Math 283 - Instanton Floer Homology

Taught by Peter B. Kronheimer
Notes by Dongryul Kim

Spring 2018

The course was taught by Peter Kronheimer, on Mondays, Wednesdays, and Fridays from 11am to 12pm. There were three assignements throughout the course.

Contents

1 January 22, 2018

The three main topics we will cover are:

- 4-manifolds
- Gauge theory (Donaldson's work in the 1980s)
- 3-manifolds, knots, trivalent spatial graphs

1.1 Topology of smooth 4-manifolds

Before Donaldson's work, nothing was know about 4-manifolds. Here, we mean smooth oriented closed 4-manifolds. The $\pi_1(X)$ can be any finitely generated group. We also have

$$H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}) \cong \{\text{complex line bundles over } X\}$$

via the Chern class $c_1(L)$. This isomorphisms also can be thought this way. Given a line bundle L and a section s, the vanishing locus $s^{-1}(0)$ is going to be an oriented 2-dimensional submanifold. Its homology class will be in H_2.

Proposition 1.1. *Every $\sigma \in H_2(X;\mathbb{Z})$ is represented by a 2-dimensional submanifold $\Sigma^2 \hookrightarrow X^4$.*

The group H_2 carries a quadratic form, the intersection product, and this is equivalent to H^2 with the cup-square

$$H^2 \times H^2 \to H^4(X;\mathbb{Z}) \cong \mathbb{Z}.$$

This is symmetric and unimodular on H^2/torsion. In H_2, this is the count of intersection points, with suitable orientation:

$$[\Sigma_1][\Sigma_2] = \sum_{\Sigma_1 \cap \Sigma_2} (\pm 1).$$

In particular, $[\Sigma][\Sigma]$ is the degree of the normal bundle, i.e., the Euler class. This can be seen by deforming Σ to Σ' along a section of the normal bundle ν.

Because this is a symmetric unimodular form, we can consider its rank and signature:

$$b_2 = b^+ + b^-, \quad \sigma = b^+ - b^-.$$

But unimodular indefinite forms over the integers is not classified by its signature. Unimodular indefinite forms are either

- odd forms $\lambda(1) \oplus \mu(-1)$ or
- even forms $\lambda\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \mu(\pm E_8)$.

Even forms have the property that $x \cdot x$ is always even.

So the fundamental question we ask are:

- Which quadratic forms arise from 4-manifolds?
- Does the quadratic form determine the 4-manifold if $\pi_1(X) = 0$?

Here are some examples of 4-manifolds.

- $X = S^4$: Here $Q = 0$.
- $X = S^2 \times S^2$: Here, the two S^2 intersect once, and each have trivial normal bundle. So $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
- $X = \mathbb{C}P^2$: The rank is 1 and $Q = (1)$.
- $X = \overline{\mathbb{C}P^2}$, with the opposite orientation: Here, $Q = (-1)$.

Now note that taking connecting sums of manifolds give the direct sum of intersection forms. This gives us all the odd forms.

To get E_8, we need something more interesting. Take X a $K3$-surface, and we are going to look at its tangent bundle. But let's take a step back. Given a general X^4, we can consider $w_2(X) \in H^2(X; \mathbb{Z}/2)$, $p_1(X) \in H^4$, $e \in H_4$. In the $\pi_1 = 0$ case, we have

$$w_2(X) \smile y = y \smile y.$$

So if $H^2(X; \mathbb{Z})$ is torsion-free, every $y \in H^2(X; \mathbb{Z}/2)$ has an integer lift. So $w_2(X) = 0$ if and only if $[\Sigma][\Sigma]$ is even for all Σ (i.e., when the quadratic form is even). Also, we have

$$p_1(X)[X] = 3\sigma, \quad e(X)[X] = \sum_0^4 (-1)^i b_i(X).$$

by the Hirzebruch signature theorem.

Going back to the $K3$-surface, we see that $c_1(TX) = 0$ and $c_2(TX) = e(X) = 24$. From $w_2 = c_1 \pmod{24}$ and $p_1 = c_1^2 - c_2$, we obtain $\operatorname{rank} Q = b_2 = 22$. Also, $\sigma = -8$. This shows that

$$Q_X = 3\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 2(-E_8).$$

Unsolved question. *Which of $\lambda\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \mu(-E_8)$ arise as Q_X? In particular, is always $\lambda \geq \frac{3}{2}\mu$?*

It can be shown that μ is even. If $w_2 = 0$ then σ is divisible by 16.

For a smooth complex surface, you can blow up at a point. This is taking the connected sum with $\overline{\mathbb{C}P^2}$. So we define

$$X' = X \# \overline{\mathbb{C}P^2}$$

as the blow-up in general. In particular, $X = K3 \# \overline{\mathbb{C}P^2}$ has rank $b_2 = 23$ and signature $\sigma = -17$. This is now odd, and

$$Q_X = 3(1) \oplus 20(-1).$$

Going back to our uniqueness question, we can ask if $K3 \# \overline{\mathbb{C}P^2} \cong 3\mathbb{C}P^2 \# 20 \overline{\mathbb{C}P^2}$. The answer is no, and the deep reason comes from gauge theory.

1.2 Connections

Now we are going to study gauge theory. The main set up is

- a n-manifold X,
- a vector bundle $E \to X$,
- and a connection A on E.

This connection is a map

$$\nabla : \Omega^0_X(E) \to \Omega^1_X(E) = \Gamma(\Lambda^1_X \otimes E)$$

satisfying $\nabla(fs) = f\nabla(s) + (df) \otimes s$. Given two vector bundles E, E' and a map $u : E' \to E$, we can pull back the connection A on E, and get a connection $u^*(A)$ on E'. From now, I will consider objects (E, A), a bundle together with a connection.

2 January 24, 2018

I'm not talking about bundles and connections because I want to start from scratch, but because I want to make sure we're talking about the same things. Let G be a Lie group and $\mathfrak{g}$ be the left-invariant vector fields. Let $\pi : P \to X$ be a principal bundle so that G acts on P on the right. We have a subbundle $V \subseteq TP$ given by $V = \ker \pi_*$.

Definition 2.1. A **connection** is a subbundle $H \subseteq TP$ such that $V \oplus H = TP$ at every point and is G-invariant.

So given any path γ on X, there is a unique horizontal lift to P. This is parallel transport. We also have H as the kernel

$$\theta : TP \to V \cong \mathfrak{g} \cdot P,$$

so where θ is a $\mathfrak{g}$-valued 1-form on P.

Consider the associated vector bundle $E \to X$, coming from a finite-dimensional representation of G. Then we get a map

$$\nabla : \Omega^0_X(E) \to \Omega^1_X(E).$$

This is another way to think about connections. A representation is a map $\mathfrak{g} \to \mathrm{End}(\mathbb{R}^N)$, so we get a map

$$\mathfrak{g}_P \to \mathrm{End}(E),$$

where $\mathfrak{g}_P$ is the associated vector bundle with fiber $\mathfrak{g}$. In particular, if $G = \mathrm{SU}(N)$ and $E = \mathbb{C}^N$ then $\mathfrak{g}_P \subseteq \mathrm{End}(E)$ is the traceless skew-symmetric endomorphisms.

Let ∇, ∇' be two connections in $E \to X$. Then $\nabla - \nabla'$ is tensorial, because

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s.$$

So $a = \nabla - \nabla'$ is in $\Omega^1_X(\mathrm{End}\, E)$, and in the principal bundle case, $\Omega^1_X(\mathfrak{g}_P)$. If we consider the space $\mathscr{A}$ of all connections in $E \to X$ with structure group G, this is an affine space over $\Omega^1_X(\mathfrak{g}_P)$, i.e.,

$$\mathscr{A} = \{\nabla^0 + a : a \in \Omega^1_X(\mathfrak{g}_P)\}.$$

2.1 Curvature

Curvature is what connections have. Let x^i be local coordinates on X. I can write a connections as

$$\nabla = \sum_i \nabla_i dx^i,$$

where ∇_i send sections of E to sections of E. Now differentiations don't commute, and we can consider

$$F_{ij} = [\nabla_i, \nabla_j] \in \Gamma(\mathrm{End}(E)).$$

We then can define **curvature** as

$$F = \sum_{i,j} F_{ij} dx^i \wedge dx^j \in \Omega^2_X(\mathfrak{g}_P).$$

The curvature is the local obstruction to the connection being trivial.

If X^n is Riemannian and oriented, then it has the **Hodge star operator** acting as $\star : \bigwedge^p \to \bigwedge^{n-p}$ or $\star : \Omega^p_X \to \Omega^{n-p}_X$. In Euclidean space, it is going to map

$$\star(e^1 \wedge \cdots \wedge e^p) = e^{p+1} \wedge \cdots \wedge e^n.$$

We can write this as

$$\alpha \wedge \star b = (\alpha, \beta) d\text{vol}_X.$$

If $n = 2m$ is even, we have the map $\star : \Omega^m \to \Omega^m$, and in particular, for $n = 4$ we have

$$\star : \Omega^2_X(\mathfrak{g}_P) \to \Omega^2_X(\mathfrak{g}_P).$$

In this particular dimension, $\star^2 = 1$, and then the eigenvalues are ± 1. So, on an Riemannian oriented X^4, we can decompose

$$\textstyle\bigwedge^2 = \bigwedge^2_+ \oplus \bigwedge^2_-$$

with respect to the eigenvalues of $\star$ being 1 and -1. We can then decompose $\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$. These are called **self-dual** and **anti-self-dual** parts.

We can do this for the curvature. That is, we can write $F = F^+ + F^-$ where $F^+ \in \omega^+_X(\mathfrak{g}_P)$.

Definition 2.2. A connection is **anti-self-dual** if its curvature has $\star F = -F$, i.e., $F^+ = 0$.

Example 2.3. There are flat connections. Here, parallel transport along the path is unchanged as the path homotopes. That is $F = 0$, and you can even take this as the definition. Here, the data that determines the connection is the homomorphism $\rho : \pi_1(X) \to G$.

Example 2.4. On S^4 with the standard orientation and round metric as a unit sphere, there is a Levi–Civita connection on TS^4, and also on $\bigwedge^2$. Take $E = \bigwedge^2$, which is going to be a rank 3 vector bundle, so have fibers $\mathbb{R}^3$. But $\mathbb{R}^3 \cong \mathfrak{so}(3)$. This is confusing. The original Levi–Civita connection has curvature in $\Omega^2_{S^4}(\mathfrak{so}(4))$, and The curvature of $E = \bigwedge^-$ is a map $\bigwedge^2 \to \text{End}(E)$, and it is going to be the projection $\bigwedge^2 \to \bigwedge^-$. In particular, F annihilates $\bigwedge^+$. todo

Every SO(3) bundle on S^4 lifts to SU(2). So we get an anti-self-dual connection for $G = \text{SU}(2)$ on S^4 as well. This is a "standard 1-instanton".

The reason for this is the following. The $\star$ operator is conformally invariant on Ω^2 on X^4. So the condition $F^+ = 0$ depends only on the conformal class. Now consider the conformal map $S^4 \to \mathbb{R}^4$. If we look at the density $|F|^2$ this is going to have a peak at a point and decay away from this point.

On S^4, we had a SU(2)-bundle by taking the double cover. These are classified by $\mathbb{Z}$ via $c_2(P)[S^4]$, i.e., $c_2(E)[S^4]$ where E has fiber $\mathbb{C}^2$. Ditto, on a connected oriented closed X^4, the Chern class is classified by $k = c_2(P)[X]$. The Chern class is also computed in terms of the curvature F. This is an example of the Chern–Weil formula. For an SU(2)-bundle, we can compute

$$k = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(F \wedge F).$$

Here, F is a 2×2-matrix-valued 2-form. So when I write $F \wedge F$, we are multiplying matrices and wedge producting the 2-forms. Then $F \wedge F$ is going to be 2×2-matrix-valued 2-form. Then taking the trace gives a 4-form.

Note that $-\mathrm{Tr}(ab)$ is the standard inner product (Killing form) on $\mathfrak{su}(2)$. Also, the Hodge star turns wedge products to inner products. So we can write

$$k = -\frac{1}{8\pi}^2 \int_X (F, \star F) d\mathrm{vol} = \frac{1}{8\pi^2} \int_X (|F^-|^2 - |F^+|^2) d\mathrm{vol}.$$

The L^2-norm of F is

$$\frac{1}{8\pi^2} \|F\|^2 = \frac{1}{8\pi^2} \int_X (|F^+|^2 + |F^-|^2) d\mathrm{vol}.$$

Corollary 2.5. *We have*

$$\frac{1}{8\pi^2} \|F\|^2 \geq k,$$

and equality holds if and only if the connection is anti-self-dual.

3 January 26, 2018

So we have a Riemannian oriented closed 4-manifold X^4. We also have a principal G-bundle $P \to X$, where often $G = \mathrm{SU}(N)$, and this will be associated to a bundle $E \to X$ with fiber $\mathbb{C}^N$. We want to look at the automorphisms $g : E \to E$, which is called the **gauge group** $\mathcal{G}$. This is going to just be the sections of $\mathcal{G}_p$.

3.1 Moduli spaces of connections

Fix a E, and let $\mathcal{A}$ be the space of all connections on E. This is an affine space, and consider $\mathcal{B} = \mathcal{A}/\mathcal{G}$. We can think of it as parametrizing pairs (E', A') up to isomorphism of pairs, with $E' \cong E$. If we write $k = c_2(E)[E]$, then we can consider it as

$$\mathcal{B} = \bigcup_k \mathcal{B}_k.$$

The anti-self-dual connections form a subset $M_k \subseteq \mathcal{B}_k$. This k is called the instanton number. For any $A \in \mathcal{A}$, we get $[A] \in \mathcal{A}/\mathcal{G}$ the equivalence class. Then

$$M_k = \{[A] : F_A^+ = 0\}$$

is the moduli space of anti-self-dual connections.

Example 3.1. On S^4, with the round metric, and $G = \mathrm{SU}(2)$, the moduli space M_k is a smooth non-compact of dimension $8k - 3$ for $k \geq 1$. What is M_0? We've seen that the Yang–Mills functional is

$$\|F\|^2_{L^2(X)} \geq 8\pi^2 k$$

with equality for anti-self-dual connections. So if $k = 0$, every anti-self-dual connection should be flat, and the bundle is trivial if $\pi_1 = 0$.

