpha_{j}\right\}$ is the orthonormal basis for $H^{0}\left(Y, \Omega_{Y / \mathbb{C}}^{1}\right)$ with $\langle\alpha, \beta\rangle=\frac{i}{2} \int_{Y} \alpha \wedge \bar{\beta}$. Note that
(1) $d \mu^{\mathrm{Ar}}$ is a smooth $(1,1)$-form independent of $\alpha_{j}$,
(2) $\int_{Y} d \mu^{\mathrm{Ar}}=\frac{1}{g}(1+\cdots+1)=1$,
(3) $d \mu^{\mathrm{Ar}}$ is positive.

On $Y \times Y$, we define the $(1,1)$-form

$$
\gamma=P_{1}^{*} d \mu^{\mathrm{Ar}}+P_{2}^{*} d \mu^{\mathrm{Ar}}=\frac{i}{2} \sum_{j=1}^{g}\left(P_{1}^{*} \alpha \wedge \overline{P_{2}^{*} \alpha_{j}}+P_{2}^{*} \alpha_{j} \wedge \overline{P_{1}^{*} \alpha_{j}}\right)
$$

We are going to show that there is a metric on $\mathscr{O}_{Y \times Y}(\Delta)$ with $\gamma$ as a curvature. Then we are going to show that the norm of 1 is the Green function. Then we are going to pull the metric along the diagonal.

## 17 March 2, 2018

Last time we talked about picking a canonical element of $\operatorname{Pic}(\hat{\mathfrak{X}})$. For a compact Riemann surface $Y$ (which we secretly think as $X_{\sigma}$ ) with genus $g \geq 1$, we defined

$$
d \mu^{\mathrm{Ar}}=\frac{i}{2 g} \sum_{i=1}^{i} \alpha_{j} \wedge \bar{\alpha}_{j}
$$

with $\left\{\alpha_{j}\right\}$ an orthonormal basis for $H^{0}\left(Y, \Omega_{Y / \mathbb{C}}^{1}\right)$.

### 17.1 Green's function from the diagonal

Our goal is to find a canonical $d \mu^{\mathrm{Ar}}$-admissible metric on $\Omega^{1}$. But we can describe

$$
\Omega^{1}=\delta^{*} \mathscr{O}_{Y \times Y}(-\Delta)
$$

where $\delta: Y \rightarrow Y \times Y$ is the diagonal and $\Delta$ is the image as a divisor. In fact, this is how Hartshorne defines, and this is what we normally have.

This is because the sheaf of ideals corresponding to $\Delta$ is $I=\operatorname{ker}(A \otimes A \rightarrow A)$. Then the map

$$
d: A \rightarrow I / I^{2} ; \quad x \mapsto 1 \otimes x-x \otimes 1
$$

Here, we are taking $I$ as a left $A$-module acting on the left component, and you can check that $\left(I / I^{2}, d\right)$ is the universal algebra. So this is $\Omega_{A / K}^{1}$.

Now if we consider $d \mu^{\mathrm{Ar}}$ a probability volume form on $Y$, we can define

$$
\gamma=p_{1}^{*} d \mu^{\mathrm{Ar}}+p_{2}^{*} d \mu^{*} d \mu^{\mathrm{Ar}}-\frac{i}{2} \sum_{j=1}^{g}\left(p_{1}^{*} \alpha_{j} \wedge \overline{p_{2}^{*} \alpha_{j}}+p_{2}^{*} \alpha_{j} \wedge \overline{p_{1}^{*} \alpha_{i}}\right) .
$$

Then you can check that $\delta^{*} \gamma=-(2 g-2) d \mu^{\mathrm{Ar}}$.
Proposition 17.1. We have $\gamma=\frac{1}{2 \pi i} \operatorname{curv}\left(\mathscr{O}_{Y \times Y}(\Delta),\|-\|_{\Delta}\right)$ for certain $\|-\|_{\Delta}$.
Proof. Here we need to use some Hodge theory.
This metric is going to be unique up to scalar.
Corollary 17.2. Let $H: Y \times Y \rightarrow \mathbb{R}$ be defined by $H(P, Q)=\|1\|_{\Delta}(P, Q)$. Then there exist $c>0$ such that $c H=G_{d \mu^{\mathrm{Ar}}}$. Note that we are using symmetry of both $H$ and $G$.

Proof. This can be checked by checking the axioms for the Green function. For the curvature condition, if we restrict $\gamma$ to any coordinate, we get just $d \mu^{\mathrm{Ar}}$. So it can be checked. Note that we are using symmetry of both $H$ and $G$.

Corollary 17.3. $G$ is $C^{\infty}$ away from $\Delta$ in $Y \times Y$, and vanishes to order 1 along $\Delta$.

Recall that $G=G_{d \mu^{\mathrm{Ar}}}$ is canonically attached to $d \mu^{\mathrm{Ar}}$, and $d \mu^{\mathrm{Ar}}$ is canonically attached to the Riemann surface $Y$.

Proposition 17.4. There exists a unique $C^{\infty}$-metric $\|-\|_{-\Delta}$ on $\mathscr{O}_{Y \times Y}(-\Delta)$ determined by the condition

$$
\|1\|_{-\Delta}(P, Q)=G(P, Q)^{-1}
$$

for all $P \neq Q$. Furthermore, it induces a metric $\|-\|_{\mathrm{Ar}}$ on $\Omega_{Y}^{1}$ by $\delta^{*}$. This metric is canonical and $d \mu^{\mathrm{Ar}^{-}}$-admissible.

Proof. First this gives a smooth metric, because $G(P, Q)$ is nonzero smooth away from $\Delta$ and vanishes to order 1 on $\Delta$. The metric is admissible because duals work well with curvature and $-\delta^{*} \gamma=(2 g-2) d \mu_{\mathrm{Ar}}$.

For a point $P$, there is a residue map

$$
\left.\mathscr{O}_{Y}(P) \otimes \Omega_{Y}^{1}\right|_{P} \xlongequal{\cong} \mathbb{C} ; \quad \frac{1}{t} \otimes d t \mapsto 1
$$

This is pushing forward along the diagonal, and then pulling back along some horizontal copy of $Y$ to get a sheaf supported at a point.

## 18 March 5, 2018

Last time we had for $Y$ a compact Riemann surface, a probability volume for $d \mu^{\mathrm{Ar}}$. We defined $\gamma$ a $(1,1)$-form, and for the diagonal $\delta: Y \rightarrow Y \times Y$ and horizontal embedding $h_{P}: Y \rightarrow Y \times Y$, had

$$
-\delta^{*} \gamma=(2 g-2) d \mu^{\mathrm{Ar}}, \quad h_{P}^{*} \gamma=d \mu^{\mathrm{Ar}}
$$

We can compute

$$
\gamma=\frac{1}{2 \pi i} \operatorname{curv}\left(\mathscr{O}_{Y \times Y}(\Delta),\|-\|_{\Delta}\right)
$$

and also checked that

$$
\|1\|_{\Delta}(-,-)=G(-,-)
$$

up to some constant $c>0$. Then $G(-,-)^{-1}$ determines a metric $\|-\|_{-\Delta}$ on $\mathscr{O}_{Y \times Y}(-\Delta)$, and this gives a metric on $\Omega_{Y / \mathbb{C}}^{1}$ which is $d \mu^{\mathrm{Ar}}$-admissible. In particular, we will have

$$
\frac{1}{2 \pi i} \operatorname{curv}\left(\Omega_{Y / \mathbb{C}}^{1},\|-\|^{\mathrm{Ar}}=(2 g-2) d \mu^{\mathrm{Ar}}\right.
$$

Let me make one additional comment. We have

$$
\Omega_{Y}^{1} \otimes \delta^{*}(\mathscr{O}(\Delta)) \cong \mathscr{O}_{Y}
$$

by just taking dual, and then pushing forward gives

$$
\delta_{*} \Omega_{Y}^{1} \otimes \delta_{*} \delta^{*} \mathscr{O}(\Delta) \cong \delta_{*} \mathscr{O}_{Y}
$$

as sheaves of $\mathscr{O}_{Y \times Y}$-modules supported on $\Delta$. Then if you pull back by $h_{P}$, we get sheaves on $Y$ supported at $P$. The fiber at $P$ is going to be

$$
\left.\left.\Omega_{Y}^{1}\right|_{P} \otimes \mathscr{O}_{Y}(P)\right|_{P} \cong \mathbb{C}
$$

Proposition 18.1. This isomorphism is the following. Choose t a uniformizer at $p$, and then

$$
\left.\left.d t\right|_{p} \otimes \frac{1}{t}\right|_{p} \mapsto 1
$$

Moreover, everything we have been talking about were isometries, and so the metrics match. That is,

$$
\|d t\|^{\operatorname{Ar}}(P) \cdot\left\|\frac{1}{t}\right\|_{\mathscr{O}(P)}(P)=1
$$

Note that the residue maps are also the adjunction isomorphisms for $P \rightarrow Y$ over $\operatorname{Spec} \mathbb{C}$.

### 18.1 The canonical sheaf

Form now on, my arithmetic surface is going to be

$$
\hat{\mathfrak{X}}=\left(\mathfrak{X}, \pi,\left\{d \mu^{\mathrm{Ar}}\right\}_{\sigma: K \rightarrow \mathbb{C}}\right)
$$

where $\mathfrak{X}$ is my usual regular semi-stable surface and $d \mu_{\sigma}^{\mathrm{Ar}}$ is $d \mu^{\mathrm{Ar}}$ for $X_{\sigma}=$ $\mathfrak{X} \otimes_{\sigma} \mathbb{C}$.

So we have this canonical sheaf

$$
\widehat{\omega}=\left(\omega_{\pi},\left\{\|-\|_{\sigma}^{\mathrm{Ar}}\right\}_{\sigma: K \rightarrow \mathbb{C}}\right)
$$

This is a relative dualizing sheaf for $\pi: \mathfrak{X} \rightarrow S$ and it is a line sheaf because $\mathfrak{X}$ is regular and semi-stable. This is the canonical class in $\operatorname{Pic}(\hat{\mathcal{X}})$.

Theorem 18.2 (Adjunction for sections, Arakelov). Let $P \in X(K)$ be a rational point. This induces a divisor $D_{P} \subseteq \mathfrak{X}$. Then the normalization is $B_{P}=\operatorname{Spec} \mathcal{O}_{K}=S$, so the normalization map $\nu: S \rightarrow \mathfrak{X}$ is a section. Then we have

$$
\left(D_{P} \cdot \overline{\mathscr{O}}\left(D_{P}\right) \otimes \widehat{\omega}\right)=0
$$

In other words, $D_{P}^{2}+D_{P} \cdot \widehat{\omega}=0$.
Proof. It suffices to show that

$$
\widehat{\operatorname{deg}}_{K} \nu^{*} \overline{\mathscr{O}}\left(D_{P}\right) \otimes \widehat{\omega}=0
$$

but the sheaf is precisely $\overline{\mathscr{O}}(S)$. (For the finite part, use the adjunction formula for $\mathscr{I}_{D_{P}}=\mathscr{O}_{\mathfrak{X}}\left(-D_{P}\right)$, and for the infinite part, we have made so that they match.)

## 19 March 9, 2018

So where are we now?

1. We have a theory of good integral models over number fields.
2. We have a notion of metrics on line sheaves, and also a notion of admissibility. From this we obtained global equations $\varphi_{P}$ and the Green function $G$.
3. We have an "intersection" paring for $\operatorname{Div}(\hat{\mathfrak{X}})$. This take values in real numbers, and sometimes take negative values even for effective divisors. This is quite general for $\hat{\mathfrak{X}}=\left(\mathfrak{X}, \pi,\left\{d \mu_{\sigma}\right\}_{\sigma}\right)$.
4. We have a canonical class $\omega=\omega_{\pi}$. But to put metrics, we need admissibility and should be uniquely normalized. So we had (on $Y=X_{\sigma}$ with $g \geq 1$ ) to restrict to a special probability volume form

$$
d \mu^{\mathrm{Ar}}=\frac{i}{2 g} \sum_{j=1}^{g} \alpha_{j} \wedge \bar{\alpha}_{j}
$$

where $\alpha_{j}$ form an orthonormal basis for $H^{0}\left(Y, \Omega^{1}\right)$. There was some canonical construction $\|-\|^{\mathrm{Ar}}$ on $\Omega_{Y}^{1}$ that is $d \mu^{\mathrm{Ar}}$-admissible. Also, it satisfied the property that the residue/adjunction map

$$
\operatorname{Res}_{P}:\left.\left.\Omega_{Y}^{1}\right|_{P} \otimes \mathscr{O}_{Y}(P)\right|_{P} \rightarrow \mathbb{C}=\left.\mathscr{O}_{Y}\right|_{P} ;\left.\left.\quad d t\right|_{P} \otimes \frac{1}{t}\right|_{P} \mapsto 1
$$

for $P \in Y$ is an isometry.

### 19.1 Outline of the theory

Theorem 19.1 (Adjunction for section). Let $P \in X(K)$ be a point and $D_{P} \subseteq$ $\mathfrak{X}$. Then

$$
\overline{\mathscr{O}}\left(D_{P}\right) \cdot\left(\overline{\mathscr{O}}\left(D_{P}\right) \otimes \hat{\omega}\right)=0 .
$$

Another consequence is that the canonical height is

$$
h_{X, K_{X}}(P)=\widehat{\operatorname{deg}} \nu_{P}^{*} \hat{\omega}=-D_{P}^{2}
$$

Here is another application. Take $J_{X}$ the Jacobian. Then there is a uniquely defined height $\hat{h}_{J_{X}, \theta}$ called a Neron-Tate height. Weil height is only defined up to constant, but then you can add points together and do some limiting process. How does it compare to the self-intersection $D_{P}^{2}$ we defined above? You choose a basepoint on $X$ and map $X \rightarrow J_{X}$. Then you can compare the self-intersection number in $\operatorname{Div}(\hat{\mathfrak{X}})$ orthogonal to fibers, with $-\hat{h}_{J_{X}, \theta}$.

Theorem 19.2 (Hodge index theorem).
(1) If $D$ is orthogonal to all fibers in $\operatorname{Div}(\hat{\mathfrak{X}})$, then

$$
-D^{2}=2[K: \mathbb{Q}] \hat{h}_{J_{X}, \theta}\left(\left.\mathscr{O}(D)\right|_{X}\right)
$$

(2) The signature of $(-,-)$ on $\operatorname{Div}(\hat{\mathfrak{X}}) /$ num is $(+,-, \ldots,-)$ where the negative part comes from (1).
(3) The number of - signs is

$$
\sum_{v \in S}\left(\#\left\{\text { components of } \mathfrak{X}_{v}\right\}-1\right)+\operatorname{rk} J_{X}(K)+1
$$

This will follow from Riemann-Roch. We should expect some alternating sum of dimension of cohomology, but cohomology are not vector spaces but $\mathcal{O}_{K}$-modules. We are going to put some metrics, but do this for both $H^{0}$ and $H^{1}$ in a way that they are defined uniquely up to some common multiple which will cancel out.

For $M$ be a finitely generated $\operatorname{rank} \mathcal{O}_{K}$-module of rank $n$, write $\Lambda M=\wedge^{n} M$. Then we can define

$$
\lambda R \Gamma(\mathfrak{X}, \mathscr{L})=\Lambda\left(H^{0}(\mathfrak{X}, \mathscr{L})\right) \otimes \Lambda\left(H^{1}(\mathfrak{X}, \mathscr{L})^{\vee}\right)
$$

and the same for $X_{\sigma}$. Note that a Hermitian metric on $\Lambda V$ with $V$ a $\mathbb{C}$-vector space is the same as a volume on $V$. Putting the metric $\lambda$ is not easy and Faltings complain about this. We want it to be:
(1) compatible with isomorphisms (of $\mathscr{L}$ ),
(2) compatible with long exact sequences.

We are going to do this on some moduli space $\mathrm{Pic}^{d}$.

## 20 March 19, 2018

Recall Haar measures on vector spaces over $\mathbb{R}, \mathbb{C}$. We are going to consider $M$ an $\mathcal{O}_{K}$-odule, finitely generated. (These will come from cohomology.) They will come with volumes at $\infty$, on $M \otimes_{\sigma} \mathbb{C}$. This is not going to be a full-rank lattice, because in the simplest case $\mathbb{Z} \subseteq \mathbb{Q}$, we get $\mathbb{Z} \subseteq \mathbb{C}$ and we can't immediately use the covolume.

We can define

$$
M_{f}=M / M_{\mathrm{tor}}=\bigoplus_{\sigma} M \otimes_{\sigma} \mathbb{C} \cong M \otimes_{\mathbb{Z}} \mathbb{C}
$$

and then $M \otimes_{\mathbb{Z}} \mathbb{R}$ inside $M \otimes_{\mathbb{Z}} \mathbb{C}$. Then it make sense to consider $\operatorname{Vol}\left(M \otimes_{\mathbb{Z}}\right.$ $\left.\mathbb{R} / M_{f}\right)$, but this is not exactly what we will use.

### 20.1 Haar measure

Let $V$ be an $\mathbb{R}$-vector space, and $n=\operatorname{dim} V$.
Definition 20.1. A Haar measure on $V$ is $\mu$ a Borel measure such that it is (i) translation invariant and (ii) $\mu(\lambda A)=|\lambda|^{n} \mu(A)$.

Proposition 20.2. Haar measures on $V$ are the same as norms in $\operatorname{det}(V)=$ $\wedge^{n} V$.

Proof. Take $v_{1}, \ldots, v_{k}$ a basis of $V$. Then we can form $v_{1} \wedge \cdots \wedge v_{n} \in \operatorname{det}(V)$. Then we define

$$
d \mu=\left\|v_{1} \wedge \cdots \wedge v_{n}\right\|_{\mu} d x_{1} \wedge \cdots \wedge d x_{n}
$$

where $x_{j}$ are the dual basis.
We the same notation, we consider $M=\left\langle v_{1}, \ldots, v_{n}\right\rangle_{\mathbb{Z}} \subseteq V$ and we can define the covolume
$\operatorname{Vol}_{\mu}(V / M)=\int_{V / M} d \mu=\int_{0}^{1} \cdots \int_{0}^{1}\left\|v_{1} \wedge \cdots \wedge v_{n}\right\|_{\mu} d x_{1} \wedge \cdots \wedge d x_{n}=\left\|v_{1} \wedge \cdots \wedge v_{n}\right\|_{\mu}$.
The complex case is a little annoying. Let $V$ be a $\mathbb{C}$-vector space and $n=$ $\operatorname{dim}_{\mathbb{C}} V$. A Haar measure can be defined similarly but with $\mu(\lambda A)=|\lambda|^{2 n} \mu(A)$. Again, $\mu$ a Haar measure corresponds to a norm on $\operatorname{det}_{\mathbb{C}} V$. Take $v_{1}, \ldots, v_{n}$ a $\mathbb{C}$-basis of $V$. Then there is a dual basis $z_{j}=x_{j}+i y_{j}$. (We're taking real an imaginary parts of $V \rightarrow \mathbb{C}$.) Then we can take

$$
d \mu=\left\|v_{1} \wedge \cdots \wedge v_{n}\right\|_{\mu}^{2} d x_{1} \wedge d y_{1} \wedge \cdots \wedge d x_{n} \wedge d y_{n}
$$

To define the volume, we can take

$$
W=\mathbb{R}\left\langle v_{1}, \ldots, v_{n}\right\rangle
$$

so that $V=W \oplus i W$. Let $M=\mathbb{Z}\left\langle v_{1}, \ldots, v_{n}\right\rangle \subseteq V$. Given a Haar measure $\mu$ on $V$, we can take $\mu_{W}$ the only Haar measure on $W$ such that $\mu$ is the product measure on $V=W \oplus i W$. Altenratively, we have the formula

$$
d \mu_{W}=\left\|v_{1} \wedge \cdots \wedge v_{n}\right\|_{\mu} d x_{1} \wedge \cdots \wedge d x_{n}
$$

Then

$$
\operatorname{Vol}_{\mu_{W}}(W / M)=\int_{W / M} d \mu_{W}=\left\|v_{1} \wedge \cdots \wedge v_{n}\right\|_{\mu}=\sqrt{\operatorname{Vol}_{\mu}(V / M \oplus i M)}
$$

### 20.2 Measurable modules

Let $K$ be a number field and $M$ a finitely generated $\mathcal{O}_{K}$-module.
Definition 20.3. A measurable $\mathcal{O}_{K}$-module is $\hat{M}=\left(M,\left\{\mu_{\sigma}\right\}_{\sigma}\right)$ where
(1) $M$ is a finitely generated $\mathcal{O}_{K^{-}}$-module,
(2) for $\sigma: K \rightarrow \mathbb{C}, \mu_{\sigma}$ is a Haar measure on $M_{\sigma}=M \otimes_{\sigma} \mathbb{C}$.

By general number theory, there is a canonical isomorphism

$$
M_{\mathbb{C}}=M \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{\sigma: K \rightarrow \mathbb{C}} M_{\sigma} ; \quad x \otimes 1 \mapsto\left(x \otimes_{\sigma} 1\right)_{\sigma}
$$

Let $M_{f}$ be the image of $M$ on the right side so that it is isomorphic to $M / M_{\text {tor }}$. Then $M \otimes_{\mathbb{Z}} \mathbb{R}$ is the $\mathbb{R}$-span of $M_{f}$ in $M \otimes_{\mathbb{Z}} \mathbb{C}$.

So we first get a measurable $\mathbb{Z}$-module from $\hat{M}$. This $M$ is a finitely generated $\mathbb{Z}$-module. Then we can embed into $M \otimes_{\mathbb{Z}} \mathbb{C}$, and then we can take the product measure from the isomorphism. Now we can define the covolume $\operatorname{Vol}_{\mu}\left(M \otimes_{\mathbb{Z}}\right.$ $\mathbb{R} / M)$.

Proposition 20.4. If you take $\widehat{\mathcal{O}}_{K}=\left(\mathcal{O}_{K}\right.$, obvious measures on $\left.\mathbb{C}\right)$, then the covolume is $\operatorname{Vol}\left(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R} / \mathcal{O}_{K}\right)=\sqrt{\left|D_{K}\right|}$.

## 21 March 21, 2018

Last time we had $V$ a $\mathbb{C}$-vector space and $\mathcal{B}=\left\{v_{j}\right\}_{j}$ a basis over $\mathbb{C}$. Then we get $W=\mathbb{R}\langle\mathcal{B}\rangle$, and from $\mu$ a measure on $V, \nu$ a measure on $W$. We had $K$ a number field and a notion of a measurable finitely $\mathcal{O}_{K^{-}}$-module $\widehat{M}=\left(M,\left\{\mu_{\sigma}\right\}_{\sigma}\right)$. For every $\sigma: K \rightarrow \mathbb{C}, \mu_{\sigma}$ is a Haar measure on $M_{\sigma}=M \otimes_{\sigma} \mathbb{C}$.

1. We defined the covolume $\operatorname{Vol}\left(M \otimes_{\mathbb{Z}} \mathbb{R} / M\right)$.
2. This construction factors through regarding $\widehat{M}$ an $\mathcal{O}_{K}$-module a as a measurable $\mathbb{Z}$-module.

Lemma 21.1. $\operatorname{Vol}\left(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R} / \mathcal{O}_{K}\right)=\sqrt{\left|D_{K}\right|}$, where $\widehat{\mathcal{O}}_{K}$ has the Euclidean measure on $\mathcal{O}_{K} \otimes_{\sigma} \mathbb{C} \cong \mathbb{C}$.

Proof. We have to understand the measure on $\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{C}=\bigoplus_{\sigma} \mathbb{C}$. This is the same as taking the usual absolute value on $\operatorname{det}_{\mathbb{C}}\left(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{C}\right)=\operatorname{deg}\left(\bigoplus_{\sigma} \mathbb{C}\right) \cong \mathbb{C}$. Let $b_{1}, \ldots, b_{n}$ be a $\mathbb{Z}$-basis for $\mathcal{O}_{K}$. Then I need to compute is

$$
\operatorname{Vol}\left(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R} / \mathcal{O}_{K}\right)=\left\|b_{1} \wedge \cdots \wedge b_{n}\right\|_{\mu}=\left|\operatorname{deg}\left[\sigma\left(b_{i}\right)\right]_{i, \sigma}\right|=\sqrt{\left|D_{K}\right|}
$$

### 21.1 Changing the module/volume

Now I would like to talk about a trick. Our goal is to show that $\operatorname{Vol}\left(M \otimes_{\mathbb{Z}} \mathbb{R} / M\right)$ is something explicit formula. To compute this, we can replace $M$ with other modules of finite index or coindex. Or we can vary the volume.

Let me look at the special case of a rank 1 projective $\mathcal{O}_{K}$-module. Consider $\bar{M}=\left(M,\left\{\|-\|_{\sigma}\right\}_{\sigma}\right)$ a metrized line sheaf on $\mathcal{O}_{K}$. Then for every $\sigma: K \rightarrow \mathbb{C}$, I have $\operatorname{det}_{\mathbb{C}}\left(M_{\sigma}\right)=M_{\sigma}$. So the norm is just a measure on $M_{\sigma}$. That is, one can go back and forth between a metrized line sheaf $\bar{M}$ and a measurable projective rank 1 module $\widehat{M}$. So what is the relation between the two real numbers $\widehat{\operatorname{deg}}_{K} \bar{M}$ and $\operatorname{Vol}(M \otimes \mathbb{R} / M)$ ?

Proposition 21.2. $\log \operatorname{Vol}\left(M \otimes_{\mathbb{Z}} \mathbb{R} / M\right)=\frac{1}{2} \log \left|D_{K}\right|-\widehat{\operatorname{deg}}_{K} \bar{M}$.
Proof. First note that projective $\mathcal{O}_{K}$-modules of rank 1 are fractional ideals up to isomorphism. So we are going to assume that $M$ is a fractional ideal. Then $M_{\sigma}=M \otimes_{\sigma} \mathbb{C} \cong K \otimes_{\sigma} \mathbb{C}$ canonically. Now I can meaningfully say that I keep the archemedian data fixed while I change $M$.

We replace $M$ by any other fractional ideal $M^{\prime} \subseteq M$, keeping the norms and the volumes fixed. The change on the left hand side is $\log \left[M: M^{\prime}\right]$. On the right hand side, in the definition of the degree, we were counting the cardinality of the quotient. So if we pick some $\eta \in M^{\prime} \subseteq M$, the archemedian part doesn't change and the cardinality decreases by $\log \left[M: M^{\prime}\right]$. So the changes match up.

But I can actually replace $M$ by any other fractional ideal, say $M=\mathcal{O}_{K}$. Now, $M_{\sigma} \cong \mathbb{C}$ canonically, and so $\|-\|_{\sigma}=\lambda_{\sigma}|-|$ with the same coefficient $\mu_{\sigma}=\lambda_{\sigma} m$. Now we can compute $\widehat{\operatorname{deg}}_{K}$ with $\eta=1$.

Now we want to base change. Ultimately we want to prove Riemann-Roch, and we usually do this by adding divisors one by one. When you put in vertical divisors, this is usually not very hard. But if we put in horizontal divisors, we need some sort of adjunction formula to see what changes. But this only works for sections. So what we are going to do is suitably bash change so that the horizontal divisor becomes a section.
Definition 21.3. Let $\widehat{M}$ be a measurable $\mathcal{O}_{K}$-module and $F / K$ a finite extension. We define

$$
\widehat{M}_{\mathcal{O}_{F}}=\left(M_{\mathcal{O}_{F}}=M \otimes_{\mathcal{O}_{K}} \mathcal{O}_{F},\left\{\nu_{\tau}\right\}_{\tau: F \rightarrow \mathbb{C}}\right)
$$

where $\left(M \otimes_{\mathcal{O}_{K}} \mathcal{O}_{F}\right) \otimes_{\tau} \mathbb{C} \cong M \otimes_{\mathcal{O}_{K},\left.\tau\right|_{K}} \mathbb{C} \cong M_{\left.\tau\right|_{K}}$ is given the volume induced from $\mu_{\left.\tau\right|_{K}}$.