3.2 Elliptic operators

Consider a subset $\Omega \subseteq \mathbb{R}^n$, and an operator on $\mathbb{C}^N$-valued or $\mathbb{R}^N$-valued functions on Ω. In other words, we consider a differential operator $D : \Gamma_\Omega(V) \to \Gamma_\Omega(W)$ where V and W are trivial bundles. Assuming that D is first-order, we can write D as

$$Dv = \sum_{i=1}^n a^i \Big(\frac{\partial}{\partial x^i} v \Big) + bv$$

where $a^i, b \in \mathrm{Hom}_\Omega(V, W)$. Replacing $\partial/\partial x^i$ with ξ_i gives a symbol P. Then the leading part is

$$\sigma_D = \sum_i a^i \xi_i : T^* \to \mathrm{Hom}(V, W).$$

Definition 3.2. D is said to be **elliptic** if $\sigma_D(\xi)$ is invertible whenever $\xi \neq 0$.

For instance, consider the Cauchy–Riemann equations $\overline{\partial} f = 0$, where $f = f_1 + \sqrt{-1} f_2$. Then

$$\sigma_j(\xi) = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}$$

is elliptic because the determinant is $\xi_1^2 + \xi_2^2$.

Definition 3.3. For $r \geq 0$ an integers, we define the **Sobolev norm** as

$$\|f\|_{L^p_r} = \sum_{|\alpha| \leq r} \|D^\alpha f\|_{L^p}.$$

Let $\Omega_1 \subseteq \Omega$ be open sets where the closure Ω_1 is in Ω. If D is a first-order elliptic operator, then we have inequalites

$$\|f\|_{L^p_{r+1}(\Omega_1)} \leq C(\|Df\|_{L^p_r(\Omega)} + \|f\|_{L^p(\Omega)}).$$

To get an intuition for this estimate, consider the Cauchy–Riemann equations. Then for a holomorphic function f, we can compute $f^{(r)}(z)$ by taking a contour integral around z. So

$$|f^{(r)}(z)| \leq C \sup_{w \in \gamma} |f(w)|.$$

If $Df = 0$ and $f \in L^p_1$, then you can show that f is smooth if D is elliptic. Let's look at connections. Consider a trivial bundle over Ω in $\mathbb{R}^4$. If we write $\nabla = d + A$, then

$$F = dA + A \wedge A \in \Omega^2(\mathfrak{g}).$$

So the self-dual part will be

$$F_A^+ = d^+ A + (A \wedge A)^+,$$

where $d^+ : \Omega^1 \to \Omega^+$ is the composition $\Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{\pi} \Omega^+$. So the equation we want to solve is

$$d^+ A + (A \wedge A)^+ = 0.$$

The problem is that d^+ is not elliptic. You can expect this, because a very large group $\mathcal{G}$ acts on solutions. So we need to impose additional "gague-fixing" conditions on A.

3.3 Linear gauge-fixing

Let us take $G = S^1$ and $\mathfrak{g} = i\mathbb{R}$. Then $\nabla = d + ia$ where $a \in \Omega^1$. The usual condition you want to impose on A is the **Coulomb gague-fixing** condition. This is given by

$$d^* a = 0$$

where $d^* = \star d \star : \Omega^1 \to \Omega^0$. Now we have the equation $d^+ a = d^* a = 0$. Then the equation

$$d^+ \oplus d^* : \Omega^1 \to \Omega^+ \oplus \Omega^0$$

is an elliptic operator. If you look at the ranks, Ω^1 has rank 4 and $\Omega^+ \oplus \Omega^0$ has rank $3+1$. This is a natural condition for ellipticity. You can actually check this using

$$\sigma_{d^+}(\xi) = \{\eta \mapsto (\xi \wedge \eta)^+\}, \quad \sigma_{d^*}(\xi) = \{\eta \mapsto \xi \cdot \eta\}.$$

We need to show that $(\xi \wedge \eta)^+ = 0$ and $\xi \cdot \eta = 0$ implies either $\xi = 0$ or $\eta = 0$. To see this, note that $|\omega^+| = |\omega^-|$ if $\omega \wedge \omega = 0$. So $(\xi \wedge \eta)^+ = 0$ means $\xi \wedge \eta = 0$, i.e., they are linearly dependent in $\bigwedge^1$.

So how do we do this impose this Coulomb gauge? A $g \in \mathcal{G}$ is an S^1-valued function. So we seek a $g : d + a \mapsto d + a'$ such that $d^* a' = 0$. Take a trivial bundle over X^4 a closed Riemannian manifold. We have

$$a' = a - g^{-1}(dg),$$

so if we write $g = \exp(i\varphi)$ then $a' = a - id\varphi$. The equation $d^* a' = 0$ means

$$-\Delta\varphi = d^* d\varphi = -id^* a.$$

The Friedholm alternative says that this can be solved if the right hand side is orthogonal to the kernel, i.e.,

$$\int_X (d^* a) d\mathrm{vol} = 0.$$

This follows from the divergence theorem. So after the gauge-transformation, we can use the elliptic package to get smoothness, regularity, etc.

For the $\mathrm{SU}(N)$ case, what should we do? If we seek $g = \exp(\varphi)$, where φ is a section of $\mathfrak{g}_P$. If this sends

$$\nabla = d_{A_0} + A \mapsto d_{A_0} + A',$$

then we can also impose the Coulomb condition relative to A_0,

$$d^*_{A_0} A' = 0.$$

The equation we want to solve becomes

$$d^*_{A_0} d_{A_0} \varphi + (\text{non-linear terms}) = (\text{given in terms of } A, A_0).$$

We need a framework in which we can say that the non-linear terms don't matter.

4 January 29, 2018

So we are working in the Sobolev spaces L^p_k with $1 < p < \infty$. Let X^n be a closed manifold and $V \to X^n$ be a vector bundle.

4.1 Sobolev embeddings

For ∇ any smooth connection and s a section we define

$$\|s\|_{L^p_k} = \sum_0^k \|\nabla^r s\|_{L^p},$$

and $L^p_k(X, V)$ is going to be the completion of $\Gamma(X; V)$. There is a Sobolev embedding

$$L^p_k \hookrightarrow C^0$$

if $k - \frac{n}{p} > 0$, and also embeddings

$$L^p_k \hookrightarrow L^q_\ell$$

if $k \geq \ell$ and $k - \frac{n}{p} \geq \ell - \frac{n}{q}$. So in dimension 4, we have $L^2_3 \hookrightarrow C^0$ and $L^2_1 \hookrightarrow L^4$.

Beyond that, we want multiplication theorems. Once $k - \frac{n}{p} > 0$ (so that all sections are continuous), we have a bilinear map

$$L^p_k \times L^p_k \to L^p_k$$

induced from any $V_1 \otimes V_2 \to V_3$. Likewise, we have

$$L^p_k \times L^q_\ell \to L^q_\ell$$

if $k - \frac{n}{p} > 0$ and $L^p_k \hookrightarrow L^q_\ell$. If you're below the borderline so that $k - \frac{n}{p} < 0$ and $\ell - \frac{n}{q} < 0$, we have

$$L^p_k \times L^q_\ell \to L^r_m$$

when $k - \frac{n}{p} + \ell - \frac{n}{q} = m - \frac{n}{r}$ and $k, \ell \geq m$. For 4-manifolds, we have $L^2_1 \times L^2_1 \to L^2$. Also, L^2_3 is an algebra and L^2_2 is an L^2_e-module. Given that $L^2_1 \hookrightarrow L^4$, it actually follows from Cauchy–Schwartz that $L^2_1 \times L^2_1 \to L^2$.

Now let $\mathcal{G}$ the L^2_3-gauge transformations of an $SU(2)$-bundle $E \to X^4$. This is a Banach Lie group, with Lie algebra

$$\mathrm{Lie}(\mathcal{G}) = L^2_3(X; \mathfrak{g}_P),$$

with the Lie algebra structure coming from the pointwise Lie algebra structures. Consider the space of connections

$$\mathcal{A} = \{A_0 + a : a \in L^2_2(X; \mathfrak{g}_P)\},$$

where A_0 is C^∞. Here, we need one less, because g acts as $g \circ (d_{A_0} + a) \circ g^{-1}$, and the $g(dg^{-1})$ term comes in. So noe $\mathcal{G}$ acts on the affine Banach space $\mathcal{A}$.

Given any A and $A + a$ with $\|a\|_{L_2^2}$ small, there exists an $g \in \mathcal{G}$ such that $g : A + a \mapsto A + a'$ with $d_A^* a' = 0$. Let me describe this pictorially. Consider

$$S = \{A + a : d_A^* a = 0\}.$$

This is a linear space in $\mathcal{A}$, and what this statement says is that every nearby $\mathcal{G}$-orbit meets S. (The intersection is actually a single point for generic A.) This is some kind of a implicit function theorem.

4.2 Moduli space of anti-self-dual connections

We were interested in the moduli space $M \subseteq \mathcal{B} = \mathcal{A}/\mathcal{G}$ of anti-self dual connections. It is quite a direct argument, but elliptic regularity states that $M \subseteq \mathcal{B}^{L_2^2}$ can be considered as $M \subseteq \mathcal{B}^{C^\infty}$. That is, every anti-self-dual L_2^2 connection is L_3^2-gauge equivalent to a smooth one.

If we assume that any $\mathcal{G}$-orbit close to A intersects S at a single point, we have that a neighborhood of $[A]$ in M is isomorphic to

$$\{A + a : A + a \in S, F_{A+a}^+ = 0, \|a\|_{L_2^2} < \epsilon\}.$$

So the equation we want to solve is

$$\begin{cases} d_A^* a = 0, \\ d_A^+ a + (a \wedge a)^+ = 0. \end{cases}$$

This is not a linear equation because of $a \wedge a$, but we can consider the direct sum

$$\delta_A = d_A^* \oplus d_A^+ : L_2^2(\textstyle\bigwedge^1 \otimes \mathfrak{g}_P) \to L_1^2(\bigwedge^0 \oplus \bigwedge^+).$$

It turns out that δ_A is an elliptic operator, and the elliptic package tells us that

- $\ker(\delta_A)$ is finite-dimensional,
- the formal adjoint δ^* is also elliptic,
- $\ker(\delta^*) \cong \operatorname{coker}(\delta)$.

For the quadratic part, we have $a \mapsto (a \wedge a)^+$ which is $L_2^2 \to L_1^2$ because $L_2^2 \times L_2^2 \to L_1^2$.

If δ_A is surjective, the implicit function theorem tells us that the solutions to $d^* a = 0$ and $F_{A+a}^+ = 0$ form a smooth manifold near a, with tangent space $\ker \delta_A$. So M is a smooth (finite-dimensional) manifold near A if

- the $\mathcal{G}$-orbits near A meet S at one point,
- $d_A^* \oplus d_A^+$ is surjective.

Consider the stabilizer

$$\Gamma_A = \{g : g \circ d_A \circ g^{-1} = d_A\} = \{g : d_A g = 0\}$$

of A. These are parallel gauge transforms. If we define

$$\mathrm{Hol}_A = \{\mathrm{hol}_\gamma(A) : \gamma\} \subseteq \mathrm{SU}(2),$$

then Γ_A is the centralizer of Hol_A in $\mathrm{SU}(2)$. The interesting case is $\Gamma_A \cong S^1$, when $\mathrm{Hol}_A(S) \subseteq S^1$ but is not in $\{\pm 1\}$. This happens when E splits as $E = L \oplus L^{-1}$ and Hol_A preserves the decomposition.

First of all, Γ_A always acts on $\mathcal{A}$, and ± 1 acts trivially. So $\Gamma_A/(\pm 1) \subseteq \mathcal{G}/(\pm 1)$ acts on $\mathcal{A}$, fixes A, and acts on S. This means that $\mathcal{G}$-orbits near A meet S in orbits of Γ_A.

Because Γ_A is finite-dimensional, its Lie algebra is

$$\mathrm{Lie}(\Gamma_A) = \{u : d_A u = 0\}$$

and $u \in L^2_3(X; \mathfrak{g}_P)$. The picture can be summarized in a complex

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^+(\mathfrak{g}_P) \to 0.$$

The statement $d_A^+ \circ d_A = 0$ is equivalent to the fact that $F_A^+ = 0$. These are smooth sections, and we take the completion to L^2_3, L^2_2, and L^2_1.

Let us compute the cohomology groups. We have

$$H^0_A = \mathrm{Lie}(\Gamma_A),$$

and this is 0 if $\Gamma_A = \pm 1$, i.e., A is irreducible. The second cohomology H^2_A vanishes if and only if $d_A^* \oplus d_A^+$ is onto, which is the hypothesis for the implicit function theorem. The first cohomology H^1_A is interesting. This is

$$H^1_A = \frac{\ker d_A^+}{\mathrm{im}\, d_A} = (\ker d_A^+) \cap (\ker d_A^*) = \ker(d_A^+ \oplus d_A^*).$$

So if $H^0 = H^2 = 0$, then this is the tangent space to M.

5 January 31, 2018

Let X^4 be a Riemannian manifold. Last time we had a complex

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^+ \to 0.$$

There are the dual maps $d^* : \Omega^+ \to \Omega^1$ and $d^* : \Omega^1 \to \Omega^0$. Here,

$$\operatorname{coker}(d^+) = \ker(d^*|_{\Omega^+}) = \ker(*d*|_{\Omega_+}) = \ker(*d|_{\Omega^+}) = \ker(d|_{\Omega_+}).$$

On Ω^2, we have the harmonic 2-forms $\mathcal{H}^2 \cong H^2_{\mathrm{dR}}$, of dimension b^2. Under $*$, this decomposes into eigenspaces $\mathcal{H}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-$. Then their dimension can be written as $b^2 = b^+ + b^-$. Then we can write

$$\Omega^1 \xrightarrow{d^* \oplus d^+} \Omega^0 \oplus \Omega^+$$

with kernel H^1 and cokernel $H^0 \oplus \mathcal{H}^+$. Its index would then be $b^1 - 1 - b^+$ if X is connected.

For an anti-self-dual connection A, we had

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^+(\mathfrak{g}_P) \to 0.$$

Then $H^0_A = \mathrm{Lie}(\Gamma_A)$ and H^2_A is the kernel of the linearization of the equation $F_A^+ = 0$ in the Coulomb gauge, and H^2 is the cokernel. If A is irreducible and surjective, we have $H^0_A = H^2_A = 0$ and

$$d = \dim H^1_A = \operatorname{index}(\delta_A).$$

If A is the trivial connection, the index is the dimension of $\mathfrak{g}$ times the $(-b^+ + b^1 - b^0)$. In the case of a general SU(2)-bundle E, by some index theoremwe get todo

$$\operatorname{index} = (\mathrm{const})c_2(E)[X] - 3(b^+ - b^1 + b^0) = 8k - 3(b^+ - b^1 + b^0).$$

Here, k is the second Chern number. In general, for a SU(N)-bundle, the index is going to be

$$4Nk - (N^2 - 1)(b^+ - b^1 + 1).$$

Example 5.1. Consider the 1-instanton on the round $X = S^4$. Here, $b^+ = 0$ and $b^1 = 0$ and so the dimension of the moduli space is $\dim = 8k - 3 = 5$. But anti-self-duality is preserved under conformal transformation on S^4. On $\mathbb{R}^4$ under stereographic projection, it is going to be determined by its center and scale. It turns out that the moduli space is the open 5-ball B^5.