Proposition 21.4. With the same notion, and $r=\operatorname{rk}_{\mathcal{O}_{K}} M=\operatorname{rk}_{\mathcal{O}_{F}}\left(M_{\mathcal{O}_{F}}\right)$,

$$
\frac{\operatorname{Vol}\left(M_{\mathcal{O}_{K}} \otimes_{\mathbb{Z}} \mathbb{R} / M_{\mathcal{O}_{K}}\right)}{\operatorname{Vol}\left(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R} / \mathcal{O}_{K}\right)^{r}}=\left(\frac{\operatorname{Vol}\left(M_{\mathcal{O}_{K}} \otimes_{\mathbb{Z}} \mathbb{R} / M_{\mathcal{O}_{K}}\right)}{\operatorname{Vol}\left(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R} / \mathcal{O}_{K}\right)^{r}}\right)^{[F: K]}
$$

Proof. We only mess around with $M$. We first kill the torsion in $M$, then take finite index free submodule, so that we can assume that $M$ is free. Then the free parts can be separated, so we assume $r=1$ and $M=\mathcal{O}_{K}$. Then both side consists of these scalars coming from the volumes, and each $K \rightarrow \mathbb{C}$ appears exactly $[K: F]$ times in the $F \rightarrow \mathbb{C}$.

## 22 March 23, 2018

Last time we looked at the trick of changing volume. But there is this discriminant coming out, which is annoying.

### 22.1 Arithmetic Euler characteristic

Let $\mathbb{E}: 0 \rightarrow M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$ be an exact sequence of $\mathbb{Z}$-modules, with volumes. I want to make sense of volume exactness. We can define $\operatorname{det} \mathbb{E}$ as

$$
\operatorname{det}_{\mathbb{C}} \mathbb{E}=\bigotimes_{i=1}^{3}\left(\underset{\mathbb{C}}{ } \operatorname{det}_{i} \otimes \mathbb{C}\right)^{\otimes(-1)^{i+1}} \cong \mathbb{C}
$$

where the isomorphism comes from exactness.
Definition 22.1. We say that the sequence $\mathbb{E}$ is volume exact if $\operatorname{det} \mathbb{E} \otimes \mathbb{C}$ with the volumes matching up.

Now we can define the Euler characterisitic

$$
\chi(\widehat{M})=-\log \operatorname{Vol}(M \otimes \mathbb{R} / M)+\log \# M_{\mathrm{tor}}
$$

Proposition 22.2. For $\mathbb{E} a$ volume exact sequence, $\chi\left(\widehat{M}_{1}\right)-\chi\left(\widehat{M}_{2}\right)+\chi\left(\widehat{M}_{3}\right)=$ 0 .

Let $K$ be a number field, and $\widehat{M}$ be a measurable $\mathcal{O}_{K}$-module. We define

$$
\chi_{K}(\widehat{M})=-\log \operatorname{Vol}\left(M \otimes_{\mathbb{Z}} \mathbb{R} / M\right)+\log \# M_{\mathrm{tor}}+\mathrm{rk}_{\mathcal{O}_{K}}(M) \log \left|D_{K}\right|^{1 / 2}
$$

Note that the last term work well with exact sequences as well.
Corollary 22.3. Note that a line sheaf $\bar{M}$ gave $\widehat{M}$. In this case, $\chi_{K}(\widehat{M})=$ $\widehat{\operatorname{deg}}_{K} \bar{M}$.
Corollary 22.4. If $\widehat{M}$ is a measurable $\mathcal{O}_{K}$-module, and $F / K$ is a finite extension, then

$$
\chi_{F}\left(\widehat{M}_{\mathcal{O}_{F}}\right)=[F: K] \chi_{K}(\widehat{M}) .
$$

Now let us go back to $\left(\widehat{\mathfrak{X}}, \pi,\left\{d \mu_{\sigma}^{\mathrm{Ar}}\right\}_{\sigma}\right)$. If $\mathcal{L}$ is a line sheaf on $\mathfrak{X}$, note that $H^{j}(\mathfrak{X}, \mathscr{L})=(0)$ for $j \geq 2$. This is because $\mathfrak{X}$ is projective. Roughly RiemannRoch should look like

$$
\operatorname{Vol}\left(H^{0}(\mathfrak{X}, \overline{\mathscr{L}}) \otimes_{\mathbb{Z}} \mathbb{R} / H^{0}(-)\right)=\text { formula involving intersection of } \overline{\mathscr{L}}
$$

But we need to put a volume on the global sections. This is something we need to do. We are sort of going to make sense of the difference between the volume on $H^{0}$ and $H^{1}$, because we need to take the difference.

Note that

$$
H^{0}(\mathfrak{X}, \mathscr{L}) \otimes_{\sigma} \mathbb{C} \cong H^{0}\left(X_{\sigma}, \mathscr{L}_{\sigma}\right)
$$

So we are good if we can put volumes on the sections of any Riemann surface.

### 22.2 Determinant of cohomology

Let $Y$ be a compact Riemann surface, and $g_{Y}=g \geq 1$. Let $\mathscr{L}$ be a line sheaf on $Y$. Then determinant of cohomology is

$$
\lambda(Y, \mathscr{L})=\operatorname{det}_{\mathbb{C}} H^{0}(Y, \mathscr{L}) \otimes \operatorname{det}_{\mathbb{C}} H^{1}(Y, \mathscr{L})^{\vee}
$$

So this is a line. Good norms on $\lambda(Y, \mathscr{L})$, depending on the given metrics on $\mathscr{L}$, are what we want.

Here are the things I want. The trick used in the usual Riemann-Roch is that if $D$ is a divisor on $Y$ with $P \in Y$, and $D^{\prime}=D+P$, we have an exact sequence

$$
\left.0 \rightarrow \mathscr{O}(D) \rightarrow \mathscr{O}\left(D^{\prime}\right) \rightarrow \mathscr{O}\left(D^{\prime}\right)\right|_{P}=\mathscr{O}_{P} \otimes \mathscr{O}\left(D^{\prime}\right) \rightarrow 0
$$

This gives a long exact sequence
$\left.0 \rightarrow H^{0}(Y, \mathscr{O}(D)) \rightarrow H^{0}\left(Y, \mathscr{O}\left(D^{\prime}\right)\right) \rightarrow \mathscr{O}\left(D^{\prime}\right)\right|_{P} \rightarrow H^{1}(Y, \mathscr{O}(D)) \rightarrow H^{1}\left(Y, \mathscr{O}\left(D^{\prime}\right)\right) \rightarrow 0$.
Then we get

$$
\lambda(Y, \mathscr{O}(D+P)) \cong \lambda(Y, \mathscr{O}(D)) \otimes \mathscr{O}\left(D^{\prime}\right)_{P}
$$

which we will call the "exactness isomorphism".
Here is one attempt for the structure sheaf. Let $Y$ as before. We have a canonical isomorphism

$$
\lambda\left(Y, \mathscr{O}_{Y}\right) \cong \operatorname{det} H^{0}\left(Y, \Omega^{1}\right)^{\vee}
$$

Then there is a hermitian norm induced by $\langle-,-\rangle$ on $H^{0}\left(\Omega_{Y}^{1}\right)$ given by

$$
\langle\alpha, \beta\rangle=\frac{i}{2} \int_{Y} \alpha \wedge \bar{\beta}
$$

(We call this $\|-\|_{F, \text { can. }}$.)
Theorem 22.5 (Faltings's volume in cohomology). Let $Y$ be a compact Riemann surface with $g=g_{Y} \geq 1$. Then there is a unique way to associate to each metrized lined sheaf $\overline{\mathscr{L}}$ on $Y$, a hermitian norm $\|-\|_{F, \overline{\mathscr{L}}}$ on $\lambda(Y, \mathscr{L})$ such that
(1) (isometry) If $\overline{\mathscr{L}}_{1} \cong \overline{\mathscr{L}}_{2}$ is an isometric isomorphism, then it induces an isomorphism on $\lambda$ with compatible $\|-\|_{F, \mathscr{L}}$.
(2) (scaling) If for $\overline{\mathscr{L}}=(\mathscr{L},\|-\|)$ and $a>0$ we define $a \overline{\mathscr{L}}=(\mathscr{L}, a\|-\|)$, then

$$
\|-\|_{F, a \overline{\mathscr{L}}}=a^{h^{0}(\mathscr{L})-h^{1}(\mathscr{L})}\|-\|_{F, \overline{\mathscr{L}}} .
$$

(3) (exactness) For $D$ a divisor on $Y$ and $P$, put unique Ar-admissible normalized metrics on $\mathscr{O}(\operatorname{div}) s$. Then the "exactness isomorphism"

$$
\left.\lambda(Y, \mathscr{O}(D+P)) \cong \lambda(Y, \mathscr{O}(D)) \otimes \mathscr{O}(D+P)\right|_{P}
$$

is an isometry for the $\|-\|_{F}$ norm.
(4) (normalization) The norm $\|-\|_{F, \overline{\mathscr{O}}}$ is equal to the one that had a moment ago on $\lambda\left(Y, \mathscr{O}_{Y}\right)$.

## 23 March 26, 2018

Last time we talked about volume exactness and the Euler characteristic $\chi_{K}$. Also, we were able to formulate Falting's theorem on volumes on " $H^{0}-H^{1 "}$.

### 23.1 Arithmetic Riemann-Roch

First, we define the Euler characteristic of metrized line sheaves. Consider a semi-stable regular model $\widehat{\mathfrak{X}}=\left(\mathfrak{X}, \pi,\left\{d \mu_{\sigma}^{\mathrm{Ar}}\right\}_{\sigma}\right)$. Define

$$
\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{L}})=\chi_{K}\left(H^{0}(\mathfrak{X}, \overline{\mathscr{L}})\right)-\chi_{K}\left(H^{1}(\mathfrak{X}, \overline{\mathscr{L}})\right)
$$

Here, we use, at each $\sigma: K \rightarrow \mathbb{C}$, Faltings' volumes on $H^{j}\left(X_{\sigma},\left.\overline{\mathscr{L}}\right|_{X_{\sigma}}\right)$. Each term alone is not well-defined, but when we take the difference, this is welldefined.

Theorem 23.1 (arithmetic Riemann-Roch). Let $\overline{\mathscr{L}}$ be an admissible line sheaf on $\widehat{\mathfrak{X}}$. Then

$$
\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{L}})=\frac{1}{2}\left(\overline{\mathscr{L}} \cdot \overline{\mathscr{L}} \otimes \widehat{\omega}^{\vee}\right)+\chi_{K}\left(\widehat{\mathfrak{X}}, \overline{\mathscr{O}}_{\mathfrak{X}}\right)
$$

Proof. First it is enough to assume that $\overline{\mathscr{L}}=\overline{\mathscr{O}}(D)$ where $D$ is some Arakelov divisor. The statement is clearly true for $\overline{\mathscr{L}}=\overline{\mathscr{O}}$. Now we show that the truth of the equation is unchanged when we add to $D$ a divisor $\alpha \cdot F_{\sigma}$. The change in the left hand side is

$$
\Delta(\mathrm{LHS})=\alpha\left(h^{0}\left(\left.\mathscr{O}(D)\right|_{X}\right)-h^{1}\left(\left.\mathscr{O}(D)\right|_{X}\right)\right)
$$

The change in the right hand side is

$$
\begin{aligned}
\Delta(\mathrm{RHS}) & =\frac{1}{2}\left(D+\alpha F_{\sigma} \cdot D+\alpha F_{\sigma}-\widehat{\omega}\right)-\frac{1}{2}(D \cdot D-\widehat{\omega}) \\
& =\alpha\left(F_{\sigma} \cdot D\right)-\frac{1}{2} \alpha\left(F_{\sigma} \cdot \widehat{\omega}\right)=\alpha\left(\operatorname{deg}_{X}\left(\left.D\right|_{X}\right)+1-g\right)=\alpha\left(h^{0}\left(\left.D\right|_{X}\right)-h^{1}\left(\left.D\right|_{X}\right)\right)
\end{aligned}
$$

by ordinary Riemann-Roch.
So we may assume that $D=D_{\text {fin }}$. It is now enough to show that both sides vary in the same way if we change $D$ to $D+C$, where $C$ is irreducible in $\mathfrak{X}$. Here, we may replace $K$ by a finite extension. Here, we need to be a bit careful, because our surface might no longer be regular. So we have to take a desingularization. For $F / K$, we can base change $\mathfrak{X}$ to $\mathfrak{Y}=\mathfrak{X} \times{ }_{S} S_{F}$ and then to $\mathfrak{Y}^{\prime}$. Now we might mess up with the canonical sheaf or cohomology. Let $f: \mathfrak{Y}^{\prime} \rightarrow \mathfrak{X}$. But in the semistable case, we checked that

1. $\widehat{\omega}_{\mathfrak{Y}^{\prime}}=f^{*} \widehat{\omega}$, and
2. $H^{j}(\mathfrak{X}, \mathscr{L}) \otimes_{\mathcal{O}_{K}} \mathcal{O}_{F} \cong H^{j}\left(\mathfrak{Y},\left.\mathscr{L}\right|_{\mathfrak{Y}}\right) \cong H^{j}\left(\mathfrak{Y}^{\prime},\left.\mathscr{L}\right|_{\mathfrak{Y}^{\prime}}\right)$.

So the measurable module doesn't know about the desingularization at all.

Going back to the proof, we are trying to replace $D$ by $D+C$. First assume that $C$ is horizontal. Then after base change, we may assume that $C$ is a section. That is, there is $\nu: S \rightarrow \mathfrak{X}$ such that $C=\nu(S)$. Now we have an exact sequence

$$
\begin{aligned}
0 & \rightarrow H^{0}(\mathfrak{X}, \mathscr{O}(D)) \rightarrow H^{0}(\mathfrak{X}, \mathscr{O}(D+C)) \rightarrow H^{0}\left(S, \nu^{*} \mathscr{O}(D+C)\right) \\
& \rightarrow H^{1}(\mathfrak{X}, \mathscr{O}(D)) \rightarrow H^{1}(\mathfrak{X}, \mathscr{O}(D+C)) \rightarrow 0
\end{aligned}
$$

This is the "exactness" exact sequence, so

$$
\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{O}}(D))+\chi_{K}\left(H^{0}\left(S, \nu^{*} \mathscr{O}(D+C)\right)\right)=\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{O}}(D+C))
$$

But the middle term is

$$
\widehat{\operatorname{deg}}_{K} \nu^{*} \widehat{\mathscr{O}}(D+C)=(C \cdot D+C)=(C \cdot D)+C^{2}=C \cdot D-C \cdot \widehat{\omega}
$$

by adjunction. The variation of the right hand side is going to be the same number.

## 24 March 28, 2018

Last time we were proving Riemann-Roch. We want to show

$$
\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{L}})=\frac{1}{2}\left(\overline{\mathscr{L}} \cdot \overline{\mathscr{L}} \otimes \widehat{\omega}^{\vee}\right)+\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{O}}) .
$$

We now only need to check the variation under $D$ replaced with $D+C$ with $C$ irreducible. Last time, we were dealing with the case when $C$ is horizontal, and after base change, we were checking for $C$ a section. We computed that

$$
\Delta(\mathrm{LHS})=C \cdot D-C . \widehat{\omega} .
$$

This used the exactness property of the $\|-\|_{F}$, basic properties of (-.-), and the adjunction formula for sections (due to Arakelov). Lang actually proves Riemann-Roch without assuming semistability, and here he uses some other complicated version of the adjunction formula that works in the regular setting. Anyways, the variation of the right hand side is

$$
\begin{aligned}
\Delta(\mathrm{RHS}) & =\frac{1}{2}\left((D \cdot D-\widehat{\omega})+(D \cdot C)+(C \cdot D)+C^{2}-(C \cdot \widehat{\omega})-(D \cdot D-\widehat{\omega})\right) \\
& =\frac{1}{2}(2(C \cdot D)-2(C \cdot \widehat{\omega})) .
\end{aligned}
$$

The last case is when $C$ is a fiber component $\mathfrak{X}_{s}$ for $s \in S$. This you can check it.

### 24.1 Constructing metrics on determinants of cohomology

So this is how Riemann-Roch is proved, after we have this metrics on determinants of cohomology. Today I'm going to give an outline of why you should expect this to be true.

Now we are working over $\mathbb{C}$. We have a Riemann surface $Y$ of genus $g \geq 1$, and we have $d \mu^{\mathrm{Ar}}$ on it. We want to put metrics $\|-\|_{F, \overline{\mathscr{L}}}$ on $\lambda(Y, \mathscr{L})$ that depend nicely on admissible $\overline{\mathscr{L}}$. Recall that "nicely" means respecting isometries, scaling, exactness, normalization at $\overline{\mathscr{O}}$. Note that if such metrics exist, they are unique.

Let me make some reductions:

1. It is enough to consider sheaves that look like $\mathscr{O}(D)$ (as long as we can show isometric isomorphisms).
2. It is enough to use the unique normalized (at 1) admissible metric on $\mathscr{O}(D)$, which we call $\overline{\mathscr{O}}(D)$.
3. The case $\overline{\mathscr{O}}$ is done.

Lemma 24.1. Under "exactness" and "normalization", every $\overline{\mathscr{O}}(D)$ receives a unique well-defined metric on $\lambda(Y, \mathscr{O}(D))$.

Proof. We're just starting from $\overline{\mathscr{O}}$ at adding or subtracting points. Here, you are going to need to check that the resulting metric does not depend on the order of adding points. If you write out, this will be coming from that $G(-,-)$ is symmetric.

Therefore it suffices to check that the metrics on $\lambda$ only depend on the linear equivalence class of $D$. The way Faltings proves this is beautiful. First we may a couple more of technical reductions.

1. It is enough to assume that $\operatorname{deg} D$ is fixed, say $g-1$.
2. It is enough to fix an arbitrarily large $r>0$, and a divisor $E$ with $\operatorname{deg} E=$ $r+g-1$, and consider only $D=E-\left(P_{1}+\cdots+P_{r}\right)$.

Sketch of proof. First, there is a "universal determinant of cohomology". This means that there is a certain line sheaf $\mathscr{N}$ on $Y^{r}$ together with canonical isomorphisms for $P=\left(P_{1}, \ldots, P_{r}\right) \in Y^{r}$,

$$
\left.\mathscr{N}\right|_{P} \cong \lambda\left(Y, \mathscr{O}\left(E-\left(P_{1}+\cdots+P_{r}\right)\right)\right) .
$$

This is roughly because determinants of cohomology can be taken on affine charts. Now there are metrics on the $\lambda \mathrm{s}$, and we can see how this varies explicitly in terms of $G$. Then we can write down the curvature of the metric induced on $\mathscr{N}$, which is a $(1,1)$-form.

Consider $\mathrm{Pic}_{g-1}(Y)$. This doesn't have a distinguished point, but it has a distinguished divisor

$$
\Theta=\left\{\text { locus of }[\mathscr{L}]: \operatorname{deg} \mathscr{L}=g-1, h^{0}(\mathscr{L})>0\right\}
$$

Then we have the following morphism

$$
\varphi: Y^{r} \rightarrow \operatorname{Pic}_{g-1}(Y) ; \quad P=\left(P_{1}, \ldots, P_{r}\right) \mapsto\left[\mathscr{O}\left(E-\left(P_{1}+\cdots+P_{r}\right)\right)\right]
$$

One can check that $\varphi^{*} \mathscr{O}(-\Theta) \cong \mathscr{N}$.
Finally, it is enough to prove that our metric $\mathscr{N}$ comes from a metric from $\mathscr{O}(-\Theta)$. You can write down a $(1,1)$-form on $\operatorname{Pic}_{g-1}(Y)$ such that the pullback is equal to the $(1,1)$-form on $Y^{n}$. Then you can find a metric such that the curvature is the $(1,1)$-form.

## 25 March 30, 2018

Let $Y$ be a compact Riemann surface. The key idea for constructing $\|-\|_{F,-}$ on $\lambda$ are
(1) there exists a universal $\mathscr{N} / Y^{r}$ with fibers being $\lambda$,
(2) there exist $\varphi: Y^{r} \rightarrow \operatorname{Pic}_{g-1}(Y)$ such that $\mathcal{N}$ is the pullback of the $\mathscr{O}(-\Theta)$, and
(3) there $\mathscr{O}(-\Theta)$ has a metric such that the pullback metric on $\mathscr{N}$ agrees with the fiberwise-defined metrics from $|-|_{F,-}$.

Today I will give some explanation for (1) and (2).

### 25.1 The universal bundle

We have the map $Y^{r} \rightarrow \operatorname{Pic}_{g-1}(Y)$ taking $P$ to $\mathscr{L}(P)=\mathscr{O}_{Y}\left(E-\left(P_{1}+\cdots+P_{r}\right)\right)$. We consider $r$ large enough so that $\varphi$ is surjective. (Dominance comes from Riemann-Roch and it is also proper.) Now we consider a auxiliary variety $Z=Y^{r} \times Y$. Look at the projection map $\pi_{j}: Y^{r} \rightarrow Y$ and $\Gamma_{j}$ the graph of $\pi_{j}$ in $\operatorname{Div}(Z)$. Then consider $D=\sum_{j=1}^{r} \Gamma_{j}$ and $Y^{r} \times E$, both divisors.

Write $Z_{P}=\{P\} \times Y$ (for $\left.P=\left(P_{1}, \ldots, P_{r}\right)\right)$. Then $Z_{P} \cong Y$ and consider the line bundle

$$
\mathscr{L}=\mathscr{O}_{Z}\left(Y^{r} \times E-D\right)
$$

Then

$$
\left.\mathscr{L}\right|_{Z_{P}} \cong \mathscr{L}(P)
$$

So moving the point $P$ around, we can recover all the line sheaves we care about.
Using this construction, we can define our determinant of cohomology bundle. Work locally on $Y^{r}$. For a point $P \in Y^{r}$, consider an affine neighborhood $U=\operatorname{Spec} A \ni P$.

Proposition 25.1 (Hartshorne, p.282). There is a bounded complex of finitely generated free $A$-modules

$$
L^{0} \xrightarrow{\delta_{0}} L^{1} \rightarrow \cdots \rightarrow L^{n}
$$

such that for every $A$-module $M$, we have " $L \bullet \otimes M$ computes cohomology $H^{\bullet}\left(Z_{U}, \mathscr{L}^{\bullet} \otimes_{A}\right.$ $M) "$ : we have that $H^{j}\left(Z_{U}, \mathscr{L} \otimes_{A} M\right) \cong H^{j}\left(L^{\bullet} \otimes_{A} M\right)$. Hence,

$$
H^{j}(Y, \mathscr{L}(P)) \cong H^{j}\left(Z_{P},\left.\mathscr{L}\right|_{Z_{P}}\right) \cong H^{j}\left(L^{\bullet} \otimes_{A} \kappa_{P}\right)
$$

Now define the $A$-module

$$
N=\bigotimes_{j=0}^{n}\left(\bigwedge^{d_{j}} L^{j}\right)^{\otimes(-1)^{j}}
$$

Here, a technicality is that $N$ is uniquely determined only canonical isomorphism. This is because $L$ is unique up to quasi-isomorphism, and they will
induced a true isomorphism on $N$. Localization behaves well, because localization of $L$ will do the same thing. This means that $\tilde{N}$ on $U$ glue to give a sheaf $\mathscr{N}$ on $Y^{r}$. This is what we want, because

$$
\left.\mathscr{N}\right|_{P}=N \otimes \kappa_{P}=\otimes_{j=0}^{n}\left(\bigwedge^{d_{j}}\left(L^{j} \otimes \kappa_{P}\right)\right)^{\otimes(-1)^{j}} \cong \lambda(Y, \mathscr{L}(P))
$$

The last canonical isomorphism is an exercise in linear algebra. This shows (1).
Now we want to have an isomorphism $\mathscr{N} \cong \varphi^{*} \mathscr{O}(-\Theta)$. So what is "1" for $\mathscr{N}$ ? We're assuming that $\varphi: Y^{r} \rightarrow \operatorname{Pic}_{g-1}(Y)$ is a surjection. Then for general $P \in Y^{r}$, we have $\varphi(p) \notin \Theta$.

Recall that $\operatorname{dim} H^{0}(Y, \mathscr{L}(P))-\operatorname{dim} H^{1}(Y, \mathscr{L}(P))=g-1+1-g=0$. So we have that $\varphi(P) \notin \Theta$ if and only if

$$
H^{0}(Y, \mathscr{L}(P))=H^{1}(Y, \mathscr{L}(P))=0
$$

So in this case, $\lambda(Y, \mathscr{L}(P))=\mathbb{C}$ is canonical. Here, there is a 1 . The question is, do they vary nicely so that they give a meromorphic section?

We want a section $s$ of $\mathscr{N}$ such that it agrees with these 1. Here it is useful to go back to how we constructed $\mathscr{N}$. Consider

$$
\left.s=\bigotimes_{j=0}^{n-1}\left(\bigwedge^{\mathrm{rk} \delta_{j}}\left(\delta_{j}\right)\right)^{\otimes(-1)^{j+1}}\right) \in N \otimes_{A} \operatorname{Frac}(A)
$$

This really can be considered as an element. The claim is that $\left.s\right|_{P} \in N \otimes_{A} \kappa_{P}$ for $P \notin \varphi^{-1} \Theta$ is 1. For $P \in \varphi^{-1} \Theta$ generic, we will have $\operatorname{dim} H^{0}(Y, \mathscr{L}(P))=$ $\operatorname{dim} H^{1}(Y, \mathscr{L}(P))=1$ and so you will be able to see that there is a pole of order 1.

## 26 April 2, 2018

Last time we looked at determinant of cohomology and a sketch of how Faltings proved existence of volumes. At the end of the day, we only care about $H^{0}$, but we have to have this $H^{1}$ term. The algebraic part of the existence was given by pulling back some line bundle along

$$
\varphi: Y^{r} \rightarrow \operatorname{Pic}_{g-1}(Y) ; \quad P \mapsto \mathscr{L}(P)=\mathscr{O}\left(E-P_{1}-\cdots-P_{r}\right)
$$

Then we get an isomorphism $\mathscr{N} \cong \varphi^{*} \mathscr{O}(-\Theta)$.

### 26.1 Analytic input to construction of metric

Now for the analytic part, we look at two metrics on both line bundles, and show that the curvature is equal to the pullback of the curvature. Consider $\mathscr{L}=\mathscr{O}(E)$ so that we have

$$
\mathscr{L}(P)=\mathscr{L} \otimes \mathscr{O}\left(-P_{1}-\cdots-P_{r}\right)=\mathscr{L}\left(-P_{1}-P_{2}-\cdots-P_{r}\right)
$$

By definition, we have an isometric isomorphism

$$
\begin{aligned}
\lambda(\overline{\mathscr{L}}) & =\left.\lambda\left(\overline{\mathscr{L}}\left(-P_{1}\right)\right) \otimes \overline{\mathscr{L}}\right|_{P_{1}}=\left.\left.\lambda\left(\overline{\mathscr{L}}\left(-P_{1}-P_{2}\right)\right) \otimes \overline{\mathscr{L}}\left(-P_{1}\right)\right|_{P_{2}} \otimes \overline{\mathscr{L}}\right|_{P_{1}} \\
& =\left.\left.\left.\lambda\left(\overline{\mathscr{L}}\left(-P_{1}-P_{2}\right)\right) \otimes \overline{\mathscr{O}}\left(-P_{1}\right)\right|_{P_{2}} \otimes \overline{\mathscr{L}}\right|_{P_{2}} \otimes \overline{\mathscr{L}}\right|_{P_{1}}=\cdots \\
& =\left.\left.\lambda\left(\overline{\mathscr{L}}\left(-P_{1}-\cdots-P_{n}\right)\right) \otimes \bigotimes_{i<j} \overline{\mathscr{O}}\left(-P_{i}\right)\right|_{P_{j}} \otimes \bigotimes_{j=1}^{r} \overline{\mathscr{L}}\right|_{P_{j}}
\end{aligned}
$$

Then if you take curvature of $\mathscr{N}$, the contribution of $\left.\overline{\mathscr{O}}\left(-P_{i}\right)\right|_{P_{j}}$ is going to be some Green function. So if you work out,

$$
\operatorname{curv}(\mathscr{N})=-\sum_{j=1}^{r} p_{j}^{*} \operatorname{curv}(\overline{\mathscr{L}})+\sum_{i \neq j} \partial \bar{\partial} \log G\left(P_{i}, P_{j}\right)
$$

But $\mathscr{L}$ is an admissible metric on $\mathscr{O}(E)$. So its curvature is the degree times $d \mu^{\mathrm{Ar}}$.