Example 5.2. Consider $X = \mathbb{C}P^2$, with $b^+ = 0$ and $b^- = 1$. Here, $\dim M_k = 8k - 3$. (By dimension I mean formal dimension $\operatorname{index}(\delta_A)$, which actually will be the dimension in nice cases.) Then M_1 is 5-dimensional and will be a cone on $\mathbb{C}P^2$ with a singularity at the vertex. The local model there will be $\mathbb{C}^3/S^1$.

5.1 Reducibles on M

In the case $\Gamma_A = \mathrm{SU}(@)$, we have that A is flat, and that holonomy is contained in $\{\pm 1\}$. Then $c_2 = 0$ automatically. This is not interesting.

In the case $\Gamma_A = S^1$, we have a splitting $E = L \oplus L^{-1}$. Here, the curvature also splits as $F_A = \left(\begin{smallmatrix} \omega_L & 0 \\ 0 & -\omega_L \end{smallmatrix}\right)$, and $\omega_L \in \Omega^2(X; i\mathbb{R})$. We need to $\omega_L \in i\mathcal{H}^-$ for F_A to be anti-self-dual. But the Chern–Weil formula states that

$$\frac{i}{2\pi}\omega_L$$

represents $c_1(L)$ with real coefficients. So the question is whether $c_1(L)$ is contained in the linear subspace $\mathcal{H}^- \subseteq \mathcal{H}^2 \cong H^2(X;\mathbb{R})$. This condition $c_1(L) \in \mathcal{H}^-$ will be equivalent to the anti-self-dual equation we are trying to solve.

Consider the space $\mathcal{M}et$ of Riemannian metrics on X and $\mathcal{G}rass$ of b^--dimensional subspaces of $H^2(X;\mathbb{R})$. There is a map given by

$$\mathcal{M}et \to \mathcal{G}rass; \quad g \mapsto \mathcal{H}^-_g.$$

Then there is a subspace $\mathcal{N} \subseteq \mathcal{G}rass$ given by subspaces $H \in \mathcal{G}rass$ that contain $c_1(L)$.

Lemma 5.3 (Donaldson)**.** *The map $\mathcal{M}et \to \mathcal{G}rass$ is transverse to $\mathcal{N}$ if $c_1(L) \neq 0$. So*

$$\text{“bad metrics”} = \{g \in \mathcal{M}et : c_1(L) \in \mathcal{H}^-_{(g)}\}$$

has codimension b^+ inside $\mathcal{M}et$.

Corollary 5.4. *If $b^+ > 0$ then for a generic Riemannian metric, there do not exist reducibles in M with $c_1(L) \neq 0$.*

If $b^+ > 0$, then it will hold for a generic path of metrics. Here, $c_1(L) = 0$ over $\mathbb{R}$ means that $c_1(L)$ is torsion, and so L and $L \oplus L^{-1}$ are flat bundles, with $c_2(E) = -c_1(L)^2 = 0$, so $k = 0$.

The other thing we need to worry about is whether $H^2_A = 0$, i.e., whether the linearization of $F^+_A = 0$ is surjective.

Theorem 5.5 (Freed–Uhlenbeck, for SU(2))**.** *Let $M^* \subseteq M$ be irreducible solutions. For a generic Riemannian metric g, δ_A is onto for all $[A] \in M^*_{(g)}$.*

Together with the previous discussion, we get the following corollary.

Corollary 5.6. *If $b^+(X) > 0$ and $k \neq 0$, then for generic g, the moduli space M is smooth, has no reducibles, and* $\dim M = 8k - 3(b^+ - b^1 + 1)$.

6 February 2, 2018

Today we are going to talk about Uhlenbeck's compactness theorem. We looked at $X = S^4$ and $k = 1$, and saw that the moduli space is $M_1 = B^5$, which is noncompact. If we look at the curvature density on S^4, these are going to look like things concentrated around a specific point. Then as we move to the boundary of M_1, we get a small concentrated distribution around a point. The theorem states roughly that this is the only way that can happen in the non-compactness.

6.1 Uhlenbeck's compactness theorem

Definition 6.1. A **ideal connection** with $c_2 = k$ on X is a configuration $([A'], x_1, \dots, x_l)$ where $[A']$ is anti-self-dual with $c_2 = k - l$ and $x_i \in X$. (Here, we consider $(x_1, \dots, x_l) \in X^l/\Sigma_l$.)

Suppose $E \to X$ has $c_2 = k$, and A_n for $n \in \mathbb{N}$ be anti-self-dual connections. We say that

$$[A_n] \to ([A'], x_1, \dots, x_k)$$

weakly converges if

- A' is an anti-self-dual (smooth) connection $E' \to X$ with $c_2 = k - l$,
- on $X^\circ = X \setminus \{x_1, \dots, x_n\}$, there are isomorphisms $g_n : E'|_{X^\circ} \to E|_{X^\circ}$ so that $g_n^*(A_n) \to A'$ in the $C^\infty(K)$-topology for all compact $K \subseteq X^\circ$,
- if $\mu_n = \frac{1}{8\pi^2}|F_{A_n}|^2$, then as measures there is a weak convergence

  $$\mu_n \to \mu' + \sum_1^l \delta_{x_i}$$

  where $\mu' = \frac{1}{8\pi}|F_{A'}|^2$.

Weak convergence here means that

$$\frac{1}{8\pi^2}\int |F_{A_n}|^2 \psi \, d\mathrm{vol}_X \to \frac{1}{8\pi^2}\int |F_{A'}|^2 \psi \, d\mathrm{vol}_X + \sum \psi(x_i)$$

for any test function $\psi \in C^0(X)$. Given a bounded sequence of holomorphic functions on the disk, there is a subsequence that converges uniformly on compact sets to a holomorphic function. These are solutions to the Cauchy–Riemann equations, and we are looking at solutions to the anti-self-duality equation. The difference is that the Cauchy–Riemann equations is elliptic, while the anti-self-duality equation is not elliptic until you put the Coulomb gauge condition.

Theorem 6.2. *Given $[A_n] \in M_k$ a sequence, there exists a subsequence that converges weakly in this sense, to some $([A'], x_1, \dots, x_k)$.*

Consider a flat ball B_1^4 and take a trivial bundle. A connection is $d + A$ for $A \in \Omega^1(\mathfrak{g})$, and then the anti-self-duality equation is

$$F_A^+ = d^+ A + (A \wedge A)^+ = 0.$$

Theorem 6.3 (Uhlenbeck)**.** *There exists a universal constant η such that if $\frac{1}{8\pi^2}\int_{B^4} |F_A|^2 < \eta$ then*

- *there exists a gauge transformation g such that if $\tilde{A} = g(A)$ then $d^*\tilde{A} = 0$,*
- *there exists a universal C such that $\|\tilde{A}\|_{L_1^2} \le C\|F_A\|_{L_2}$ (and there are also bounds higher derivatives because it solves an elliptic equation).*

Suppose we had a $d^*A = 0$. How will we get the estimate on $|A|_{L_1^2}$? Say we do this on X with $H^1(X) = 0$ and $E = X \times \mathbb{C}^2$. Then

$$\delta = d^* + d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$$

has no kernel. If we look at the anti-self-duality equation $\delta A + (A \wedge A)^+ = B$, its linear equation is already $\delta A = B$. So $\|A\|_{L^1} \le C\|B\|_{L^2}$ and then

$$\|A\|_{L_1^2} \le C(\|B\|_{L^2} + \|(A \wedge A)^+\|_{L^2}) \le C'(\|B\|_{L^2} + \|A\|_{L_1^2}^2)$$

by the Sobolev multiplication. So if $\|A\|_{L_1^2} \le \frac{1}{2C'}$ (which is a universal bound), then we can say

$$\frac{1}{2}\|A\|_{L_1^2} \le C'\|B\|_{L^2}.$$

So this gives a local solution for the Coulomb gauge fixing problem. To make this global, we use the method of continuity. On B^4, We can always find a family A_t of connections such that $A_0 = 0$ and $A_1 = A$. This can be done by looking at the map $\times t : B^4 \to B^4$ and pulling back the connection.

We can solve our problem for $t = 0$, so we aim to show that the space $\{t : \text{can solve}\}$ is both open and closed in $[0, 1]$. To show that open, we use the implicit function theorem. To show that it is closed, we use estimates.

Recall that we have said that for any $[A_n] \in M_k$, there exists a subsequence weakly converging $([A'], x_1, \dots, x_l)$. To prove this, consider the measures

$$\mu_n = \frac{1}{8\pi^2}\|F_{A_n}\|^2 d\mathrm{vol}_X.$$

It is a general fact that there exists a weak-$*$ convergent subsequence $\mu_{n'} \to \mu_\infty$. If $x_0 \in X$ has a ball neighborhood B_ϵ^4, then $\mu_\infty(B_\epsilon^4) < \eta$, then we get a convergence on this neighborhood from the local result, because

$$\frac{1}{8\pi^2}\int_{B_\epsilon^4} |F_n|^2 < \eta.$$

(Here, note that $\|F\|_{L^2}$ is scale-invariant, so η does not depend on the radius.) Now where this fails cannot be infinite, because the total energy we have is always $\mu_\infty(X) = k$, which is the second Chern class. So there is just a finite number of points where we can't do this.

Now outside these points, we locally have these connections, and we need to patch them together. This is not fun, because there can be some horrible gauge transformations. So we use a removability of singularities statement.

Proposition 6.4. *One B^4, given a A on $B^4 \setminus \{0\}$ with $|F_A|^2 \leq \eta_1$ and $F_A^+ = 0$, there exists a gauge transformation g on $B^4 \setminus \{0\}$ such that $\tilde{A} = g(A)$ extends smoothly to B^4.*

7 February 5, 2018

7.1 Uhlenbeck compactification

On a 4-manifold X^4, we were looking at the moduli space M_k of anti-self-dual connections on a vector bundle with $c_2 = k$. When we compactify, we need to attach the ideal

$$([A'], x_1, \dots, x_l) \in M_{k-l} \times \mathrm{sym}^l(X).$$

So the compactifiaction of M_k is going to be given by

$$M_k \subseteq M_k \cup (M_{k-1} \times X) \cup (M_{k-2} \times \mathrm{sym}^2(X)) \cup \dots \cup (M_0 \times \mathrm{sym}^k(X)).$$

The right hand side becomes compact, and the closure $\overline{M}_k$ in this space is called the **Uhlenbeck compactification**. In most cases, the closure is going to contains all factors $M_i \times \mathrm{sym}^{k-i}(X)$ for $i > 0$. If you think about the dimension, formal dimension goes down by 8 from M_i to M_{i-1} and then gains 4 by multiplying by X. But at the last term $M_0 \times \mathrm{sym}^k(X)$, we have that M_0 is a point and so actual dimension of $M_0 \times \mathrm{sym}^k(X)$ is $4k$. This means that the dimension doesn't match, so that we shouldn't expect $\overline{M}_k$ to be everything.

Example 7.1. For the round sphere S^4, the Uhlenbeck compactification $\overline{M}_1$ is the closed 5-ball. For $\overline{\mathbb{C}P}^2$ with the Fubini–Study metric, its Uhlenbeck compactification is the cone $\overline{\mathbb{C}P}^2 \times [0,1]/\sim$.

Example 7.2. If we have a manifold with $b^+ = 1$ and $b^1 = 0$, then $\dim M_1 = 8k - 6 = 2$, but $\dim M_0 \times X = \dim X = 4$. So the space $M_1 \cup (M_0 \times X)$ is going to look like a 2-dimensional thing attached on a 4-dimensional thing. For the Uhlenbeck compactification $\overline{M}_1$, we only need to add the boundary part of the 2-dimensional thing.

This picture can be established more generally.

Theorem 7.3 (Taubes, Donaldson, Taubes's collar theorem)**.** *If $\pi_1(X) = 0$ (so that $M_0 = *$) and $b^+(X) = 0$, then a neighborhood of $M_0 \times X \subseteq \overline{M}_1$ is a 5-manifold with boundary X. That is, it contains a collar.*

The idea is to actually construct solutions. In particular, this theorem tells us that M_1 is nonempty.

Corollary 7.4. *There is no smooth compact oriented 4-manifold with $\pi_1 = 1$ and $Q_X = -rE_8$ with $r > 0$ (or indeed any nonzero even form with $b^+ = 0$).*

Proof. In this case, we can show that M_1 has no reducibles. Suppose that E with $c_2(E)[X] = 1$ can be reduced to $E = L \oplus L^{-1}$. Then we have

$$1 = c_2(E) = c_1(L) \cdot c_1(L^{-1}) = -c_1(L)^2,$$

but $c_1(L)^2$ should be even because the form is even.

So this shows that there are no reducibles, and then M_1 is a smooth 5-manifold. By Taubes's collar theorem, this shows that X is a boundary of a smooth 5-manifold. Now cobordism theory shows that a 4-manifold X is an oriented boundary if and only if $\sigma(X) = 0$, i.e., $b^+ - b^0 = 0$. But this can't happen unless $b^+ = b^- = 0$. □

This is essentially Donaldson's thesis.

7.2 Three-manifolds and the Chern–Simons functional

I want to think a bit about 3-manifolds. Let Y^3 be a closed, oriented, Riemannian manifold. In this case, we make it into a 4-manifold by taking the product $\mathbb{R} \times Y^3$. The coordinates will be denoted (x_0, x_1, x_2, x_3) or (t, y_1, y_2, y_3). If $E \to \mathbb{R} \times Y$ is a SU(2)-bundle, it is always trivial because $E \to Y$ is trivial.

For A a connection on $E \to \mathbb{R} \times Y$, we can think of it as $d + A$ with

$$A \in \Omega^1(\mathfrak{su}(2)).$$

Let us write this as

$$A_0 dt + \sum_1^3 A_i dy^i.$$

We say that A is in **temporal gauge** if $A_0 = 0$. Given A, there always exists some g so that $\tilde{A} = g(A)$ is in temporal gauge. The way to do this is to trivalize the bundle by parallel transport along the t-line.