Note that if a certain metric on $\mathscr{O}(-\Theta)$ works for given choice $(E, r)$, the same works for all $(E, r)$. To see this, consider $\left(E_{1}, r_{1}\right)$ and $\left(E_{2}, r_{2}\right)$. Then for $E=\max \left(E_{1}, E_{2}\right)$, we will have something like

$$
\mathscr{O}\left(E_{1}-\sum P_{j}^{(1)}\right)=\mathscr{O}\left(E-\sum P_{j}\right)=\mathscr{O}\left(E_{2}-\sum P_{j}^{(2)}\right)
$$

### 26.2 Averages of $G$

Let $Y$ be a compact Riemann surface with $g \geq 1$. The Green functions is sort of measuring distance. Suppose we have $x_{1}, \ldots, x_{n} \in Y$ all different. How big can the sum

$$
\sum_{i \neq j} \log G\left(x_{i}, x_{j}\right)
$$

be? The trivial upper bound is $c_{Y} n^{2}$ because the Green function is globally bounded above. Naively, we expect to see some cancellation because there can't be too many points away from each other.
Theorem 26.1 (Faltings, Elkies). There is a constant $c>0$, only depending on $Y$, such that for all $n \geq 2$ and $x_{1}, \ldots, x_{n} \in Y$, we have

$$
\sum_{i \neq j} \log G\left(x_{i}, x_{j}\right) \leq c n \log n
$$

Proof. Recall that $\Delta: C^{\infty}(Y, \mathbb{C})^{0} \rightarrow \mathbb{C}^{\infty}(Y, \mathbb{C})^{0}$ was defined as $(\pi i)^{-1} \partial \bar{\partial} \varphi=$ $(\Delta \varphi) d \mu^{\mathrm{Ar}}$. Note that we computed

$$
\partial \bar{\partial} f=\frac{-i}{2}\left(f_{x x}+f_{y y}\right) d x \wedge d y
$$

So in reality, our Laplacian is negative of the ordinary Laplacian. Thus the eigenvalues of our $\Delta$ are positive, $0<\lambda_{1} \leq \lambda_{2} \leq \cdots$. Let us write the eigenfunctions as $\varphi_{1}, \varphi_{2}, \ldots$ Recall also that we had the operator $\Gamma: C^{\infty}(Y, \mathbb{C})^{0} \rightarrow C^{\infty}(Y, \mathbb{C})^{0}$ given by

$$
(\Phi \psi)(p)=\int_{Y}(-g(P, Q)) \psi(P) d \mu^{\mathrm{Ar}}(Q)
$$

and checked that this is the inverse of $\Delta$. Now we can express the kernel using the eigenfunctions. As distributions, we should have

$$
g(P, Q)=-\sum_{n \geq 1} \frac{1}{\lambda_{n}} \varphi_{n}(P) \overline{\varphi_{n}(Q)}
$$

But distributions, this is not good enough, so for $t>0$ we define

$$
g_{t}(P, Q)=-\sum_{n \geq 1} \frac{e^{-t \lambda_{n}}}{\lambda_{n}} \varphi_{n}(P) \overline{\varphi_{n}(Q)}
$$

For $t>0$, this gives an honest $C^{\infty}$ function, because there are estimates on the growth of $\lambda_{n}$. Now we state some facts.
(1) There exists an $A=A(Y)>0$ such that for all $x \neq y$ and $t>0$, we have $g_{t}(x, y)+A t \geq g(x, y)$.
(2) There exist $B=B(Y)$ and $C=C(Y)$ such that for all $x \in Y$ and $t>0$, we have $g_{t}(x, x) \leq B \log t+C$.
(3) There exist $b=b(Y)$ and $c=c(Y)$ such that for all $x \in Y$ and $t>0$, we have $g_{t}(x, x)>b \log t-c$.
(In fact, $g_{t}(x, x)=c^{\prime} \log t+O_{x}(1)$, but we don't need this.) So we have

$$
\begin{aligned}
\sum_{i \neq j}\left(g\left(x_{i}, x_{j}\right)-A t\right) & \leq \sum_{i \neq j} g_{t}\left(x_{i}, x_{j}\right)=-\sum_{i \neq j} \sum_{k \geq i} \frac{e^{-\lambda_{k}}}{\lambda_{k}} \varphi_{k}\left(x_{i}\right) \overline{\varphi_{k}\left(x_{j}\right)} \\
& =\sum_{k \geq 1} \frac{e^{-t \lambda_{k}}}{\lambda_{k}}\left(\sum_{i=1}^{n}\left|\varphi_{k}\left(x_{i}\right)\right|^{2}-\left|\sum_{i=1}^{n} \varphi_{k}\left(x_{i}\right)\right|^{2}\right) \leq-\sum_{i=1}^{n} g_{t}\left(x_{i}, x_{i}\right) .
\end{aligned}
$$

Then we have

$$
\sum_{i \neq j} g\left(x_{i}, x_{j}\right) \leq \sum_{i=1}^{m}\left(-g_{t}\left(x_{i}, x_{i}\right)+n A t\right) \leq n(-b \log t+c+n A t)
$$

Take $t=\frac{1}{n}$.
Next time we are going to make use of this.

## 27 April 4, 2018

If $\mathbb{R}^{n}$, if there is a norm $\|-\|$ and a lattice $\Lambda$, there is a numerical criterion for the ball $B=\left\{x \in \mathbb{R}^{n}:\|x\| \leq 1\right\}$ to satisfy $B \cap \Lambda \supsetneq\{0\}$. In particular,

$$
\operatorname{Vol}(B) \geq 2^{n} \operatorname{Vol}\left(\mathbb{R}^{n} / \Lambda\right)
$$

is fine. We are going to use this to get a numerical criterion for line sheaves to be effective.

Let $Y$ be a compact Riemann surface with $g=g_{Y} \geq 1$ and $\overline{\mathscr{L}}$ be an admissible metrized line sheaf. We have a norm on $H^{0}(Y, \overline{\mathscr{L}})$ defined by

$$
\|s\|_{L^{2}}^{2}=\int_{Y}\|s\|^{2}(P) d \mu^{\mathrm{Ar}}(P)
$$

We can define (Falting's) volume

$$
V(\overline{\mathscr{L}})=\operatorname{Vol}_{F}\left(\left\{x \in H^{0}(Y, \mathscr{L}):\|s\|_{L^{2}} \leq 1\right\}\right)
$$

Lemma 27.1. If $\operatorname{deg}_{Y}(\mathscr{L}) \geq 2 g-1$, then $V(\overline{\mathscr{L}})$ is well-defined and is independent of the norm of $\overline{\mathscr{L}}$.

We write this $V(\mathscr{L})$.
Theorem 27.2. There exists a $c>0$ depending only on $Y$ such that for all $\overline{\mathscr{L}}$ of degree $d \geq 2 g-1$,

$$
V(\mathscr{L})>\exp (-c d \log d)
$$

Proof. Assume $\mathscr{L}=\mathscr{O}(E)$, and write $d=r+g-1$ with $r \geq g$. We have this map

$$
\varphi: Y^{r} \rightarrow \operatorname{Pic}_{g-1}(Y) ; \quad P \mapsto \mathscr{O}(E) \otimes \mathscr{O}\left(-\sum P_{j}\right)
$$

that is a surjection. We define $U=Y^{r}-\varphi^{*} \Theta$. Then for $P \in U$, the determinant of cohomology is $\lambda\left(Y, \mathscr{L} \otimes \mathscr{O}\left(-\sum P_{j}\right)\right)=\mathbb{C}$ contains 1. Define

$$
\omega(\overline{\mathscr{L}}, P)=\|1\|_{F, \mathscr{L} \otimes \mathscr{O}\left(-\sum_{j} P_{j}\right)} .
$$

Note that $\omega(\mathscr{L}, P)>c_{1}>0$ is bounded below, with $c_{1}$ independent of $\overline{\mathscr{L}}, r, P$, because we had $\|-\|_{F, \mathscr{N}}=\varphi^{*}\|-\|_{-\Theta}$. So the norm of 1 is given by pullback of 1 , and $\operatorname{div}(1)=-\Theta$. This shows that there are poles at $\Theta$, and no zeros.

What we want to know is $\lambda(Y, \mathscr{L})$. So we write

$$
\lambda(Y, \mathscr{L})=\lambda\left(Y, \mathscr{L} \otimes \mathscr{O}\left(-\sum_{j} P_{j}\right)\right) \otimes_{\mathbb{C}}\left(\left.\bigotimes_{i<j} \overline{\mathscr{O}}\left(-P_{j}\right)\right|_{P_{j}}\right) \otimes_{\mathbb{C}}\left(\left.\otimes_{i=1}^{r} \mathscr{L}\right|_{P_{i}}\right)
$$

Here, evaluation gives an isomorphism

$$
\mathrm{ev}:\left.H^{0}(Y, \mathscr{L}) \cong \bigoplus_{j=1}^{r} \mathscr{L}\right|_{P_{j}}
$$

and taking determinant gives $\left.\operatorname{det} H^{0}(Y, \mathscr{L}) \cong \bigotimes_{i=1}^{r} \mathscr{L}\right|_{P_{i}}$. But there is some distortion in volume, and the other terms is what measures this.

Now we can estimate

$$
\mu_{F}(B)=\frac{w(\overline{\mathscr{L}}, P)^{2}}{\prod_{i \neq j} G\left(P_{i}, P_{j}\right)} \nu(\operatorname{ev}(B))>\exp \left(-c_{2}-c_{3} r \log r\right) \nu(\operatorname{ev}(B))
$$

Then

$$
\nu(\operatorname{ev}(B))>\exp \left(-c_{5} r \log r\right)\left\|f_{1} \wedge \cdots \wedge f_{r}\right\|_{r}^{2}(P)
$$

where $s_{1}, \ldots, s_{r}$ is an $L^{2}$-orthonormal basis over $H^{0}(\mathscr{L})$ and $f_{j}=\left(\pi_{1}^{*} s_{j}, \ldots, \pi_{n}^{*} s_{j}\right) \in$ $H^{0}\left(Y^{r}, \bigoplus \pi_{j}^{*} \mathscr{L}\right)$.

We now average this inequality over $P_{i}$, i.e., take the integral

$$
\int_{Y^{r}}(-) d \mu^{\mathrm{Ar}}\left(P_{1}\right) \cdots d \mu^{\mathrm{Ar}}\left(P_{r}\right)
$$

Then we actually have $\int_{Y}\left\|f_{1} \wedge \cdots \wedge f_{r}\right\|^{2}(P) d P=r!>1$.
Theorem 27.3. Let $\overline{\mathscr{L}}$ is an admissible line sheaf on $\widehat{\mathfrak{X}}$. Suppose $\overline{\mathscr{L}}^{2}>0$ and for any fiber $F$ (or there exists an $F$ such that) $\overline{\mathscr{L}} . F>0$. Then there exist $n_{0}$ such that for all $n \geq n_{0}$, the line sheaf $\overline{\mathscr{L}}^{\otimes n}$ is effective, i.e., there exists an Arakelov divisor $D_{n} \geq 0$ such that $\overline{\mathscr{L}}^{\otimes n} \cong \overline{\mathscr{O}}\left(D_{n}\right)$.

Proof. We want for $n \gg 1$, some $0 \neq s \in H^{0}\left(\mathfrak{X}, \mathscr{L}^{\otimes n}\right)$ such that for all $\sigma: K \rightarrow$ $\mathbb{C}$, the integral $\int_{X_{\sigma}} \log \|s\|_{\sigma}(P) d \mu_{\sigma}^{\mathrm{Ar}}(P) \leq 0$ as well. Since $\log$ is concave, it is enough to get

$$
\int_{X_{\sigma}}\|s\|_{\sigma}^{2}(P) d \mu^{\mathrm{Ar}}(P) \leq 1
$$

Now we can ensure this by Minkowski. Take the lattice $\Lambda=H^{0}\left(\mathfrak{X}, \mathscr{L}^{\otimes n}\right)$. Use the Faltings's volume. The volume of the unit ball is then greater than $-\exp (c n \log n)$ (because degree is linear in $n$ ). We also have $H^{1}\left(X, \mathscr{L}^{\otimes n}\right)=0$ for $n \gg 1$ because $\overline{\mathscr{L}} . F>0$. This means that $H^{1}$ only contributes as torsion. Therefore the covolume of $\Lambda$ is at most

$$
<\exp \left(-\chi_{K}(\widehat{\mathfrak{X}}, \overline{\mathscr{L}})\right)<\exp \left(-\frac{n^{2}}{2} \overline{\mathscr{L}}^{2}+c n\right)
$$

Here, $\overline{\mathscr{L}}^{2}>0$ so we can apply Minkowski.

## 28 April 6, 2018

Today we study $D^{2}$.

### 28.1 Fibered divisors

For $s \in S$, let $V_{s}$ be the group generated by the components of $\mathfrak{X}_{s}$, as a subgroup of $\operatorname{Div}(\widehat{\mathfrak{X}})$. For $\sigma: K \rightarrow \mathbb{C}$, we will have $V_{\sigma}=\mathbb{R} F_{\sigma}$. For $s \in S$, we write $F_{s}$ the fiber corresponding to $\mathfrak{X}_{s}$.

Write $\widehat{S}=S \cup\{\sigma: K \rightarrow \mathbb{C}\}$. Then we consider

$$
V=\bigoplus_{p \in \mid \hat{S}} \mid V_{p} .
$$

(We're ignoring the 0 ideal.)
Proposition 28.1. $(-,-)$ is negative semi-definite on each $V_{p}$, and on $V$. Also, $V_{p}$ are orthogonal to each other. In each $V_{p}$, the only Arakelov divisors with square equal to 0 are multiples of $F_{p}$.

Proof. This follows from the intersection parings and properties of $i_{s}$.
Lemma 28.2. Let $D$ be a (Arakelov) divisor with $\operatorname{deg}_{X} D>0$. Then given any $F_{p}$, for $n \gg 1$ we have

$$
\left(D+n F_{p}\right)^{2}>0
$$

Proof. We have $\left(D+n F_{p}\right)^{2}=D^{2}+2 n D F_{p}+n^{2} F_{p}^{2} \geq D^{2}+2 n\left(\operatorname{deg}_{X} D\right) \log 2$. Here, the $\log 2$ term comes from the residue term. (This is true even for $p$ infinity, because $1>\log 2$.)