If there is no A_0 terms, we can think of this as a connection on Y with t-dependence. That is, I take it as

$$B(t) \in \Omega^1_Y(\mathfrak{su}(2)).$$

Or I can just regard them as connections on $\mathbb{R} \times Y$ in the temporal gauge. We denote by the time-dependent connection on Y by $d + \underline{B}$. This distinction matters, because when I look at the 4-dimensional curvature on $\mathbb{R} \times Y$, we get

$$\begin{aligned}(d_X + \underline{B})(d_X + \underline{B}) &= d_X \underline{B} + \underline{B} \wedge \underline{B} \\ &= dt \wedge \frac{dB}{dt} + d_Y B + B \wedge B = dt \wedge \frac{dB}{dt} + F^Y_B.\end{aligned}$$

Now when is $\underline{B}$ an anti-self-dual connection? We need to think about the Hodge star. If $dt, \eta_1, \eta_2, \eta_3$ is an oriented orthonormal frame in $\Omega^1(Y)_p$, then

$$\star_4(dt \wedge \eta_1) = \eta_2 \wedge \eta_3, \quad \star_3 \eta_1 = \eta_2 \wedge \eta_3.$$

So we get

$$\star_4(dt \wedge \alpha) = \star_3 \alpha.$$

We can use this to compute

$$\star_4 F_{\underline{B}} = \star_3 \Big(\frac{dB}{dt}\Big) + dt \wedge \star_3 F^Y_B.$$

The conclusion is that $\underline{B}$ is anti-self-dual if and only if

$$\frac{dB}{dt} = - *_Y F_B^Y.$$

This can be also thought of as an evolution equation, if you'd like.

One issue is that F_B and B sits inside difference spaces. The connection B lies in the Banach space $B \in L^2_3(\Lambda^1 \otimes \mathfrak{su}(2))$, but the curvature sits inside $F_B \in L^2_2$, not in L^2_3. This is formally a gradietn flow equation. For $B \in \Omega^1_Y(\mathfrak{su}(2))$, I want to define the **Chern–Simons functional**

$$CS(B) = -\int_Y \operatorname{tr}\Big(\frac{1}{2} B \wedge dB + \frac{1}{3} B \wedge B \wedge B\Big).$$

This will satisfy

$$\begin{aligned}\frac{d}{dt}(CS(B+t\beta))|_{t=0} &= -\int \frac{1}{2}\operatorname{tr}(\beta \wedge dB + B \wedge d\beta) - \int \beta \wedge B \wedge B \\ &= -\int \beta \wedge (dB + B \wedge B) = -\int \beta \wedge F_B \\ &= \int \langle \beta, \star_3 F_B \rangle d\mathrm{vol}_Y = \langle \beta, \star F_B \rangle_{L^2}.\end{aligned}$$

So the gradient of CS is $\star F_B$.

8 February 7, 2018

We were looking at the Riemannian manifold Y^3, a bundle $\mathbb{C}^2 \times Y$ and a SU(2)-connection $d + B$. We introduced the Chern–Simons functional

$$CS(B) = -\frac{1}{2}\int_Y \operatorname{tr}(B \wedge dB + \frac{2}{3} B \wedge B \wedge B).$$

Here, we interpret $B \in \Omega^1_Y(\mathfrak{su}(2))$ as something with a L^2 inner product. We compute

$$\operatorname{grad}(CB)_B = \star F_B$$

where $\star$ is with respect to Y.

8.1 Chern–Simons functional under gauge transformations

This relates to anti-self-duality on the cylinder $[t_0, t_1] \times Y$. Given a path $B(t) = \underline{B}$, the anti-self-duality equation $\star F_{\underline{B}} = -F_{\underline{B}}$ can be also written as

$$\frac{d}{dt} B(t) = -\operatorname{grad}(CS)_{B(t)}.$$

For $B(t)$ any path (not necessarily the gradient flow), we can compute

$$CS(B(t_1)) - CS(B(t_0)) = \int \Big\langle \frac{dB}{dt}(t), \star F_{B(t)} \Big\rangle dt \, d\mathrm{vol} \quad = -\int \operatorname{tr}\Big(\frac{dB}{dt} \wedge F_{B(t)}\Big) dt.$$

But because $F^X_{\underline{B}} = dt \wedge \frac{dB}{dt} + F^Y_B$, we get

$$CS(B(t_1)) - CS(B(t_0)) = -\frac{1}{2}\int \operatorname{tr}(F_{\underline{B}(t)} \wedge F_{\underline{B}}) dt.$$

If the connection $\underline{B}$ is anti-self dual, we get fill in

$$CS(t_0) - CS(t_1) = \frac{1}{2}\int \operatorname{tr}(F_{\underline{B}} \wedge F_{\underline{B}}) = \frac{1}{2}\int |F_{\underline{B}}|^2.$$

This makes sense because B is a downward gradient flow.

Denote by $\mathcal{A}(Y) = \Omega^1(\mathfrak{g})$ the connections on $\mathbb{C}^2 \times Y$ and $\mathcal{G}(Y) = \operatorname{Map}(Y, \mathrm{SU}(2))$ the gauge group. We note that

$$\pi_0 \mathcal{G}(Y) = [Y, \mathrm{SU}(2)] = [Y, S^3] \cong \mathbb{Z}.$$

The Chern–Simons form $\mathscr{A}, \mathbb{R}$ is not $\mathcal{G}$-invariant, but it is invariant under the identity component of $\mathcal{G}$. Here is why. Consider a bath from B to B' that have the same connected orbit of $\mathcal{G}$. Note that the gradient $\operatorname{grad}(CB)_B = *F_B$ lies in the Coulomb slice, because we can compute

$$d^*_B(\star F_B) = \star d_B \star (\star F_B) = \star(d_B F_B) = 0.$$

Now let's see why the Chern–Simons form is not invariant under non-identity component gauge transformations. Take a path $B(t)$ on $[t_0, t_1] \times Y$, with $B(0) = B_0$ and $B(t) = B_1$. Because B_1 is a gauge transformation of B_0, we can glue both sides together to form a torus $S^1 \times Y$, and a connection $\underline{B}$ on a bundle over this. But then

$$CS(t_1) - CS(t_0) = -\frac{1}{2}\int_{S^1\times Y} \operatorname{tr}(F_{\underline{B}} \wedge F_{\underline{B}})dt = -\frac{1}{2}(8\pi^2 c_2(E)[S^1 \times Y]) \in 4\pi^2\mathbb{Z}.$$

From this discussion, we see that the Chern–Simons form descends to

$$\overline{CS} : \mathcal{B} = \mathcal{A}/\mathcal{G} \to \frac{\mathbb{R}}{4\pi^2\mathbb{Z}}.$$

In the literature, you will normally see the normalization $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/8\pi^2\mathbb{Z}$.

8.2 Representation variety

Now we have this map $\mathcal{B} \to S^1$, and the critical points of CS in $\mathcal{A}(Y)$ will be

$$\{B : F_B = 0\} = \text{flat connections}.$$

These are flat connections modulo $\mathcal{G}$, and so

$$\mathcal{R}(Y) = \frac{\operatorname{Hom}(\pi_1(Y), \mathrm{SU}(2))}{\mathrm{SU}(2)}$$

where SU(2) acts by conjugation. This is called the **representation variety**.

Example 8.1. For $Y = S^3$, the representation variety is $\mathcal{R} = *$, which is the trivial flat connection.

Example 8.2. Consider $Y = T^3$. Here, $\pi_1 \cong \mathbb{Z}^3$, where we use γ_i as the generators. So up to conjugation, we can simultaneously diagonalize so that

$$\rho(\gamma_n) = \begin{pmatrix} e^{i\theta_n} & 0 \\ 0 & e^{-i\theta_n} \end{pmatrix}.$$

But we also have $\rho_{(\theta_1,\theta_2,\theta_3)} \sim \rho_{(-\theta_1,-\theta_2,-\theta_3)}$ and so the representation variety is

$$\mathcal{R} = T^3/(x \sim -x).$$

The stabilizer of a point in SU(2) is generically S^1 and SU(2) at special points.

Example 8.3. Let P be the Poincaré homology sphere. Consider the symmetry group of a icosahedron $A_5 \cong I \subsetneq \mathrm{SO}(3)$, and lift it to $\tilde{I} \subseteq \mathrm{SU}(2)$ with $|\tilde{I}| = 120$. Now $\tilde{I} = [\tilde{I}, \tilde{I}]$ and $P = \mathrm{SU}(2)/\tilde{I}$ is a homology sphere.

Now let us look at is representation variety

$$\mathcal{R} = \frac{\operatorname{Hom}(\tilde{I} \to \mathrm{SU}(2))}{\sim}.$$

There is the trivial connection θ, the inclusion $\alpha : \tilde{I} \to \mathrm{SU}(2)$, and the map $\beta : \tilde{I} \to \mathrm{SU}(2)$ coming from the outer automorphism of $\tilde{I}$. In $\mathcal{B}$, θ is going to have nontrivial stabilizer and α and β is going have trivial stabilizer.

Question. *If Y is a homology sphere ($H_1 = 0$) and $Y \neq S^3$, does $\mathcal{R}$ always contain a nontrivial $\rho \neq 0$?*

The answer to this is not known in general. The answer is yes for a rather large class of manifolds, obtained by surgery on a knot.

9 February 9, 2018

9.1 Gradient flows and PDE

Given a path $\gamma : S^1 \to \mathbb{C}$, you can write this in terms of a Fourier series

$$\gamma(\theta) = \sum_n a_n e^{in\theta}.$$

We can define the energy as

$$\varepsilon(\gamma) = \frac{1}{2}\int_{S^1} \langle \gamma, \Delta\gamma \rangle d\theta = \frac{1}{2}\int_{S^1} |\dot{\gamma}|^2 d\theta.$$

If we follow the downward gradient flow of this functional ε, it would be given by the heat equation

$$\frac{\partial \gamma}{\partial t} = \frac{d^2}{d\theta^2}\gamma = -\Delta\gamma.$$

Then it would shrink to the origin a_0, because

$$a_n(t) = e^{-n^2 t} a_n(0).$$

Here n^2 are the eigenvalues of Δ, and is positive. This is why the forward direction is well-defined, while for backward direction to exist for even short time, you need rapidly decaying Fourier coefficients.

Compare this with the following example: for $\gamma : S^1 \to \mathbb{C}$, consider

$$S(\gamma) = \frac{1}{2}\int \Big| \gamma, i\frac{d}{d\theta}\gamma \Big| d\theta = \frac{1}{2}\langle \gamma, D\gamma \rangle d\theta.$$

Then the downward gradient is

$$\frac{\partial \gamma}{\partial t} = -\frac{\partial \gamma}{\partial \theta},$$

which is the Cauchy–Riemann equation for $[a, b] \times S^1 \to \mathbb{C}$. If you look at the spectrum $\mathrm{Spec}(D)$, it is $\mathbb{Z}$ which is neither positive not negative. So it doesn't make sense to talk about the forward flow or even the backward flow. Still, we can think of bounded solutions on $[0, \infty) \times S^1$ as holomorphic functions on the disk, with $f(0) = 0$.

Let us look at the abelian case. The Chern–Simons function for $d+b$ is given by

$$CS = \frac{1}{2}\int_Y \langle d, \star db \rangle \mathrm{dvol}_Y.$$

If you think about $\star d : \Omega^1 \to \Omega^1$, this is symmetric because

$$\int \langle a, \star db \rangle \mathrm{dvol} = \int a \wedge db = \int da \wedge b.$$

Here $\star d$ has a big kernel, and $\ker d$ is comparable to $\operatorname{im} d$. But if we restrict $\star d$ to $\ker(d^*)$, its kernel is finite-dimensional.

The linear "flow" is on $\ker(d^*)$, but we have neither forward or backward existence, even on $[0, \epsilon)$. Floer's insight was that we can really do Morse theory even though we don't have the flow. Another problem is that the representation variety has some singularities.

Half the idea is to pass from SU(2) to SO(3). Consider a SO(3)-bundle P on Y^3. If we fix $w_2(P) = \omega$, this determines P up to isomorphism. Then we can look at the representation variety

$$\mathcal{R}^{\omega}_{\mathrm{SO}(3)}(Y) = \frac{(\text{flat connections})}{\mathcal{G}_P^{\mathrm{SO}(3)}} = \frac{\rho : \pi_1 \to \mathrm{SO}(3)}{\mathrm{SO}(3)}.$$

Example 9.1. Take $Y = S^2$ (or $S^1 \times S^2$ if you really want it to be a 3-manifold). Let ω non-zero. Then $\mathcal{R}^{\omega}_{\mathrm{SO}(3)}(Y) = \emptyset$ while $\mathcal{R}_{\mathrm{SU}(2)}(Y) = \theta$.

Example 9.2. Take $Y = T^2$. Here, we need to consider $r_\theta : \mathbb{Z}^2 \to \mathrm{SO}(3)$. They can either be rotations about the same axis, in which case they have $w_2 = 0$ and hence lift to SU(2). The other possibility is

$$\rho(\gamma_1) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

So we get $\mathcal{R}^{\omega}_{\mathrm{SO}(3)}(T^2) = *$.

Let $\mathcal{B}$ be the flat SO(3)-connection on T^2. In $\mathcal{G}_P^{\mathrm{SO}(3)}$, the stabilizer of the connected component $\Gamma_B^{\mathrm{SO}(3)}$ of $\mathrm{Hol}(B)$. This is the commutant of the image of ρ, which is the Klein four group V_4. So things get more interesting than in SU(2).

Here is the other half-idea. Consider

$$\mathcal{G}_P^{\mathrm{SO}(3)} = \mathrm{Aut}(P) = \text{sections of } \underline{G} \to Y.$$

For example, look at $Y = T^2$. Here, π_1 is generated by two loops γ_1, γ_2, and this already gives a map

$$\mathcal{G}_P^{\mathrm{SO}(3)} \to \pi_1(\mathrm{SO}(3))^2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Note that is an SO(3)-action on SU(2) by conjugations, so there is a bundle $\underline{G}^{\mathrm{SU}(2)} \to Y$ with fiber SU(2). So we can define

$$\mathcal{G}_P^1 = \mathrm{SU}(2)\text{-gauge transformations} = \{\text{sections of } \underline{G}^{\mathrm{SU}(2)} \to Y\}.$$

So we get a map

$$\mathrm{Map}(Y, \pm 1) \to \mathcal{G}_P^1 \to \mathcal{G}_P^{\mathrm{SO}(3)}.$$

Here, the image is the automorphisms of $P \to Y$ which lift to a section of $\underline{G}^{\mathrm{SU}(2)} \to Y$.