Theorem 28.3 (special case of Ragnaud). There is an $S$-scheme $\mathcal{P} i c_{g-1}(\mathfrak{X} / S)$ locally of finite type with a universal line sheaf $\mathscr{L}$ on $\mathfrak{X} \times_{S} \mathcal{P} c_{g-1}(\mathfrak{X} / S)$ which is the moduli space of line sheaves of $\mathfrak{X}$ of degree $\operatorname{deg}_{X}=g-1$.

This can only be locally of finite type, because if your curve has many components, the only condition is that the degrees on each of the components add up to $g-1$. So there are infinitely many discrete possibilities. The reason we are only looking at $g-1$ is because the Faltings volume is going to be independent of the metric. By Riemann-Roch, we have $h^{0}-h^{1}=\operatorname{deg}_{X}+1-g=0$.
(1) There exists a $\tilde{\Theta} \in \operatorname{Div}\left(\mathcal{P} i c_{g-1}(\mathfrak{X} / S)\right)$ which extends the $\Theta$ on $X_{\sigma}$ and $\operatorname{Pic}_{g-1}\left(X_{\sigma}\right)$.
(2) Consider $\mathscr{F}$ a line sheaf on $\mathfrak{X}$ with $\operatorname{deg}_{X} \mathscr{F}=g-1$. Then there is a section $x \in \mathcal{P} c_{g-1}(\mathfrak{X} / S)(S)$ such that $\left.\mathscr{L}\right|_{\mathfrak{X} \times_{S} x}$ gives $\mathscr{F}$ on $\mathfrak{X} \cong \mathfrak{X} \times_{S} x$.
(3) There is a universal determinant of cohomology using $\mathscr{L}$. Then $\lambda(\mathscr{L}) \cong$ $\mathscr{O}(-\tilde{\Theta})$ with the Riemann-Roch metrics. Moreover, for all $\mathscr{F}$ and $x$ as in (2), we have

$$
\widehat{\operatorname{deg}}_{K} x^{*} \lambda(\mathscr{L})=\chi_{K}(\widehat{\mathfrak{X}}, \mathscr{F})
$$

(Here, we can use any admissible metric on $\mathscr{F}$ because the degree is $g-1$.)

### 28.2 Intersection pairing and Néron-Tate heights

Consider $\operatorname{Jac}(X / K)=\operatorname{Pic}^{0}(X / K)$. Note that $F_{p} \perp F_{p}$ for all $p$, and also $V+V^{\perp} \subsetneq \operatorname{Div}(\widehat{\mathfrak{X}})$ because $D \in V^{\perp}$ implies $\operatorname{deg}_{X} D=0$.

Choose $E \in \operatorname{Div}(\widehat{\mathfrak{X}})$ of degree $\operatorname{deg}_{X} E=g-1$. Take $D \in V^{\perp}$ and let $\mathcal{P}$ be those with the same intersection pattern as before for $E$. Then for all $n \geq 1$, we have $[E+n D] \in \mathcal{P}(S)$. So we have sections $x_{n}: S \rightarrow \mathcal{P}$. We have

$$
\widehat{\operatorname{deg}}_{K} x_{n}^{*} \lambda(\mathscr{L})=\chi_{K}(\widehat{\mathcal{X}}, \mathscr{O}(E+n D))=\cdots=\frac{D^{2}}{2} n^{2}+O(n)
$$

by Riemann-Roch. Also, this degree is

$$
\begin{aligned}
\widehat{\operatorname{deg}}_{K} x_{n}^{*} \lambda(\mathscr{L}) & =\widehat{\operatorname{deg}}_{K} x_{n}^{*} \overline{\mathscr{O}}(-\tilde{\Theta}) \\
& =-[K: \mathbb{Q}] h_{\operatorname{Pic}_{g-1}(X / K), \Theta}\left(x_{n}\right)+O(1) \\
& =-[K: \mathbb{Q}] \hat{h}_{J_{X}, \Theta}([D]) n^{2}+O(n)
\end{aligned}
$$

where $h$ is the Weil height and $\hat{h}$ is the canonical Néron-Tate height.
Theorem 28.4. For all $D \in V^{\perp}$, (so that $\operatorname{deg}_{X} D=0$ and $[D] \in \operatorname{Jac}_{X / K}(K)$ ) we have

$$
D^{2}=-2[K: \mathbb{Q}] \widehat{h}_{J_{X}(\theta)}([D]) \leq 0
$$

## 29 April 9, 2018

Last time we proved this theorem.
Theorem 29.1 (Faltings-Hriljac). For all $D \in V^{\perp}$, we have

$$
D^{2}=-2[K: \mathbb{Q}] \hat{h}_{J_{X}, \Theta}([D])
$$

### 29.1 Hodge index theorem

Consider

$$
\operatorname{Num}(\widehat{\mathfrak{X}})_{\mathbb{R}}=\operatorname{Div}(\widehat{\mathfrak{X}})_{\mathbb{R}} / \text { numerical equivalence }
$$

where $\operatorname{Div}(\widehat{\mathfrak{X}}])_{\mathbb{R}}$ is the Arakelov divisors of $\mathfrak{X}$ with $\mathbb{R}$-coefficients. Here, the intersection pairing extends linearly. Then $\operatorname{Num}(\widehat{\mathcal{X}})_{\mathbb{R}}$ is a $\mathbb{R}$-vector space with with a non-degenerate bilinear form. By no means it is clear that it is finitedimensional.

Now consider

$$
V_{s, \mathbb{R}}=V_{s} \otimes_{\mathbb{Z}} \mathbb{R}, \quad V_{\sigma, \mathbb{R}}=V_{\sigma}, \quad V_{\mathbb{R}}=\bigoplus_{\bullet} V_{\bullet, \mathbb{R}} \subseteq \operatorname{Div}_{\mathbb{R}}(\widehat{\mathfrak{X}})
$$

for $s \in|S|$. This maps to $\operatorname{Num}(\widehat{\mathfrak{X}})_{\mathbb{R}}$. Similarly, we have $V_{\mathbb{R}}$.
Fix $E \in \operatorname{Div}(\widehat{\mathfrak{X}})$ with $E^{2}>0$ and $\operatorname{deg}_{X}(E)>0$. (For example, take $E=$ $D_{P}+n F_{\sigma}$ for $P \in X\left(K^{\text {alg }}\right)$ and $n \gg 1$.) We want to see the contribution of $E$ in Num. This cannot be in $V$ because $\operatorname{deg}_{X}(E)>0$ and it cannot be in $V^{\perp}$ because its self-intersection is positive.

Take any $D \in \operatorname{Div}(\widehat{\mathfrak{X}})_{\mathbb{R}}$. We see that $D$ modulo $\mathbb{R} E$ can be represented in the following way:

$$
E \equiv D^{\prime}+(\text { fiber components }) \text { with } \operatorname{deg}_{X}=0, \quad D^{\prime} \in V^{\perp}
$$

This is because we can first look at $D-\frac{\operatorname{deg}_{X} D}{\operatorname{deg}_{X} E} E$, and then this is some linear algebra thing. So we have the following surjectivity:

$$
\nu: \mathbb{R} E+V_{\mathbb{R}}+V_{\mathbb{R}}^{\perp} \rightarrow \operatorname{Num}(\widehat{\mathfrak{X}})_{\mathbb{R}}
$$

Theorem 29.2 (Hodge index theorem). $\operatorname{dim}_{\mathbb{R}} \operatorname{Num}(\widehat{\mathfrak{X}})_{\mathbb{R}}$ is finite and the intersection pairing $(-,-)$ has signature $(1,-1, \ldots,-1)$. Moreover, the number of -1 is

$$
1+\operatorname{rank} J_{X}(K)+\sum_{s \in S}\left(\operatorname{dim}_{\mathbb{R}} V_{s, \mathbb{R}}-1\right)
$$

Proof. First we see that $E$ gives +1 . Now it is enough to understand the image of $V+V^{\perp}$. We see that

$$
\nu\left(V_{\mathbb{R}}\right)=\bigoplus_{s} V_{s, \mathbb{R}}^{0} \oplus \mathbb{R} F_{\bullet}
$$

for your favorite $F_{\bullet}$. On $\nu\left(V_{\mathbb{R}}^{\perp}\right)$, it follows from $D^{2}=-2[K: \mathbb{Q}] \hat{h}_{\mathrm{NT}}([D])$. Because the intersection paring on $\nu\left(V_{\mathbb{R}}^{\perp}\right)$ is non-degenerate, its should come from $J_{X}$. Then the intersection pairing is negative-definite on a codimension 1 space, and is positive at some point.

Looking back, the non-trivial part of this is the $V_{\mathbb{R}}^{\perp}$, but this is just the Jacobian. Here are some applications. Consider $g_{X} \geq 2$ and let us look at $\widehat{\omega}$ again.

Theorem 29.3. Let $D \in \operatorname{Div}(\widehat{\mathfrak{X}})$ and suppose $D \geq 0$. Then

$$
D \cdot \widehat{\omega} \geq \frac{\widehat{\omega}^{2}}{4 g(g-1)} \operatorname{deg}_{X}(D)
$$

Proof. We may assume that $D$ is irreducible and not formal. First consider the case when $D$ is a fiber component. Then $\operatorname{deg}_{X}(D)=0$. By geometric adjunction, we have

$$
-1 \leq g_{D}-1=\frac{1}{2 \log \# \kappa_{S}}\left(D^{2}+D \cdot \widehat{\omega}\right)
$$

If $D . \widehat{\omega}<0$, then we would have $i_{s}(D, D)>-2$. This means that either $D$ is $F_{s}$ (in which case $g_{D}>0$ and is not possible) or $D^{2}=-1$ (in which case it contradicts that $\mathfrak{X}$ is semi-stable).

Now take $D$ horizontal. Say (up to base change) that $D$ is a section. Then

$$
\operatorname{deg}_{X}((2 g-2) D-\widehat{\omega})=0
$$

implies that it comes from $V+V^{\perp}$ and so $((2 g-2) D-\widehat{\omega})^{2} \leq 0$. Expanding and using $D^{2}=-D . \widehat{\omega}$ gives a contradiction.

Provided that $\widehat{\omega}^{2} \geq 0$, this is going to be a highly nontrivial arithmetic information.

## 30 April 11, 2018

Last time we had this theorem that

$$
\widehat{\omega} \cdot D \geq \frac{\widehat{\omega}^{2}}{4 g(g-1)} \operatorname{deg}_{X} D
$$

For the horizontal divisor, we had $\operatorname{deg}_{X}((2 g-2) D-\widehat{\omega})=0$ and so we get that $(2 g-2) D-\widehat{\omega}$ has self-intersection number not positive. Then we can collect terms by adjunction, we get

$$
4 g(g-1) D \cdot \widehat{\omega} \geq \widehat{\omega}^{2}
$$

### 30.1 Positivity of $\widehat{\omega}^{2}$

Theorem 30.1 (for $g \geq 2$ ). We have $\widehat{\omega}^{2} \geq 0$.
Proof. We have $\operatorname{deg}_{X} \widehat{\omega}=2 g-2>0$. Taking $F=F$ • some fiber, we know that for rational $\lambda \gg 1$ we have

$$
\operatorname{deg}_{X}(\widehat{\omega}+\lambda F)>0, \quad(\widehat{\omega}+\lambda F)^{2}>0
$$

By application of Riemann-Roch, we know that $\widehat{\omega}+\lambda F$ is linearly equivalent to some effective $\mathbb{Q}$-Arakelov divisor. (We did this by using Elkies's result and Minkowski and stuff.) So by the previous theorem,

$$
\widehat{\omega}^{2}+\lambda \widehat{\omega} \cdot F=(\widehat{\omega}, \widehat{\omega}+\lambda F) \geq \frac{\widehat{\omega}^{2}}{4 g(g-1)} \operatorname{deg}_{X}(\widehat{\omega}+\lambda F)=\frac{\widehat{\omega}^{2}}{2 g}
$$

We want a good choice of $\lambda \in \mathbb{Q}$ with $(\widehat{\omega}+\lambda F)^{2}>0$, because this is all we used. This condition is equivalent to

$$
0<(\widehat{\omega}+\lambda F)^{2}=\widehat{\omega}^{2}+2 \lambda(\widehat{\omega} \cdot F) .
$$

So it is enough to take $\lambda \in \mathbb{Q}$ such that $\lambda(\widehat{\omega} \cdot F)>-\frac{1}{2} \widehat{\omega}^{2}$. This implies

$$
\widehat{\omega}^{2}-\frac{1}{2} \widehat{\omega}^{2} \geq \frac{\widehat{\omega}^{2}}{2 g}
$$

and so $\widehat{\omega}^{2} \geq 0$.

### 30.2 More on fiber divisors

Let me give more details on the fiber divisors we talked about last time. Recall that

$$
V_{s}^{0}=\left\{\sum \alpha_{j} C_{j}: F_{s}=\sum C_{j}, \sum \alpha_{j}=0\right\}
$$

We can also consider $V_{s, \mathbb{Q}}^{0}$. We also defined $V^{0}=\bigoplus_{s} V_{s}^{0}$ and similarly $V_{\mathbb{Q}}^{0}$. WE also denote

$$
\operatorname{Div}(\widehat{\mathfrak{X}})^{0}=\left\{D \in \operatorname{Div}(\widehat{\mathfrak{X}}): \operatorname{deg}_{X}(D)=0\right\}
$$

Proposition 30.2. The following equivalent properties hold:
(1) for all $D \in \operatorname{Div}(\widehat{\mathfrak{X}})^{0}$, there exists a unique $E \in V_{\mathbb{Q}}^{0}$ such that $D-E \in V^{\perp}$. Furthermore, the rule $\Phi: \operatorname{Div}(\widehat{\mathfrak{X}})^{0} \rightarrow V_{\mathbb{Q}}^{0}$ given by $D \mapsto E$ is a group homomorphism.
(2) Let $s \in|S|$. For all $D \in \operatorname{Div}(\widehat{\mathcal{X}})^{0}$, there exists a unique $E \in V_{s, \mathbb{Q}}^{0}$ such that $D-E \in V_{s, \mathbb{Q}}^{\perp}$. Furthermore, $\Phi_{s}: \operatorname{Div}(\widehat{\mathfrak{X}})^{0} \rightarrow V_{s, \mathbb{Q}}^{0}$ is a group homomorphism.