10 February 12, 2018

10.1 SO(3)-bundle as a SU(2)-bundle

Given a SU(2)-bundle $\tilde{P} \to Y$, you get a SO(3)-bundle $P \to Y$ by a 2-to-1 covering map. If a bundle lifts, then the space of connections are the same:

$$\mathcal{A}(\tilde{P}) = \mathcal{A}(P)$$

are both affine spaces of $\Omega^1(Y, \mathbb{R}^3)$ where I identify $\mathfrak{su}(2) = \mathfrak{so}(3) = \mathbb{R}^3$.

But there are bundles that don't lift, so modulo gauge transformations, they can be different. Even for flat connections, they can be different. Consider Σ a genus g surface. A flat connection on a SO(3)-bundle is going to be given by a_i, b_i satisfying

$$\prod_i a_i^{-1} b_i^{-1} a_i b_i = 1 \in \mathrm{SO}(3).$$

When we lift this to SU(2), this can be either -1 or $+1$. If it is -1, then $w_2 \neq 0$ and if it is $+1$, then $w_2 = 0$. If -1, this can be thought of as a flat connection on $\Sigma \setminus \{p\}$ with holonomy -1 around p.

Let Y be a 3-manifold with $\omega = w_2(P)$ with $P \to Y$ a SO(3)-bundle. Let ω be the Poincaré dual of $[w]$ where w is a closed curve in Y. Then on $Y \setminus w$, ω restricts to 0 and so we can lift P to $\tilde{P}$. Then flat connections in P can be thought of as flat connections $\tilde{P}$ over $Y \setminus w$ with holonomy $-1 \in \mathrm{SU}(2)$ around the link of w.

Example 10.1. Let us consider the example $Y = T^2 \times S^1$ and $w = p \times S^1$. Let a, b be the two generators of $\pi_1(T^2)$ and c be the generator of $\pi_1(S^1)$. Then our condition becomes

$$\alpha^{-1}\beta^{-1}\alpha\beta = -1, \quad \alpha^{-1}\gamma^{-1}\alpha\gamma = 1, \quad \beta^{-1}\gamma^{-1}\beta\gamma = 1.$$

Up to conjugation, there are only the solutions

$$\alpha = i, \quad \beta = j, \quad \gamma = \pm 1$$

as quaternions. Then the representation variety $\mathcal{R}^w(Y)$ two points. These are two irreducible representations.

Consider

$$\mathcal{B}^w(Y) = \mathcal{A}^w(Y)/\mathcal{G}^{\mathrm{SU}(2)}.$$

Then the Chern–Simons functional is going to have some reducible solutions and critical points.

Lemma 10.2. *If there exists an orientable $\Sigma_g \hookrightarrow Y$ and $w \cdot \Sigma_g$ is odd, then $R^w(Y)$ consists of irreducibles.*

Proof. This is because $\prod \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = -1$. □

So the reducibles stay away from the critical points of the Chern–Simons form in this case.

10.2 Morse theory

Let f be a Morse function on a Riemannian manifold B. At $\alpha \in \mathrm{Crit}(f)$, we define the **index**

$$\begin{aligned}\mathrm{index}(\alpha) &= \text{dim. of unstable manifold}\\ &= \#\text{ of negative eigenvalues of } \mathrm{Hess}(f) \text{ at } \alpha.\end{aligned}$$

For α and β critical points, we define the space

$$\begin{aligned}M(\alpha,\beta) &= \text{space of flow lines from } \alpha \text{ to } \beta\\ &= U(\alpha)\cap S(\beta).\end{aligned}$$

This flow is called **Morse–Smale** if $U(\alpha)\cap S(\beta)$ is a transverse intersection. In this case,

$$\begin{aligned}\dim(M(\alpha,\beta)) &= \dim U(\alpha) + \dim S(\beta) - \dim B\\ &= \dim U(\alpha) - \dim U(\beta) = \mathrm{index}(\alpha) - \mathrm{index}(\beta).\end{aligned}$$

You can think this in terms of differential equations. Let $V = \nabla f$. The space of flow lines is the space of solutions to

$$\frac{d\gamma}{dt} + V(\gamma(t)) = 0.$$

If u is a vector field along γ, then the line $\gamma + \exp(\epsilon u)$ is going to be another flow line if

$$\frac{du}{dt} + (\nabla_u V)|_{\gamma(t)} = 0.$$

This can be phrased as $\nabla_{\dot\gamma(t)}(u) + H(u) = 0$ where H is the Hessian. So the Morse–Smale condition is equivalent to the surjectivity of

$$L : u \mapsto \nabla_{\dot\gamma(t)} u + H(u).$$

So let us look at the space

$$L : u \mapsto \frac{du}{dt} + H(t)u$$

for $u : \mathbb{R} \to \mathbb{R}^b$ and $H(t)$ self-adjoint $b\times b$ matrices, which is bounded and C^∞. When is this operator $L^2_1 \to L^2$ Fredholm? The kernel is clearly finite-dimensional because it is determined by value at a point. But is the cokernel finite-dimensional? If we can check that $\mathrm{im}(L)$ is closed, then we can just check that $\mathrm{im}(L)^\perp$ is finite-dimensional, which is the same as checking that the kernel of

$$u \mapsto -\frac{du}{dt} + H(t)u$$

is finite-dimensional.

Now is $\operatorname{im}(L)$ closed? Let us take $H \equiv 0$ for instance. Here, the operator is

$$L^2_1 \to L^2; \quad u \mapsto \frac{d}{dt}u.$$

If we look at $v = \frac{t}{1+t^2}$, then this can be approximated by things that are derivatives, but it is not a derivative of a L^2 function, because the integral of t^{-1} blows up. That is, this operator is not Fredholm.

Now let us look at $H \equiv \epsilon$ where ϵ is a nonzero 1×1 matrix. Then you can solve the equation

$$\frac{du}{dt} + \epsilon u = v$$

explicitly by convoluting $u = G * v$ where G is the Green's function

$$G(t) = \begin{cases} \exp(-\epsilon t) & t > 0 \\ 0 & t < 0. \end{cases}$$

So the operator is invertible, and in particular, Fredholm.

11 February 14, 2018

Let $D : \Gamma_Y(E) \to \Gamma_Y(E)$ be a first-order operator on Y. Assume that this is elliptic and formally self-adjoint. In particular, consider a situation in which

$$D : L^2_1(Y; E) \to L^2(Y, E)$$

has no kernel, and hence no cokernel. In this case, $\|Du\|_{L^2(Y)} \sim \|u\|_{L^2_1(Y)}$.

Example 11.1. Let $D = i\frac{d}{d\theta}$ on S^1. Its inverse is

$$D^{-1} : L^2 \to L^2_1 \hookrightarrow L^2$$

is a compact operator by Rellich's lemma. This means that $D^{-1} : L^2 \to L^2$ is a compact self-adjoint operator, and we can apply the spectral theorem. Then $L^2(Y)$ has a complete orthonormal system e_n and $\lambda_n \in \mathrm{Spec}(D)$. These eigenspaces are going to be finite-dimensional, with $|\lambda_n| \to \infty$ as $|n| \to \infty$.

11.1 Operator on a cylinder

Now consider the operator

$$Q = \frac{d}{dt} + D : L^2_1(X; E) \to L^2(X; E).$$

on $\mathbb{R} \times Y = X$. (Here, we are assuming that D is translation-invariant in the t-direction.) I claim that this is Fredholm and invertible. Given any u, we can write it as

$$u = \sum u_n(t) e_n$$

for $u_n : \mathbb{R} \to \mathbb{C}$. Then the equation we are looking at can be written as

$$\Big(\frac{d}{dt} + \lambda_n\Big) u_n = v_n.$$

If we look at the L^2-norm of Qu, we can take the Fourier transform in t and then write

$$\sum_n \int_{\mathbb{R}} |i\tau + \lambda_n|^2 |\hat{u}_n|^2 = \sum_n \int_{\mathbb{R}} (|\tau|^2 + |\lambda_n|^2)|\hat{u}_n|^2.$$

Then the L^2-norm of Qu is

$$\|Qu\|^2_{L^2_X} \simeq \int_{\mathbb{R}} \Big|\frac{du}{dt}\Big|^2 + \int_{\mathbb{R}} \|u(t)\|^2_{L^2_1} \simeq \|u\|^2_{L^2_1}.$$

This shows that D is invertible.

We are interested in the operator, on $X = \mathbb{R} \times Y$,

$$-d^*_X \oplus d^+_X : \omega^1(X) \to \Omega^0 \oplus \Omega^+.$$

For $a = cdt + b$ with c a t-dependent function and b a t-dependent 1-form on Y, we can write

$$\begin{aligned} -d_X^* a &= \dot{c} + d_Y^* b, \\ 2d^+ a &= (\dot{b} - dc + \star db) \wedge dt + \star(\dot{b} - dc + \star db). \end{aligned}$$

So this operator is

$$\begin{pmatrix} c \\ b \end{pmatrix} \mapsto \begin{pmatrix} \dot{c} \\ \dot{b} \end{pmatrix} + \begin{pmatrix} 0 & -d^* \\ -d & -\star d \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} c \\ b \end{pmatrix} + D \begin{pmatrix} c \\ b \end{pmatrix}.$$

Here, D is formally self-adjoint. So for this operator

$$Q : L_1^2 \to L^2,$$

$Q + \epsilon$ is invertible if and only if $0 \notin \operatorname{Spec}(D + \epsilon)$.

Let us first look at the case $D = D^0 + h(t)$ where 0 is a 0th order operator (which is just a multiplication), and D and D^0 are symmetric. Let us assume that $h(t) = h^\infty$ for $|t| > R$, and that $0 \notin \operatorname{Spec}(D^0 + h^\infty)$. If we write $D^\infty + D^0 + h^\infty$, then $D^\infty - D$ is a 0th order operator supported in $[-R, R]$. Then

$$D^\infty - D : L_1^2 \to L^2$$

is compact, and because D^∞ is Fredholm, we get that D is Fredholm with $\operatorname{index}(D) = 0$. (Deformations don't change index.)

Now let us relax the condition so that $h(t) = h^{+\infty}$ for $t > R$ and $h(t) = h^{-\infty}$ for $t < -R$, with $D^0 + h^{\pm\infty}$ both $0 \notin$ Spec. The strategy is to solve it locally using a partition of unity, and then gluing them back. Of course, there are errors, but the point is that the errors are compact.

Consider functions $h^+(t)$ and $h^-(t)$ with

$$h^+(t) = h^{+\infty} \text{ for } |t| > s, \quad h^-(t) = h^{-\infty} \text{ for } |t| > s.$$

Then

$$Q^+ = \frac{d}{dt} + D^0 + h^+(t), \quad Q^- = \frac{d}{dt} + D^0 + h^-(t)$$

falls into the first case. When you try to get the inverse, you have

$$Pv = \gamma_+ P^+ \eta_+ v + \gamma_- P^- \eta_- v$$

up to some compact contribution from the partitions of unity.

Example 11.2. Let A be a connection on $\mathbb{R} \times Y$, that is translation invariant on $[R, \infty)$ (call it A^+) and translation invariant on $(-\infty, R]$ (call it A^-). We then can say that $\delta_A = -d_A^* \oplus d_A^+$ is Fredholm provded that the 3-dimensional operators don't have kernel (when 0 is not in the spectrum of A^+ and A^-).

12 February 16, 2018

Last time we looked at the standard theory of a differential operator on the cylinder. Let $D = D(t) : L^2_1(Y, E) \to L^2(Y, E)$ be a differential operator on Y with $D = D^0 + h(t)$, where D^0 and h are symmetric. Write $Q = \frac{d}{dt}$.

Assume that $h = h^-$ for $t \le -R$ and $h = h^+$ for $t \ge R$. Assume that $0 \notin \mathrm{Spec}(D^\pm) = \mathrm{Spec}(D^0 + h^\pm)$. Then what we said last time is that $Q : L^2_1 \to L^2$ is Fredholm.

12.1 Spectral flow

Theorem 12.1. *The index of Q is the spectral flow of $D(t)$ for t from $-\infty$ to ∞.*

As we flow form D^- to D^+, we will sometimes hit 0. So we should look at the set of h such that $D^0 + h$ has 0 in the spectrum. If we suppose that $(D^0 + h)U = 0$ with $\|U\|_{L^2} = 1$, then the eigenvalue will move as we change H, so we will have

$$(D^0 + H + \epsilon h)(U + \epsilon u) = \epsilon\lambda U + \epsilon h).$$

Let H be a real Hilbert space. If $\ker(D^0 + H$ is 1-dimensional, then in a neighborhood Ω of H, there exists a $\Lambda : \Omega \to \mathbb{R}$ with $d\Lambda \neq 0$, such that $\Lambda(H+h)$ is an eigenvalue of $H + h$ with $\Lambda = 0$ at H. (We're not worrying about 0 being a repeated eigenvalue, because this is a codimension 4 condition and a generic path won't have such a case.)

So to prove this theorem, we only need to check it in the case of one eigenvalue moving from negative to positive, and check that the index of Q is 1.

12.2 Negative gradient flow

A Morse function is a function with nondegenerate Hessian at critical points. The equation can be written as

$$\dot{\gamma} + \nabla f = 0.$$

Then

$$\frac{d}{dt} f(\gamma(t)) = -\|\nabla f|_{\gamma(t)}\|^2.$$

Suppose that $\gamma(t) \to b$ and for simplicity assume $f(b) = 0$. If f is nondegenerate at that point, then

$$\|\nabla f\|^2 \ge c\|\gamma(t) - b\|^2$$

and so $\frac{d}{dt} f \le -cf$. This shows that we have exponential decay in the distance. This is not true for degenerate critical points. You can image a bowl-shaped LP record with the groove infinitely extending toward the center. Then the negative gradient can be made to converge as slow as you want.