Proof. The two statements are equivalent because we can take $\Phi=\sum_{s} \Phi_{s}$ and $V_{\mathbb{Q}}^{0}=\bigoplus_{s} V_{s, \mathbb{Q}}^{0}$. So let us prove (2).

If $s$ is such that $\mathfrak{X}_{s}$ is irreducible, then $V_{s, \mathbb{Q}}^{0}=(0)$. If this is not the case, take $F_{s}=\sum C_{j}$ where $C_{j}$ are irreducible components. We can use $i_{s}$ instead of $(-,-)$, and this is integer-valued. Consider

$$
W_{\mathbb{Q}}^{0}=\left\{a \in \mathbb{Q}^{r}: \sum a_{j}=0\right\} \cong V_{s, \mathbb{Q}}^{0}
$$

For $D \in \operatorname{Div}(\widehat{\mathfrak{X}})^{0}$, define

$$
b(D)=\left(b_{j}(D)\right)_{j}, \quad b_{j}(D)=\left\langle C_{j}, D\right\rangle \in \mathbb{Z}
$$

But we have $D \cdot F_{s}=0$ and so $b(D) \in W_{\mathbb{Q}}^{0}$. Now we want to define $\Phi(D)=a(D)$ such that for all $\gamma \in W_{\mathbb{Q}}^{0}$,

$$
\gamma^{t}\left[\left\langle C_{i}, C_{j}\right\rangle\right]_{i j} a(D)=\gamma^{t} b(D)
$$

Now the key observation is that $\langle-,-\rangle$ is negative definite on $V_{s, \mathbb{Q}}^{0}$ which is isomorphic to $W_{\mathbb{Q}}^{0}$. So we can look at the equation as defined on $V_{s, \mathbb{Q}}^{0}$ and invert the matrix $[\langle-,-\rangle]$.

## 31 April 13, 2018

We have covered the basics of Arakelov theory.

### 31.1 Small points

Let $X / K$ be a curve with $g_{X} \geq 2$. Then $\operatorname{dim} J_{X}=g$ and let $\mathscr{L}$ be a line sheaf on $X$. Assume that $\operatorname{deg}_{X} \mathscr{L}=n \geq 1$. Then we get $j_{\mathscr{L}}: X \rightarrow J_{X}$ and consider $P \mapsto[\mathscr{L} \otimes \mathscr{O}(-n P)]$. If $\mathscr{L}=\Omega_{X / K}^{1}$, we can the canonical map

$$
j=j_{\Omega_{X / K}^{1}}: X \rightarrow J_{X}
$$

These are not in general embeddings.
On the Jacobian, there is the Néron-Tate height

$$
\hat{h}=\hat{h}_{J_{X}, \Theta}: J_{X}(\bar{K}) \rightarrow \mathbb{R}
$$

that satisfies

- $\hat{h}(P) \geq 0$, with equality if and only if $P \in J_{X}(\bar{K})$ is torsion,
- $\hat{h}$ is quadratic and so there is a pairing $\langle-,-\rangle_{\mathrm{NT}}$.

Because of Mordell-Weil, we get that

$$
J_{X}(L) \otimes_{\mathbb{Z}} \mathbb{R}
$$

becomes a Euclidean finite-dimensional space with $\hat{h}$.
Here are some conjecture that have been proven.
Conjecture 31.1 (Manin-Mumford). $j_{\mathscr{L}}(X) \cap J_{X}(\bar{K})_{\text {tor }}$ is finite.
This is a bold conjecture, because torsion points are everywhere (dense in the analytic topology). Here is something stronger.

Conjecture 31.2 (Bogomolov). There exists an $\epsilon_{0}=\epsilon_{0}(X / K, \mathscr{L})>0$ such that $j_{\mathscr{L}}(X) \cap\left\{P \in J_{X}(\bar{K}): \hat{h}(P) \leq \epsilon_{0}\right\}$ is finite.

The Manin-Mumford conjecture was proved by Raynaud in 1983 p-adically. For the Bogomolov conjecture, Szpiro figured in 1990 that one can use Arakelov theory to do this. First assume $X$ has potentially good reduction, with $\widehat{\omega}^{2}>0$. Then the Bogomolov conjecture holds for $X$. In 1993, S-W Zhang, who was a student of Szpiro, generalized Szpiro's proof to all possible reductions and proved that if $X$ does not have potentially good reduction, then $\widehat{\omega}^{2}>0$. In 1998, Ullmo proved Bogomolov's conjecture and more using Arakelov theory and equidistribution of small points. So we now know that $\widehat{\omega}^{2}>0$.

Theorem 31.3 (Szpiro). Assume $g_{X} \geq 2$ and $X / K$ has an everywhere good reduction. Let $\mathfrak{X} / S$ be the minimal regular model of $X$ and let $\widehat{\omega}=\widehat{\omega}_{\mathfrak{X} / S}^{\mathrm{Ar}}$. Let $\mathscr{L}$ be a line sheaf on $X / K$ of degree $n \geq 1$. Define $c_{\mathscr{L}}=\hat{h}\left(\left[\left(\Omega_{X / K}^{1}\right)^{\otimes n} \otimes\right.\right.$ $\left.\left.\left(\mathscr{L}^{\vee}\right)^{\otimes(2 g-2)}\right]\right) \geq 0$.
(1) If $c_{\mathscr{L}}>0$, then Bogomolov conjecture for $j_{\mathscr{L}}: X \rightarrow J_{X}$ holds.
(2) If $c_{\mathscr{L}}=0$ and $\widehat{\omega}^{2}>0$, then the Bogomolov conjecture for $j_{\mathscr{L}}$ holds.

Therefore for curves with good reduction, $\widehat{\omega}^{2}>0$ implies Bogomolov's conjecture.

Note that $c_{\mathscr{L}}=0$ just means that there are $m, \ell \geq 0$ such that $\mathscr{L}^{\otimes n} \cong$ $\left(\Omega^{1}\right)^{\otimes l}$. Also, Bogomolov's conjecture holds for $j_{\mathscr{L}}$ if and only if it holds for $j_{\mathscr{L} \otimes \iota}$ for some $\ell \geq 1$. So for (2), I can only focus on $\mathscr{L}=\Omega^{1}$.

Proof of (2). Let's get started. Suppose that the Bogomolov conjecture fails for $j=j_{\Omega^{1}}$. We want to show that $\widehat{\omega}^{2}=0$, or just $\widehat{\omega}^{2} \leq 0$. Actually for suitable $L / K$ we will bound $\widehat{\omega}_{L}^{2}$. By Cauchy-Schwartz, we note that $\left|\langle x, y\rangle_{\mathrm{NT}}\right| \leq$ $\hat{h}(x) \hat{h}(y)$.

Let $\epsilon>0$. Take $P \in X(\bar{K})$ with $\hat{h}(j(P))<\epsilon$. By Mordell-Weil, we know that as $\epsilon \rightarrow 0$ we have $\left[\kappa_{P}: K\right] \rightarrow \infty$. Let $P_{1}=P, P_{2}, \ldots, P_{N}$ be the Galois conjugates of $P / K$. Take $L$ to be the field of definition of $P_{i}$. Note that $\hat{h}\left(j\left(P_{i}\right)\right)=\hat{h}(j(P))$. The key idea is that if $P, Q \in X_{L}(L)$ so that they are sections, we have

$$
-\left(D_{P}, D_{Q}\right) \leq-\left(D_{P}, D_{P}\right)_{\infty}=\sum_{\sigma: L \rightarrow \mathbb{C}} \log G\left(P^{\sigma}, Q^{\sigma}\right)
$$

But this can't be too small by the Faltings-Elkies bound.

## 32 April 16, 2018

We were talking about small points. We were doing part (2).

### 32.1 Bolgomorov's conjecture for $\mathscr{L}=\Omega^{1}$

(2) If $c_{\mathscr{L}}=0$ (we may assume $\mathscr{L}=\Omega^{1}$ ) and $\widehat{\omega}^{2}>0$ then Bogomolov's conjecture holds.

Proof. We assume that Bogomolov's conjecture fails for $j=j_{\Omega^{1}}$. Let $\epsilon>0$ be small and consider $P=P_{1}, \ldots, P_{N} \in X(\bar{K})$ Galois orbit with

- $N$ as large as we want,
- $\widehat{h}(j(P))<\epsilon$ (i.e., $\hat{h}\left(j\left(P_{k}\right)\right)<\epsilon$ for all $k$ ).

We will show that $\widehat{\omega}^{2}=\widehat{\omega}_{L}^{2} /[L: K]$ is small, where $L / K$. We have

$$
\begin{aligned}
-\epsilon<-\hat{h}(j(P)) & =\frac{1}{2[L: \mathbb{Q}]}\left(\widehat{\omega}_{L}-(2 g-2) D_{P, L}\right)_{L}^{2} \\
& =\frac{1}{2[L: \mathbb{Q}]}\left(\widehat{\omega}_{L}^{2}-4 g(g-1) D_{P, L} \cdot \widehat{\omega}_{L}\right)
\end{aligned}
$$

by adjunction on $\mathfrak{X}_{\mathcal{O}_{L}}$. This shows that

$$
-(2 g-2) D_{P, L} \widehat{\omega}_{L}>-\frac{[L: \mathbb{Q}] \epsilon}{g}-\frac{\widehat{\omega}_{L}^{2}}{2 g} .
$$

By Cauchy-Schwartz, we have

$$
\left|\left\langle j\left(P_{i}\right), j\left(P_{k}\right)\right\rangle_{\mathrm{NT}}\right| \leq \epsilon
$$

Then

$$
\begin{aligned}
\epsilon 2[L: \mathbb{Q}] & >-\left\langle j\left(P_{i}\right), j\left(P_{k}\right)\right\rangle_{\mathrm{NT}} 2[L: \mathbb{Q}] \\
& =\left(\widehat{\omega}_{L}-(2 g-2) D_{P_{i}, L} \cdot \widehat{\omega}_{L}-(2 g-2) D_{P_{k}, L}\right)_{L} \\
& =\widehat{\omega}_{L}^{2}-(2 g-2)\left(D_{P_{i}, L}+D_{P_{k}, L}\right) \cdot \widehat{\omega}_{L}+(2 g-2)^{2} D_{P_{i}, L} \cdot D_{P_{j}, L}
\end{aligned}
$$

By the bound we had before, we get

$$
\left(1+\frac{1}{g}\right) 2[L: \mathbb{Q}] \epsilon>\left(1-\frac{1}{g}\right) \widehat{\omega}_{L}^{2}+(2 g-2)^{2} D_{P_{i}, L} \cdot D_{P_{k}, L}
$$

If $D_{P_{i}, L} \cdot D_{P_{k}, L}$ are always nonnegative, we will be able to get $\widehat{\omega}_{K}^{2} \rightarrow 0$ as $[L: \mathbb{Q}] \rightarrow \infty$ and then we are done. But Arakelov intersection doesn't work this way, and so we average over $i \neq k$. Then

$$
\begin{aligned}
\left(1+\frac{1}{g}\right) 2 \epsilon & >\left(1-\frac{1}{g}\right) \widehat{\omega}^{2}+\frac{(2 g-2)^{2}}{N(N-1)[L: \mathbb{Q}]} \sum_{i \neq k}\left(D_{P_{i}, L} \cdot D_{P_{k}, L}\right)_{L} \\
& >\left(1-\frac{1}{g}\right) \widehat{\omega}^{2}+\frac{(2 g-2)^{2}}{N(N-1)[L: \mathbb{Q}]} \sum_{i \neq k} \sum_{L \hookrightarrow \mathbb{C}}-\log G_{\sigma}\left(P_{i}^{\sigma}, P_{k}^{\sigma}\right)
\end{aligned}
$$

Note that $X_{L} \otimes_{\sigma} \mathbb{C}$ really only depends on $\left.\sigma\right|_{K}$, because it is biholomorphic to $X_{K} \otimes_{\left.\sigma\right|_{K}} \mathbb{C}$. By Faltings-Elkies, we can estimate

$$
\begin{aligned}
& =\left(1-\frac{1}{g}\right) \widehat{\omega}^{2}-\frac{(2 g-2)^{2}}{N(N-1)[L: \mathbb{Q}]} \sum_{\sigma: L \rightarrow \mathbb{C}} \sum_{i \neq k} \log G_{\left.\sigma\right|_{K}}\left(P_{i}^{\sigma}, P_{k}^{\sigma}\right) \\
& >\left(1-\frac{1}{g}\right) \widehat{\omega}^{2}-\frac{(2 g-2)^{2}}{N(N-1)[L: \mathbb{Q}]} \sum_{\sigma: L \rightarrow \mathbb{C}} c_{\left.\sigma\right|_{K}} N \log N \\
& >\left(1-\frac{1}{g}\right) \widehat{\omega}^{2}-c_{K} \frac{(2 g-2)^{2}}{N(N-1)} N \log N
\end{aligned}
$$

for some constant $c_{K}$ depending only on $K$. As $N \rightarrow \infty$, we get $\widehat{\omega}^{2} \leq 0$ and so $\widehat{\omega}^{2}=0$.

For (1), let's just look at the case when $\mathscr{L}=\mathscr{O}(P)$ for $P \in X(K)$ (after base change). For all $Q \in X(\bar{K})$, we have

$$
j_{\Omega}(Q)=(2 g-2) j_{P}(Q)+j_{\Omega}(P)
$$

So if $\hat{h}\left(j_{P}(Q)\right) \rightarrow 0$ for some sequence of $Q$, then the Néron-Tate distance between $j_{\Omega}(Q)$ and $j_{\Omega}(P)$ goes to 0 .

Note that $c_{\mathscr{L}}>0$ just means that $\hat{h}(j(P))>0$. Let $Q=Q_{1}, \ldots, Q_{N}$ be Galois conjugates with $\hat{h}\left(j_{P}(Q)\right)$ very small. Then we have

$$
-\langle j(P), j(Q)\rangle 2\left(1+\frac{1}{g}\right)[L: \mathbb{Q}]=\left(1-\frac{1}{g}\right) \widehat{\omega}_{L}^{2}+4(g-1)^{2}\left(D_{P} \cdot D_{Q}\right)_{L}
$$

But because $j(P)$ and $j(Q)$ are very close, the pairing on the left hand side is essentially $\hat{h}(j(P))>0$. We'll pick up next time.