Now let us look at the space $B = \mathcal{A}/\mathcal{G}$ on Y, of connections module gauge equivalences. Assume that β is a critical point that is irreducible. Formally, we can say that

$$TB_\beta = \text{“Coulomb slice”} = \ker d_B^* \subseteq \Omega^1(Y, \mathfrak{g}).$$

We also have $\text{Hess}(CS) = \star d_B$, on $TB|_\beta$. For this to be nondegenerate, we want $0 \notin \text{Spec}(\star d_B)$. Last time, we saw that

$$Q = \frac{d}{dt} + \begin{pmatrix} 0 & d_B^* \\ -d_B & \star d_B \end{pmatrix}.$$

todo

So for flat irreducible connections $\alpha, \beta \in \mathcal{B}_Y$, we can formally at the gradient flow lines converging to α, β at $-\infty, \infty$. This connection can be thought of as a connection on A on $\mathbb{R} \times Y$ and $F_A^+ = 0$. If we suppose that

$$\int_{\mathbb{R}\times Y} |F_A|^2 < \infty,$$

we can slice them up into connections $A^{(n)}$ on $I \times Y$. By Uhlenbeck's theorem, there then exists a converging subsequence

$$[A^{(n')}] \to [A^\infty]$$

on $I^\circ \times Y$ as $n' \to \infty$. But by the finite energy condition, we have $\|F_{A^{(n)}}\| \to 0$. This shows that A^∞ is flat. Moreover, if it has finite energy, it can't move from one neighborhood of a critical point to another neighborhood of a critical point infinitely often. Then $[A(t)] \to \beta$ in B_Y, as $t \to \infty$. If β is nondegenerate, then $\star d_b$ has no kernel in $T\mathcal{B}_\beta$ and $CS(A(t)) - CS(B)$ has exponential decay.

So we can describe the space of flowlines $M(\alpha, \beta)$ as the following. Let α, β be irreducible nondegenerate. Choose any connection A^0 on $\mathbb{R} \times Y$ that is in temporal gauge on $(-\infty, -R]$ and $[R, +\infty)$ and constant $[A^0(t)] = \alpha$ and $[A^0(t)] = \beta$ on the two intervals. Then

$$M_{[A^0]}(\alpha, \beta) = \{A = A^0 + a : F_A^+ = 0, \quad \|a\|_{L_3^2, A^0} < \infty\}/\mathcal{G}.$$

13 February 21, 2018

13.1 Morse homology

Let B be a compact finite-dimensional manifold and $f : B \to \mathbb{R}$ be a Morse function. This means that $\mathrm{Hess}(f)_\alpha$ is non-degenerate for all critical α, and this implies that there are finitely many critical points. For two critical points α, β we defined the space of flowlines as

$$M(\alpha, \beta) = \{\zeta : \mathbb{R} \to B \text{ with } \dot{\zeta} = -\nabla f\} = U_\alpha \cap S_\beta$$

There is a free action of $\mathbb{R}$ on $M(\alpha, \beta)$ by translations, and we can define

$$\overline{M}(\alpha, \beta) = M(\alpha, \beta)/\mathbb{R}.$$

We said that the function f is Morse–Smale if the intersection $U_\alpha \cap S_\beta$ is transverse for all α, β.

The transversality condition can be thought of as a linearization equation. An infinitesimal change gives a linear operator

$$\delta : L^2_1(\mathbb{R}; \zeta^*(TB)) \to L^2(\mathbb{R}; \zeta^*(TB)),$$

given by

$$\delta u = \nabla_{\partial/\partial t} u - Hu$$

where $H : TB \to TB$ is the Hessian.

The Morse–Smale condition implies that the dimension of the intersection is

$$\dim M(\alpha, \beta) = \mathrm{ind}(\alpha) - \mathrm{ind}(\beta)$$

This can be thought of as the change of the number of negative eigenvalues. So it is the spectral flow of H along ζ.

Using this, we can define Morse homology. Let C_* be the $\mathbb{F}_2$-vector space with basis $\mathrm{Crit}(f)$ (which is a finite-dimensional vector space). The grading is such that if $\alpha \in \mathrm{Crit}(f)$ has index i, then $\alpha \in C_i$. The differential here is given by

$$\partial : C_i \to C_{i-1}; \quad \partial \alpha = \sum_{\mathrm{ind}(\alpha, \beta) = 1} n_{\alpha\beta} \beta$$

where $n_{\alpha\beta}$ is the number of elements in the 0-dimensional manifold $\overline{M}(\alpha, \beta)$. We need to check that this number $n_{\alpha\beta}$ is finite, using some compactness. We also need to check that $\partial^2 = 0$. Then

Proposition 13.1. $H_i(C_*, \partial) \cong H_i^{\mathrm{sing}}(B; \mathbb{F}_2)$.

The easiest way to check this is to use the Morse flow to get an actually cell decomposition of B. This also shows that the Morse homology is independent of the choice of the Morse function, and also the Riemannian metric. In fact, give a triangulation of a manifold, you can construct a Morse function such that the

0-simplices are local minima, the barycenters of 1-simplices are index 1 critical points, the barycenters of 2-simplices are index 2 critical points, and so on.

What about in over $\mathbb{Z}$? We first need to choose orientations for each U_α. Then each S_α gets a co-orientation from U_α, i.e., an orientation of the normal. This gives an orientation on $U_\alpha \cap S_\beta$. In particular, $M(\alpha, \beta)$ acquires an orientation, and so does $\overline{M}(\alpha, \beta)$. Because it is 0-dimensional, it's a collection of points with signs. So we just need to modify the definition of $n_{\alpha\beta}$ so that it is the number of points with sign in $\overline{M}(\alpha, \beta)$.

13.2 Morse theory on the space of connections

So what can we say in the instanton case? Let Y^3 be a oriented closed Riemannian manifold, and fix $P \to Y^3$ an SO(3) bundle with $\omega = w_2(P) = PD[w]$ where $w \subseteq Y$ is 1-dimensional. Let $\mathcal{A}$ be the space of connections on P, and let $\mathcal{G}^{\mathrm{SU}(2)}$ be the SU(2)-gauge group transformations. This maps into the total gauge group $\mathrm{Aut}(P)$. Then we defined

$$\mathcal{B}^w(Y) = \mathcal{A}/\mathcal{G}^{\mathrm{SU}(2)}.$$

There is a Chern–Simons functional

$$CS : \mathcal{B}^w(Y) \to \mathbb{R}/(4\pi^2\mathbb{Z}) = S^1,$$

and the critical points $\mathrm{Crit}(CS) = \mathcal{R}^w(Y)$ are the flat connections. Suppose that w is admissible, that is, there exists a surface $\Sigma^2 \subseteq Y$ such that $w\Sigma$ is odd. Then the critical points are irreducible:

$$\mathcal{R}^w(Y) \subseteq \mathcal{B}^w(Y)^*.$$

Suppose that the Chern–Simons functional is Morse. (We will later get rid of this condition by adding a small perturbation.) In other words, assume that $\mathrm{Hess}_\alpha(CS)$ has no kernel for all $\alpha \in \mathrm{Crit}(CS)$. If we write $\alpha = [B]$, then this means that

$$\star d_B : \ker(d_B^*) \to \ker(d_B^*)$$

has no kernel. This also means that

$$\begin{pmatrix} 0 & d_B^* \\ -d_B & \star d_B \end{pmatrix}$$

is elliptic and invertible. Now we can compute the index difference between α and β as

$$\mathrm{ind}_\zeta(\alpha, \beta) = \text{spectral flow of } \mathrm{Hess}(CS) \text{ along } \zeta.$$

We might worry about $\mathrm{ind}_\zeta(\alpha, \beta)$ being dependent on ζ. We know that $\mathcal{G}^{\mathrm{SU}(2)} \simeq \mathbb{Z}$, and $\mathcal{A}$ is contractible, with the reducible being finite codimension. So

$$\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}^{\mathrm{SU}(2)} \simeq B(\mathcal{G}^{\mathrm{SU}(2)}/(\pm 1)).$$

This shows that $\pi_1(\mathcal{B}^*) \cong \mathbb{Z}$.

So what is the spectral flow of a closed loop in $\mathcal{B}^*$? Let η be a nontrivial loop. We should look at

(a) the spectral flow of $\mathrm{Hess}(CS)$, and

(b) the total drop $-\int_0^1 \frac{d}{dt}(CS)dt$ in the Chern–Simons functional.

If we cut this path η at a point, we can think of this of, in $\mathcal{A}^*$, as moving from one connected component of a $\mathcal{G}$-orbit to another connected component of the same $\mathcal{G}$-orbit. Then it can be considered as a bundle $P_\eta \to S^1 \times Y$. The spectral flow is the index

$$\begin{aligned}\mathrm{index}(-d^*_{A_\eta} \oplus d^+_{A_\eta}) &= \mathrm{index}(\text{linearized ADS with Coulomb gauge}) \\ &= 8c_2(P_\eta) - 3(b^+ - b^1 + 1) = 8c_2(P_\eta).\end{aligned}$$

The total drop in the Chern–Simons functional is

$$\frac{1}{2}\int \mathrm{tr}(F_{A_\eta} \wedge F_{A_\eta}) = 4\pi^2 c_2(P_\eta).$$

The upshot is that $\mathrm{ind}_\zeta(\alpha, \beta)$ makes sense, as well as the change in the Chern–Simons functional.

What is the Morse–Smale condition here? Let $\zeta \in M(\alpha, \beta)$ be a formal gradient flow line. Near ζ, what is the structure of $M(\alpha, \beta)$? The flow line ζ is some connection A_ζ on $\mathbb{R} \times Y$ in temporal gauge. The Morse–Smale condition is the surjectivity of the linearized opeartor

$$\delta u = \frac{d}{dt}u + D_{A_\eta}u = \frac{d}{dt}u + \begin{pmatrix} 0 & d^*_{A-\eta} \\ -d_{A_\eta} & \star d_{A_\eta} \end{pmatrix}$$

as $L^2_{k,A_\eta} \to L^2_{k-1,A_\eta}$.

The Morse–Smale condition implies that near ζ, the space $M(\alpha, \beta)$ is smooth, and its dimension is $\mathrm{ind}_\zeta(\alpha, \beta)$, which is defined as the spectral flow.

14 February 23, 2018

For Y a 3-manifold, we said that $[w] = \mathrm{PD}[w_2(P)]$ is admissible if there exists an oriented Σ^2 such that $w \cdot \Sigma$ is odd. Then the representation variety $\mathcal{R}^w(Y)$, the set of critical points of CS, is contained in the irreducibles $\mathcal{B}^*$. We want to look at the condition that CS is Morse, and the flow is Morse–Smale.

14.1 Instanton Floer homology

Let ζ be a formal gradient-flow line in $\mathcal{B}^*$ from α to β. For a fixed $z \in \pi_1(Y; \alpha, \beta) = \pi_0(\text{paths } \alpha \to \beta)$, we have

$$M(\alpha, \beta) = M_z(\alpha, \beta).$$

Here we can define $\mathrm{ind}(\alpha, \beta) =$ spectral flow. We showed last time that

$$\kappa(z) = \frac{1}{4\pi^2}(\text{drop in } CS \text{ along } z) = \frac{1}{8\pi^2} \int_{\mathbb{R}\times Y} \mathrm{tr}(F_A \wedge F_A).$$

Both of them makes sense for η a loop instead of a path. In this case,

$$\mathrm{ind}(\eta) = 8\kappa = 8c_2(\tilde{P})$$

where $\tilde{P}$ is on $S^1 \times Y$. Because $\pi_1(\mathcal{B}^*) = \mathbb{Z}$, we can consider $z + 1$ for $z \in \pi_1(Y, \alpha, \beta))$, and we will have $\kappa(z + 1) = \kappa(z)$ so that

$$\mathrm{ind}(z + 1) = \mathrm{ind}(z).$$

This shows that

$$\mathrm{Ind}(\alpha, \beta) = \mathrm{ind}_z(\alpha, \beta) \pmod 8$$

is well-defined in $\mathbb{Z}/8$. So we can think of index $\mathrm{ind}(\alpha) \in \mathrm{Gr}(Y, w)$ as inside some $(\mathbb{Z}/8)$-torsor.

The Morse–Smale condition can be written as

$$\dim M_z(\alpha, \beta) = \mathrm{ind}_z(\alpha, b),$$

and so

$$\dim \overline{M}_z(\alpha, \beta) = \mathrm{ind}_z(\alpha, \beta) - 1.$$

In this case, we can again define the Morse complex

$$C_* = \mathbb{F}_2\text{-vector space with basis } \mathcal{R}^w(Y)$$

with $\mathbb{F}_2$-coefficients. Then we have a differential

$$\partial : C_* \to C_*; \quad \partial\alpha = \sum_\beta n_{\alpha\beta}\beta$$

where again we define

$$n_{\alpha,\beta} = \sum_{z\in\pi_1(\mathcal{B}^*;\alpha,\beta),\mathrm{ind}(z)=1} \#\overline{M}_z(\alpha,\beta).$$

Certainly $\partial : C_* \to C_{*-1}$ for $* \in \mathrm{Gr}(Y,w)$. After justifying all this definition, we will be able to define the **instanton Floer homology** as

$$I^w(Y) = H_*(C_*,\partial).$$

We will first need to show that $n_{\alpha,\beta}$ is finite. Then we want to show that $\partial^2 = 0$. We also need to show that this is independent of the Riemannian metric, and that for a generic metric the Chern–Simons functional actually is Morse and Morse–Smale. Let's first deal with the compactness problem.

14.2 Uhlenbeck compactification of the space of flow lines

Suppose α and β are critical points. An **ideal trajectory** ζ^+ from α to β is the data of

$$([A], x_1, \ldots, x_\ell)$$

where $[A]$ is an anti-self-dual connection on $\mathbb{R}\times Y$ corresponding to a path ζ_A from α to β, and the bubbles $x_i \in \mathbb{R}\times Y$. We can then define

$$M_z^{\mathrm{Uhl}}(\alpha,\beta) = \{\zeta^+ : z = [\zeta_A] + \ell\}.$$

We can also formally define $\kappa(\zeta^+) = \kappa(\zeta_A) + \ell$ and $\mathrm{ind}(\zeta^+) = \mathrm{ind}(\zeta_A) + 8\ell$.

If I have a sequence $[A_n] \in M_z(\alpha,\beta)$, we say that $[A_n]$ converges weakly to some $\zeta^+ = ([A], x_1, \ldots, x_\ell)$ if $g_n(A_n) \to A$ on compact subsets of $\mathbb{R}\times Y \setminus \{x_1,\ldots,x_\ell\}$.