## 33 April 18, 2018

Last time we had the map $j_{P}: X \rightarrow J_{X}$ for $P \in X(K)$ and $\mathscr{L}=\mathscr{O}(P)$. We have $0<c_{\mathscr{L}}=\hat{h}(j(P))$ where $j=j_{\Omega^{1}}$. First we have

$$
j(Q)=(2 g-2) j_{P}(Q)+j(P)
$$

and we have

$$
\begin{aligned}
-2[L: \mathbb{Q}]\left\langle j\left(Q_{1}\right), j\left(Q_{2}\right)\right\rangle_{\mathrm{NT}} & =\left(\widehat{\omega}_{L}-(2 g-2) D_{Q_{1}}, \widehat{\omega}_{L}-(2 g-2) D_{Q_{2}}\right)_{L} \\
& =\widehat{\omega}_{L}^{2}+4(g-1)^{2} D_{Q_{1}} \cdot D_{Q_{2}}-2(g-1)\left(D_{Q_{1}}+D_{Q_{2}}\right) \cdot \widehat{\omega}_{L}
\end{aligned}
$$

On the other hand, $-2[L: \mathbb{Q}] \hat{h}(j(Q))=\widehat{\omega}_{L}-4 g(g-1) D_{Q} \cdot \widehat{\omega}_{L}$ by adjunction. Then

$$
\begin{aligned}
2[L: \mathbb{Q}] & \left\{\hat{h}\left(j\left(Q_{1}\right)\right)+\hat{h}\left(j\left(Q_{2}\right)\right)-2 g\left\langle j\left(Q_{1}\right), j\left(Q_{2}\right)\right\rangle_{\mathrm{NT}}\right\} \\
& =2(g-1) \widehat{\omega}_{L}^{2}+8 g(g-1)^{2}\left(D_{Q_{1}} \cdot D_{Q_{2}}\right)_{L} \\
& \geq 8 g(g-1)^{2}\left(D_{Q_{1}} \cdot D_{Q_{2}}\right)_{L, \infty} .
\end{aligned}
$$

Take $N$ large so that $[L: K] \rightarrow \infty$. Choose $Q_{1}, \ldots, Q_{N} \in X(L)$ (not necessarily Galois conjugates!) different with $d_{\mathrm{NT}}\left(Q_{i}, P\right)<\epsilon$. Then for suitable $\epsilon>0$, we get
$2 g\left\langle j\left(Q_{i}\right), j\left(Q_{l}\right)\right\rangle-\hat{h}\left(j\left(Q_{i}\right)\right)-\hat{h}\left(j\left(Q_{l}\right)\right)>\left(1-\epsilon^{\prime}\right)(2 g-2) \hat{h}(j(P))>(g-1) c_{\mathscr{L}}>0$.
Then

$$
c_{\mathscr{L}}<-\frac{4 g(g-1)}{[L: \mathbb{Q}]}\left(D_{Q_{i}} \cdot D_{Q_{l}}\right)_{L, \infty}
$$

Averaging over $Q_{i}$ as before gives $c_{\mathscr{L}} \leq 0$.
You can ask about the essential minimum of $\{\hat{h}(P): P \in X(\bar{K})\}$. One can get estimates for this, but the sharp number is not very accessible.

### 33.1 The computation revisited

Now let us do the same computation again, but not with Bogomolov's conjecture in mind. Here, we don't assume everywhere good reduction. Let $X / K$ be our curve and let $\mathfrak{X} / \mathcal{O}_{K}$ be the semi-stable regular model. Again, assume $g_{X} \geq 2$. Take $P, Q \in X(K)$ be rational points and look at

$$
\widehat{\omega}-(2 g-2) D_{P}-\Phi_{P} \in V_{\mathbb{Q}}^{\perp}, \quad \Phi_{P}=\Phi\left(\widehat{\omega}-(2 g-2) D_{P}\right) \in V_{\mathbb{Q}}^{0} .
$$

This is the correct thing to look at because we might not have everywhere good reduction. Then because of orthogonality,

$$
\begin{aligned}
-2[K: \mathbb{Q}]\langle j(P), j(Q)\rangle & =\left(\widehat{\omega}-(2 g-2) D_{P}-\Phi_{P}, \widehat{\omega}-(2 g-2) D_{Q}-\Phi_{Q}\right) \\
& =\widehat{\omega}^{2}+4(g-1) D_{P} \cdot D_{Q}-2(g-1)\left(D_{P}+D_{Q}\right) \cdot \widehat{\omega}-\Phi_{P} \cdot \Phi_{Q} .
\end{aligned}
$$

For $P=Q$, by adjunction

$$
-2[K: \mathbb{Q}] \hat{h}(j(P))=\widehat{\omega}^{2}-4 g(g-1) D_{P} . \widehat{\omega}-\Phi_{P}^{2}
$$

We can cancel the $D_{P} . \widehat{\omega}$ terms by adding the two equations together, and then

$$
\begin{aligned}
& 2[K: \mathbb{Q}]\{\hat{h}(j(P))-\hat{h}(j(Q))-2 g\langle j(P), j(Q)\rangle\} \\
& \quad=2(g-1) \widehat{\omega}^{2}+8 g(g-1)^{2} D_{P} \cdot D_{Q}+\left(\Phi_{P}^{2}+\Phi_{Q}^{2}-2 g \Phi_{P} \cdot \Phi_{Q}\right)
\end{aligned}
$$

Here, we can't say much, but we have some idea of what the components look like. We have $2(g-1) \widehat{\omega} \geq 0$, and $8 g(g-1)^{2} D_{P} . D_{Q}$ is bounded below by some $-\gamma_{1}(X / K)$ because the Green's function is bounded. Also, the value $\Phi_{P}^{2}+\Phi_{Q}^{2}-2 g \Phi_{P} . \Phi_{Q}$ can take only finitely values. So we get

$$
\hat{h}(j(P))+\hat{h}(j(Q))-2 g\langle j(P), j(Q)\rangle \geq-\gamma(X / K)
$$

for all $P \neq Q$ in $X(K)$.
Theorem 33.1 (Mumford inequality). $\hat{h}(j(P))+\hat{h}(j(Q))-2 g\langle j(P), j(Q)\rangle \geq$ $-\gamma(X / K)$ for all $P \neq Q$ in $X(K)$.

## 34 April 20, 2018

Last time we had a small computation about sections. Let $P \neq Q$ be rational points in $X / K$, and consider $D_{P}, D_{Q}$.

Theorem 34.1. Define

$$
M(P, Q)=2 g \Phi_{P} \cdot \Phi_{Q}-\Phi_{P}^{2}-\Phi_{Q}^{2}-2(g-1) \widehat{\omega}^{2}-8 g(g-1)^{2} D_{P} \cdot D_{Q}
$$

where $\Phi_{P}=\Phi\left(\widehat{\omega}-(2 g-2) D_{P}\right) \in V_{\mathbb{Q}}^{0}$. Then
(1) $M(P, Q)$ can be bounded above by "Arakelovian" invariants of $\widehat{\mathfrak{X}}$, independently of $P, Q$. (This seems to have no content if you invoke Faltings's theorem, but you can do a computation even for algebraic points. It is going to be complicated so I will not do this here, but then the statement has content if we don't know Faltings.)
(2) $\hat{h}(j(P))+\hat{h}(j(Q))-2\langle j(P), j(Q)\rangle_{\mathrm{NT}}=-\frac{1}{2[K: \mathbb{Q}]} M(P, Q)$.

Let $n \geq 1$ and $g \geq 2$. Let $\langle-,-\rangle$ be an inner product on $\mathbb{R}^{n}$, which induces a norm. Consider $c \geq 0$ fixed, and suppose $x, y \in \mathbb{R}^{n} \backslash\{0\}$ satisfy

$$
\|x\|^{2}+\|y\|^{2}-2 g\langle x, y\rangle \geq-c .
$$

Let $\lambda>0$ be such that $\|y\|=\lambda\|x\|$, and consider $\theta$ the angle between $x$ and $y$. Then dividing by $\|x\|\|y\|$ gives

$$
\frac{1}{\lambda}+\lambda-2 g \cos \theta \geq-\frac{c}{\|x\|\|y\|}=-\frac{c}{\lambda\|x\|^{2}}
$$

If we assume $\|x\|^{2} \geq c$, then we get

$$
2 g \cos \theta \leq \lambda+\frac{2}{\lambda}
$$

You can do the algebra and you get the following: there exists a $\theta_{0}>0$, uniform, such that

$$
y \notin \mathcal{R}_{x}=\left\{z \in \mathbb{R}^{n}:\|x\| \leq\|z\| \leq 2\|x\|, \angle(x, z) \leq \theta_{0}\right\}
$$

These observations are called Mumford's gap principle.

### 34.1 Counting points in $j(X(K))$ of bounded height in $J_{X}$

Define the counting function

$$
\left.N(x)=\#\{P \in X(K)): \hat{h}(j(P)) \geq-\frac{\gamma(\mathfrak{X})}{2[K: \mathbb{Q}]}, \hat{h}(j(P)) \leq x\right\}
$$

Proposition 34.2. For constants $\gamma_{1}(\mathfrak{X})$ and $\gamma_{2}(\mathfrak{X})$, we have

$$
N(x) \leq \gamma_{1}(\mathfrak{X})+\gamma_{2}(\mathfrak{X})^{\operatorname{rk}\left(J_{X}(K)\right)} \log x .
$$

Proof. Let $R$ by any constant with $R \geq-\frac{\gamma(\mathfrak{X})}{2[K: \mathbb{Q}]}$. Then the number of points that can be in

$$
\left\{P \in X(K): R \leq \hat{h}(j(P))^{1 / 2} \leq 2 R\right\}
$$

is going to be at most $B^{\mathrm{rk} J_{X}(K)}$ for some uniform constant $B$. This is because we know that any two such points should have different $\theta$. Then we can put something like $\log x$-many annuli we need.

We can compare this is with the ambient space. We have

$$
\#\left\{Q \in J_{X}(K): \hat{h}(Q)^{1 / 2}<x\right\} \sim \zeta \cdot x^{\mathrm{rk} J(K)}
$$

for some regulator $\zeta$, just by counting points in regions.

## 35 April 23, 2018

Today I am going to give you some context of small points on the Jacobian, although it is not Arakelov theory.

### 35.1 Curve with infinitely many torsion points

Theorem 35.1 (Ihara-Serre-Tate). (A reference is Lang, Division points on curves) Let $X \subseteq \mathbb{G}_{m}^{2}$ over $\overline{\mathbb{Q}}$ be an irreducible algebraic curve, passing through the neutral element $1=(1,1)$. If $X$ contains infinitely many torsion points of $\mathbb{G}_{m}^{2}$, then $X$ is a subgroup of $\mathbb{G}_{m}^{2}$ (which in particular implies that $g_{\tilde{X}}=0$ ).

Proof (after Tate). If $X$ contains infinitely many torsion points, then $X / k$ for some $K=\mathbb{Q}\left(\zeta_{m}\right)$. Write $X=\{F(x, y)=0\}$ for $F \in K[x, y]$ irreducible over $K$. Let $\epsilon=\left(\epsilon_{1}, \epsilon_{2}\right) \in X$ be a torsion point and let $N$ be the order of $\epsilon$. Of course, $N$ can be arbitrarily large. If we look $\langle\epsilon\rangle$, this contains the Galois conjugates of $\epsilon$. So $L=L(\epsilon)$ is Galois over $K$ and

$$
\# \operatorname{Gal}(L / K)=[L: K] \geq \frac{\varphi(N)}{m}
$$

Moreover, because we understand Galois theory of cyclotomic extensions very well, we see that if $d>1$ satisfies $(d, N)=1$ and $d \equiv 1 \bmod m$ then there exists a $\sigma_{d} \in \operatorname{Gal}(L / K)$ such that $\sigma_{d}(\epsilon)=\epsilon^{d}$.

Let's count. Note that the Galois orbit of $\epsilon$ is contained in $X \cap X_{d}$, where

$$
X_{d}=\left\{F\left(x^{d}, y^{d}\right)=0\right\}
$$

Consider $\operatorname{deg} X=\operatorname{deg} F$. By Bezout, either $X \subseteq X_{d}$ which means $F \mid F_{d}$, or $\#\left(X \cap X_{d}\right) \leq(\operatorname{deg} X)^{2} d$. In the latter case,

$$
(\operatorname{deg} X)^{2} d \geq \# X \cap X_{d} \geq \# \operatorname{Gal}(L / K) \geq \frac{\varphi(N)}{m}
$$

We have a lower bound and an upper bound and we want to get a contradiction. By Dirichlet, there exist infinitely many $p$ such that $p \equiv 1 \bmod m$ and moreover

$$
\sum_{p \equiv 1 \bmod m, p \leq x} \log p \sim \frac{1}{\varphi(m)} x
$$

If $N$ is divisible by all these primes, then

$$
\log N \geq \sum_{p \equiv 1 \bmod m, p \leq x} \log p \geq \frac{1-\epsilon}{\varphi(m)} x
$$

and this implies $N>\exp \left(\frac{1-\epsilon}{\varphi(m)} x\right)$. For $N$ large, we can let $x=2 \varphi(m) \log N$ and this shows that there is some prime $p \equiv 1 \bmod m$ with $p \leq x$ such that $p$ does not divide $N$. Then

$$
(\operatorname{deg} X)^{2} 2 \varphi(m) \log N \geq \frac{\varphi(n)}{m}
$$

is a contradiction as $N \rightarrow \infty$.
This shows that we should have $X \subseteq X_{d}$, i.e., $F(x, y) \mid F\left(x^{d}, y^{d}\right)$. From this one gets that $X$ is a subgroup, because if you test on complex points you can choose something with dense orbit and so on.

Consider $S=X(1) \times X(1)$. The Andu-Oort conjecture states that if $Y \subseteq S$ is an irreducible point with infinitely many CM points, then $Y$ is special. Here, special means
(1) $Y$ is vertical/horizontal
(2) $Y=\Gamma\left(j, j_{N}\right)$ is the image of some higher level modular curves.

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