Let us think about what this means. Suppose a flow line $\alpha \to \beta$ approaches a broken flow line $\alpha \to \alpha' \to \beta' \to \beta$. Then it is spending more and more time near α' and also β'. Depending on the parametrization, this can be thought of as converging to $\alpha' \to \beta'$, or $\alpha \to \alpha'$. Define the broken trajectories (without parametrization)

$$\mathrm{Br}\,\overline{M}_z^{\mathrm{Uhl}}(\alpha,\beta)$$

as the collection of $(\overline{\zeta}^{+,1}, \ldots, \overline{\zeta}^{+,s})$ where these are

$$\zeta^{+,r} \in M_{z_r}^{\mathrm{Uhl}}(\alpha_{r-1},\alpha_r), \quad \alpha_0 = \alpha, \quad \alpha_s = \beta,$$

and $\sum_{r=1}^s z_r = z$. Note that here we have bubbles, whereas in the finite dimensional case we don't

Now what does it mean for $[A_n]$ with paths $\zeta_n \in M_z(\alpha,\beta)$ and $\overline{\zeta}_n \in \overline{M}_z(\alpha,\beta)$ to converge to some broken trajectory? We say that it converges to

$$(\overline{\zeta}^{+,1}, \ldots, \overline{\zeta}^{+,s}) \in \mathrm{Br}\,M_z^{\mathrm{Uhl}}(\alpha,\beta)$$

if there exists a sequence of translation $\mathcal{T}_{n,r} : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{T}_{n,r}^*(\zeta_n) \to \zeta^{+,r}$ weakly. To make sure I'm not cheating by looking at the same thing multiple times, I need to make sure that $\mathcal{T}_{n,r+1}(0) - \mathcal{T}_{n,r}(0) \to \infty$ as $n \to \infty$.

Theorem 14.1. *If $\overline{\zeta}_n \in \overline{M}_z(\alpha,\beta)$, then there exists a subsequence $\overline{\zeta}_{n'}$ which converges in this sense to $(\overline{\zeta}^{+,1}, \ldots, \overline{\zeta}^{+,s}) \in \mathrm{Br}\,\overline{M}_z^{\mathrm{Uhl}}(\alpha,\beta)$.*

This is not true of a non-Morse function, even in the finite-dimensional case. Imagine a circle being a critical set, and a sequence of flow lines that just circles around this critical set a long time and then flowing down. They might be converging to a flow line $\alpha \to \alpha'$ and $\beta' \to \beta$, where α', β' lie on this circle. Then there is no flow line from α' to β'.

The first application is that $n_{\alpha,\beta}$ is well-defined. This is because $\overline{M}_z(\alpha,\beta)$ is compact if it is 0-dimensional. If $1 = \mathrm{ind}(z) = \sum_{r=1}^{s} \mathrm{ind}(\zeta^{+,r})$ then $\mathrm{ind}(\zeta^{+,r}) \leq 0$ for some some r and $\overline{M}_{z_r} = \emptyset$.

15 February 26, 2018

We were talking about the downward gradient flow for $f : \mathcal{B} \to \mathbb{R}/\mathbb{Z}$. We had a compactness theorem for the flowlines $\gamma_n : \mathbb{R} \to \mathcal{B}$ with $\dot{\gamma}_n = -\nabla f$. Consider the hot regions $|\nabla f| \geq \epsilon$ and the cold regions $|\nabla f| \leq \epsilon$. Given a sequence γ_n, we can find a subsequence $(n') \subset (n)$ such that $\gamma_{n'}$ have the same hot/cold sequence and have the same cold regions. This can be used to show that $n_{\alpha,\beta}$ are finite numbers.

15.1 Proof of compactification

A more interesting statement is that $\partial^2 = 0$ on our chain complex. We have

$$\partial\partial\alpha_0 = \partial\biggl(\sum_{\alpha_1} n_{\alpha_0\alpha_1}(\alpha_1)\biggr) = \sum_{\alpha_2}\sum_{\alpha_1}(n_{\alpha_1\alpha_2}n_{\alpha_0\alpha_1}\alpha_2.$$

The coefficient of α_2 is just

$$\sum_{\alpha_1} n_{\alpha_1\alpha_2}n_{\alpha_0\alpha_1} = \#\text{broken flow lines} \equiv 0 \pmod 2.$$

This is because $\overline{M}(\alpha_0,\alpha_2) = \bigcup_{\mathrm{ind}(z)=2}\overline{M}_z(\alpha_0,\alpha_2)$ is a smooth 1-manifold, with a compactification

$$\mathrm{Br}\,\overline{M}(\alpha_0,\alpha_2)_2.$$

Lemma 15.1. $\mathrm{Br}\,\overline{M}(\alpha_0,\alpha_2)$ *is a compact* 1*-manifold with boundary. Its boundary is exactly*

$$\bigcup_{\alpha_1}\overline{M}(\alpha_0,\alpha_1)_1 \times \overline{M}(\alpha_1,\alpha_2)_1,$$

which is a 0*-manifold.*

This is not a tautology. For instance, it includes the statement that every broken trajectory is a limit of an unbroken trajectory. This is an example of finite-dimensional gluing. Near a Morse critical point $0 \in \mathbb{N}^N$ of $f : \mathbb{R}^N \to \mathbb{R}$, At 0, the Hessian $\mathrm{Hess}(f)$ has a $+$-eigenspace K^+ and a $-$-eigenspace K^-. Let S be the stable manifold, and U the unstable manifold. Take a neighborhood Ω of 0. The claim is that the apace

$$M(T) = \{\text{flow-lines defined on } [-T,T] \text{ which remains in } \Omega\}$$

approximates $S \times U$ for T large.

Proposition 15.2. *Let $k^+ \in K^+$ and $k^- \in K^-$. If T is large and $k^\pm$ are small, then there exists a unique $u \in M(T)$ such that*

$$\pi_{K^+}u(-T) = k^+, \quad \pi_{K^-}u(+T) = k^-.$$

So for large T, $M(T)$ is parametrized by $K^+ \times K^-$, and so by $S \times U$. We can use this to show that the compactification of $\overline{M}(\alpha_0, \alpha_2)$ is what we expect.

Let me also say a bit about perturbing the Chern–Simons functional. If this is not Morse, we don't have Morse theory. Given Y, w, we seek a function

$$f : \mathcal{B} \to \mathbb{R}$$

or $f : \mathcal{A} \to \mathbb{R}$ invariant under gauge transformations, such that

- our analysis for the PDE $\dot{B} = -\nabla CS|_B$ becomes an equation with similar analytic properties, like its linearization being Friedholm, having Uhlenbeck's compactification, etc,
- there is a large enough choice for f to achieve that $CS + f$ is formally Morse and Morse–Smale.

In SU(2)-gauge theory, over a 3-manifold Y with B a connection, we have $\mathrm{Hol}_q(B) \in \mathrm{Aut}(E_{q(0)})$ for any loop q. Then we can consider a map $B \mapsto \mathrm{tr}(\mathrm{Hol}_q(B))$. Next time I will try to explain how to make this idea useful.

16 February 28, 2018

I want to return to the question of perturbing Morse functions.

16.1 Good space of perturbations

Let me begin with ordinary Morse theory. Let B be a Riemannian manifold (mostly finite-dimensional), and $g : B \to \mathbb{R}$ not a Morse function. We seek a perturbation $g + f$ which is Morse. Let Π be a big linear space of functions,

$$\Pi \ni \pi \mapsto f_\pi \in C^\infty(B).$$

We want there to exist $\pi \in \Pi$ such that $g + f_\pi$ is Morse.

Let us denote $W = \mathrm{grad}(g)$ and $V_\pi = \mathrm{grad}(f_\pi)$. Then we can consider

$$\underline{W} : \Pi \times B \to TB; \quad (\pi, b) \mapsto (W + V_\pi)(b),$$

with the trivial points

$$\underline{\mathrm{Crit}} = \underline{W}^{-1}(0).$$

The slices of $\underline{\mathrm{Crit}}$ are the critical points of $g + f_\pi$.

The idea is that

(A) if Π is big, then $\underline{\mathrm{Crit}}$ is a submanifold,

(B) $\underline{\mathrm{Crit}} \to \Pi$ is a projection, and

(C) if π is a regular value, then $g + f_\pi$ is Morse.

For (A), we want $D\underline{W}$ to be onto at all $(\pi, b) \in \underline{W}^{-1}(0)$. For instance, consider $(0, b)$. Then b is a critical point of g and the derivative is

$$D\underline{W} : `P \times T_b B \to T_b B; \quad (p, \beta) \mapsto V_p(b) + H(\beta).$$

If it is not surjective, then there exists a $U \in T_b B$ such that $U \perp \mathrm{im}(H)$ and $U \perp V_p(b)$ for all b. This does not happen if Π is big enough. That is, if for all $b \in B$ and $U \in T_B$ there exists $p \in \Pi$ such that $\nabla_U f_p|_b \neq 0$.

For (B), both spaces $\underline{\mathrm{Crit}}$ and Π are Banach spaces.

Theorem 16.1. *If $P : \mathcal{E} \to \mathcal{F}$ is a map of Banach manifolds (with $\mathcal{E}$ and $\mathcal{F}$ modeled on separated Banach spaces) and*

(i) P is smooth (C^∞),

(ii) P is Fredholm (i.e., $DP|_e$ is a Fredholm operator for all $e \in \mathcal{E}$)

then regular values exist.

So we are going to apply this theorem to the projection $\underline{\mathrm{Crit}} \to \Pi$. Near the critical points, this $\underline{W}$ needs to be smooth. That is,

$$(\pi, b) \mapsto V_\pi(b) + W(b)$$

should be smooth. Here, we need V_π to be a smooth vector field, and so for all π, f_π should be a smooth function.

We also want Π to be a Banach space as well. We can take Π to be the Banach space of C^∞ functions. Here is the idea. Choose a countable a collection of functions f_i so that for all b, U there exists an i with $\nabla_U f_i \neq 0$ at b. Then we are going to write

$$f_\pi = \sum_{i=1}^{\infty} \pi_i f_i$$

for $\pi_i \in \mathbb{R}$. Then we define

$$\Pi = \{\pi = (\pi_i) : \sum C_i |\pi_i| < \infty\}$$

where C_i is a rapidly increasing sequence, so that all such f_π is C^∞.

In our gauge theory situation, recall that we want (at (a, b))

$$(p, \beta) \mapsto V_p(b) + H_b(\beta)$$

to be onto. But note that $\text{im}(H_b)$ already has finite codimension. So for $D\overline{W}$ to be onto, we want there not to exist U with $U | V_p(b)$ for all $p \in \Pi$.

For $\mathcal{B} = \mathcal{A}/\mathcal{G}$, let us work on $\mathcal{A}$ for a moment. Consider the Banach space $\mathcal{A}_\ell$ which is the L^2_ℓ-completion. Take $W = \text{grad}(CS)$ and identify $\mathcal{A} \to \Omega^1(Y, \mathfrak{g})$. Then it can be considered as

$$\mathcal{A}_\ell \to L^2_{l-1}(Y, \textstyle\bigwedge^1 \otimes \mathfrak{g}).$$

Example 16.2. Take the function $f : C^\infty([-1,1]) \to \mathbb{R}$ given by $a \mapsto a(0)$. If we compute the L^2-gradient of this function, it is going to be the delta function δ. This is bad.

16.2 Holonomy perturbation

So we use the idea of holonomy. Let $q : D^2 \times S^1 \to Y$ be an embedding, and consider $q(z, -)$ a loop. Given a connection B, a group G, in a bundle P we can take the holonomy

$$\text{Hol}_{q(z,-)}(B) \in \text{Aut}(P_{q(z,0)}).$$

This in G is well-defined up to conjugation. So if we take a conjugation-invariant function $h : G \to \mathbb{R}$, we get a well-defined

$$h(\text{Hol}_{q(z,-)}(B)).$$

So given a bump 2-form on D^2, we can integrate and get

$$B \mapsto \int_{D^2} h(\text{Hol}_{q(z,-)}(B))\mu.$$

Take $G = S^1$ and an arbitrary $h : G \to \mathbb{R}$. Consider this as a periodic function $\mathbb{R} \to \mathbb{R}$. Then we pull back to $D^2 \times S^1$ via q. Given a connection 1-form b on $D^2 \times S^1$, we are taking

$$f(b) = \int_{D^2} h\biggl(\int_{z\times S^1} b\biggr)\mu.$$

The derivative of f at b can be computed as

$$\beta \mapsto \int_{D^2} H(b,z)\biggl(\int_{z\times S^1} \beta\biggr)\mu,$$

where

$$H(b,z) = h'\biggl(\int_{z\times S^1} b\biggr).$$

We can write this as

$$\langle \beta, \tilde{H}(b)\rangle_{L^2} \quad \text{where } \tilde{H}(b) = (H(b,-)) * \mu \in \Omega^1(S^1 \times D^2).$$

So this is really a nice L^2 function. This is how it works in the abelian group S^1 case.

17 March 2, 2018

Last time we looked at how a holonomy perturbation looks like in the abelian case $G = S^1 = \mathbb{R}/\mathbb{Z}$. On the trivial bundle, let b be a connection 1. We also needed an embedding $q : D^2 \times S^1 \hookrightarrow Y$ and $h : G \to \mathbb{R}$ or $h : \mathbb{R} \to \mathbb{R}$ a conjugation-invariant function. Then we were able to define

$$f(b) = \int_{D^2} h\biggl(\int_{z\times S^1} b\biggr)\mu$$

for $\mu \in \Omega^2(D^2)$ compactly supported. This gave a function

$$f : \mathcal{A} \to \mathbb{R}$$

with $\mathrm{grad}(f) : \mathcal{A} \to T\mathcal{A} = \Omega^1(Y; \mathfrak{g})$ given by

$$V|_b = \mathrm{grad}(f)|_b = \star\biggl(h'\biggl(\int_{z\times S^1} b\biggr)\mu\biggr).$$

17.1 Critical points of a holonomy perturbation

Let us look at the critical points of the perturbation. This is going to be

$$\mathrm{grad}(CS + f)|_b = \star F_b + V|_b = 0.$$

That is, we need $F_b = \star V(b)$. We know that $F_b = 0$ means flat connection. But how do we understand this? First it is going to be flat outside the torus $q(D^2 \times S^1)$. Given two points $z, z' \in D^2$, the difference between the holonomy around $z \times S^1$ and $z' \times S^1$ can be computed by integrating the curvature on a surface joining to two loops, and this is zero because the curvature is pulled back from $\star V = H(b)\mu$. So

$$\mathrm{Hol}_{q(z,-)} = \mathrm{Hol}_{q(z',-)} .j$$

Now consider a loop c going around the torus like $\partial D^2 \times 0$. Also, write $e = * \times S^1$ be the other loop. In this case, we have

$$\mathrm{Hol}_c(b) = -\int_{D^2\times 0} H(b)\mu = -H(b)|_z = -h'\biggl(\int_e b\biggr) = h'(\mathrm{Hol}_e(b)).$$

Therefore solutions to $F_b + \star V(b) = 0$ are flat on $Y \setminus (D^2 \times S^1)$ and satisfy

$$\mathrm{Hol}_c(b) = h'(\mathrm{Hol}_e(b)).$$

If we write $\mathrm{Hol}_c(b) = e^{i\psi}$ and $\mathrm{Hol}_e(b) = e^{i\varphi}$, then the equation becomes

$$\psi = h'(\varphi).$$

This generalizes to the non-abelian case. For a conjugation-invariant function $h : G \to \mathbb{R}$ and so $h' : G \to \mathfrak{g}$, we can define

$$f(B) = \int_{D^2} h(\mathrm{Hol}_{q(z,-)}(B))\mu.$$

Then $H(B)$ is a now section of $\mathfrak{g}_P$ supported on $D^2 \times S^1 \subseteq Y$. We get $h'(\mathrm{Hol}(B)) \in \Gamma(D^2 \times S^1; \mathfrak{g}_P)$ and so

$$V(B) = \star(h'(\mathrm{Hol}(B))\mu) \in \Omega^1(Y; \mathfrak{g}_P).$$

This is going to be C^0-bounded in B.

17.2 Checking the bigness criterion

If you remember, our criterion for "big" was that for any tangent vector U, there exists a $\pi \in \Pi$ such that $D_U f_\pi \neq 0$. This is something like distinguishing points, so for $[B_1] \neq [B_2]$ we want a f_π such that $f_\pi(B_1) = 0$ and $f_\pi(B_2) = 1$.

Let $\underline{q} = (q_1, \dots, q_r)$ be a number of loops in Y. For each loop, we can look at the holonomy, and get

$$(\mathrm{Hol}_{q_1}(B), \dots \mathrm{Hol}_{q_r}(B)) \in G_{\eta_0} \times \cdots \times G_{\eta_0}.$$

Pick a function

$$h : G \times \cdots \times G \to \mathbb{R}$$

that is invariant under simultaneous conjugation. Because we want smooth perturbation, we fatten $s = 1, \dots, r$ to $q_s : D^2 \times S^1 \to Y$, and we require $q_s = q_{s'}$ on $D^2 \times [-\epsilon, \epsilon]$. Now we define

$$f(B) = \int_{D^2} h(\mathrm{Hol}_{q_1(z,-)}(B), \dots, \mathrm{Hol}_{q_r(z,-)}(B))\mu.$$

If two connections are not gauge equivalent, there is going to be some loop where the holonomy differ.

If we only require h to be invariant under separate conjugation, it might not be able to distinguish even flat connections. Let us consider the free group $\langle g_1, \dots, g_r \rangle = \pi_1$, and let $\rho, \rho' : \pi_1 \to G$ be two representations. Suppose that for all words w, then $\rho(w)$ is conjugate to $\rho'(w)$. Does it follow that $\rho \sim \rho'$ independent of w? That is, can we simultaneous conjugate? The answer is yes if $G = \mathrm{SU}(N)$, but no for general Lie groups G.

Anyways, given data $\underline{q} = (q_1, \dots, q_r)$ and $h : G \times \cdots \times G \to \mathbb{R}$ we get an $f : \mathcal{A} \to \mathbb{R}$ that is gauge-invariant. A countable collection of such $(\underline{q}, h)$ giving f_i is enough to separate the points on any tangent vectors in $\mathcal{A}/\mathcal{G}$. Then we can define

$$\Pi = \{f_\pi = \textstyle\sum_i \pi_i f_i\}$$

where π_i are rapidly decreasing so that f_π is smooth.

$\mathcal{A}$ completes to $\mathcal{A}_l^2$, which is the L_l^2-connections, and $f_\pi : \mathcal{A}_l \to \mathbb{R}$ is C^∞. If we consider the formal L^2-gradiant

$$V_\pi = \text{grad}(f_\pi) : \mathcal{A}_l \to T\mathcal{A}_l,$$

it is a C^∞ vector field on $\mathcal{A}_l$ and invariant under $\mathcal{G}$.

18 March 5, 2018

We have seen that holonomy perturbations have nice formal properties. But we haven't talked much about mapping properties of perturbations. Consider $V = \mathrm{grad}(f_\pi)$ be a holonomy perturbation. Then our modified gradient flow was

$$\dot{B} = -\star F_B + V(B).$$

We then had that $-\star F_B + V(B) = 0$ are critical points. But is it going to be Morse?

18.1 Morse–Smale condition of perturbations

Before perturbation, we had the operator

$$H = \begin{pmatrix} 0 & -d_B^* \\ -d_B & \star d_B \end{pmatrix}$$

acting on $(\bigwedge^0 \oplus \bigwedge^1)(Y; \mathfrak{g})$. This is a 1st order elliptic symmetric operator $L_1^2 \to L^2$ on Y, and is Fredholm. If we perturb, we have to deal with the operator

$$\begin{pmatrix} 0 & -d_B^* \\ -d_B & \star d_B + L \end{pmatrix}$$

where L is the derivative at V at $\mathcal{A}(Y)$.

Considering V as a map $\mathcal{A} \to \Omega^1(Y; \mathfrak{g})$, we can complete it to $\mathcal{A}_l$ and differentiate respect to B. Then we will get from $V : \mathcal{A}_l \to L_l^2(Y; \bigwedge^1 \otimes \mathfrak{g})$, its derivative $DV : \mathcal{A}_l \to \mathrm{Hom}(L_l^2, L_l^2)$ and then so on.

You can calculate DV and see that it extends to a smooth map $\mathcal{A}_l \to \mathrm{Hom}(L_k^2 \to L_k^2)$ for $k \leq l$. Let us look at the abelian case with $r = 1$. For $b \in L_l^2$ a 1-form, consider $b + \epsilon\beta$ for $\beta \in L^2(Y)$. Then we compute

$$(\bar{b} + \epsilon\bar{\beta})(z) = \int_{z \times S^1} (b + \epsilon\beta).$$

Then the derivative with respect to β is $h'(\bar{b})\beta$.

We may write the perturbation as $(H + L) : L_1^2 \to L^2$, where L is the DV term. Here, $L : L_1^2 \to L^2$ is compact, because it factors through a bounded operator $L^2 \to L^2$. That is, it is a compact perturbation of Fredholm operator.

So we can make a perturbation so that the resulting function is Morse. Another question is whether we can make its Morse–Smale. We first make a perturbation so that the function is Morse, and then make another perturbation $V_{\pi'}$ is Morse–Smale. Here, we let $V_{\pi'} = \mathrm{grad}(f_{\pi'})$ where $f_{\pi'} = 0$ near critical points. Recall that Morse–Smale is surjectivity of the linearized equation. So for any b on a flowline satisfying

$$-\frac{d}{dt} b + H(t) b = 0,$$

we need to find a π' such that $\langle V_{\pi'}, b\rangle_{L^2(\mathbb{R}\times Y)} \neq 0$.

What happens to Uhlenbeck compactness for the perturbed Chern–Simons functional? Flowlines corresponded to connections B on $\mathbb{R}\times Y$ in temporal gauge satisfying the 4-dimensional equation $F_B^+ = 0$. If we perturb the functional, we get $F_B^+ = \hat{V}(B)$, where

$$\hat{V} : \mathcal{A}(\mathbb{R}\times Y) \to \Omega^+(\mathbb{R}\times Y; \mathfrak{g}); \quad \hat{V} = dt \wedge V + \star V.$$

Take a sequence B_n of connections, and assume that $|F(B_n)|$ grows near the point (t, y_0). If this is not on the holonomy loop, then we know what is happening. If it lies inside this tube, then we will have

$$\frac{d}{dz}\,\mathrm{Hol}(z, -) \gg 0$$

for $z \in D^2$. Because $L_2^p \hookrightarrow C^0$ for $p \geq 2$, we can take todo

18.2 Comparing Floer homology

So we have holonomy perturbations f_π such that the functional becomes Morse, Morse–Smale, Fredholm, and with Uhlenbeck perturbation. Let Y be a 3-manifold with w a 1-cycle with $w_2(P) = PD[w]$. Consider $\mathcal{A}$ the connections and $\mathcal{G}$ the SU(2)-gauge transformations. From the Morse theory of $CS + f_\pi$, we get a Morse complex

$$C_* = \bigoplus_{\text{crit }\alpha} \mathbb{F}_2, \quad \partial : C_* \to C_*.$$

Then we can define homology

$$I^w(Y, g, \pi) = H_*(C_*, \partial),$$

which depends on the Riemannian metric g and a perturbation π.

In the finite-dimensional case, why was $H_*(f) = H_*(C_*, \partial)$ independent of f and g? One reason was that this was isomorphic to singular homology. But this is not available to us now. Another thing to do is change f and see how this changes the Morse complex. There are going to be some cancellations or generations of critical points, and you can compare homology by chain homotopies, or do handle decompositions or whatever. A smarter thing to do, which is what Floer did, is given f_- and f_+ two Morse–Smale functions on B, pick a interpolating family f_t for $t \in \mathbb{R}$ such that

$$f_t = \begin{cases} f_- & t \ll 0 \\ f_+ & t \gg 0. \end{cases}$$

Consider the non-autonomous differential equation for a path in B:

$$\frac{d}{dt}\gamma(t) = -\,\mathrm{grad}(f_t).$$

Note that this is not translational invariant. Such a flowline will start from a critical point α_- of f_- and end at a critical point α_+ of f_+. So we have the space

$$M(\alpha_-, f_*, \alpha_+)$$

of solutions. If α_- and α_+ have the same index, and transversality holds, then $\dim M(\alpha_-, f_*, \alpha_+) = 0$. Then we set $m_{\alpha_- \alpha_+}$ to be the number modulo 2. This can be considered as a matrix, which is a chain map $C_*(f_-) \to C_*(f_+)$.

19 March 7, 2018

Let Y^3 be w-admissible, where w is a 1-cycle of a 1-manifold with $w\Sigma$ odd for some w. Then we can look at the instanton homology groups $I^w(Y, \pi, g)$.

19.1 Cobordism

Let X be a cobordism between (Y_0, w_0) and (Y_1, w_1). This is going to be a compact 4-manifold X with boundary

$$\partial X = (-Y_0) \amalg Y_1$$

and $v \subseteq X$ a smooth 2-manifold with $\partial v = w_0 \amalg w_1$. More formally, we can ask that we are given a (orientation-preserving) diffeomorphism $r : (-Y_0) \amalg Y_1 \to \partial X$, so that we can think of diffeomorphisms as cobordisms with $X = [0,1] \times Y$. We say that (X, v, r) is isomorphic to (X', v', r') if there exists a diffeomorphism $(X, v) \to (X', v')$ commuting with r and r'.

Consider the category $\mathcal{C}$ with

- objects (Y, w) with (π, g) with π a perturbation and g a Riemannian metric,
- isomorphism classes of cobordisms (X, v).

Theorem 19.1. *Instanton Floer homology gives a functor*

$$\mathcal{C} \to \mathsf{F}_2\text{-vector spaces.}$$

This includes the statement that instanton Floer homology does not depend on the choice of perturbation and metric. If (Y, w) is given, and π_0, g_0 and π_1, g_1 are two perturbations, then there is a morphism $(X, v) = (I \times Y, I \times v)$ in both ways.

So let X be a cobordism between Y_0 and Y_1 (really (Y_0, w_0, π_0, g_0) and (Y_1, w_1, π_1, g_1)). Attach manifolds on both ends to get a non-compact cylindrical manifold

$$X^+ = (-\infty, 0] \times Y_0 \cup X \cup [0, \infty) \times Y_1.$$

Choose Riemannian metrics g^X on X, equal to $dt^2 + g_0$ and $dt^2 + g_1$ on ends. We want to look at the perturbed anti-self-duality equation for connections on X^+. For the SO(3)-bundle P with $w_2 = PD[v]$, we can consider a SU(2)-bundle $\tilde{P}$ on $X \setminus v$. Then $\mathcal{A}$ a connection in P is the same as a connection in $\tilde{P}$ with holonomy -1 around the link of v.

On X, consider the perturbation $F_A^+ + V^X(A) = 0$ where on $[0, \infty) \times Y_1$, it looks like

$$V^X(A) = \beta(t) V_{\pi_1}(A)$$

with $\beta(t)$ a cut-off function, and on $(-\infty, 0] \times Y_2$, it looks like some cut-off times V_{π_2}. Let α_0 and α_1 be critical points on Y_0, Y_1 for the perturbed Chern–Simons form. The moduli

$$M^v(\alpha_0, X, \alpha_1)$$

will be flowlines with finite energy (finite change in $CS + f_\pi$) such that $[A|_t] \to \alpha_0, \alpha_1$ as $t \to -\infty, +\infty$. At a solution A, the linearized equation will look like

$$\delta_A : L^2_{\ell,A}(X^+, \bigwedge^1_X \otimes \mathfrak{g}) \to L^2_{\ell,A}(X^+, \bigwedge^+_X \otimes \mathfrak{g}).$$

So the moduli space is "cut out transversely" (we call this **regular**) if this δ_A is surjective. This will imply that $M^v(\alpha_0, X, \alpha_1)$ is smooth and its dimension is equal to the index. Because $\mathcal{B}(Y)$ is not always simply connected, we are going to have $d + 8k$-dimensional parts

$$M^v(\alpha_0, X, \alpha_1)_{d+8k}$$

for $k \geq 0$.

To make this all work, we need to make sure that we can achieve regularity. So far, we just made a cut-off perturbation on both ends, so we need to do more. If we have $g + f_+$ and $g + f_-$ Morse functions on B, we want to interpolate them by $g + f_t$. We first used a cut-off function, but this might not be enough, so we can just introduce small perturbations in small time intervals.

20 March 9, 2018

Right now the objects in my category are (Y, w, π, g) decorated with a perturbation and a metric. Given a morphism $(Y_-, w_-, \pi_-, g_-) \to (Y_+, w_+, \pi_+, g_+)$ which is a cobordism (X, v), we equip this with an extra little perturbation π_0 and g_X. We can form the moduli space

$$M(\alpha_-, X, \alpha_+)_d$$

of d-dimensional moduli space, regular on account of π_0.

20.1 Compactification of flowlines

We need to think about what broken flowlines should mean in this context. Take $[A_n] \in M(\alpha_-, X, \alpha_+)_d$ a sequence of connections. If there are no bubbles, convergence should mean

$$[A_n] \to [A_\infty]$$

on compact subsets of X_+, with $[A_\infty] \in M(\alpha'_-, X^+, \alpha'_+)_{d'}$ for $d' \le d$. In the limit, we will recover some critical points

$$\alpha_-^0 = \alpha_-, \alpha_-^1, \dots, \alpha_-^k = \alpha'_-, \quad \alpha'_+ = \alpha_+^l, \alpha_+^{l-1}, \dots, \alpha_+^0 = \alpha_+,$$

and the broken flowlines can be thought of as

$$\overline{M}_{d_-^0}(\alpha_-^0, \alpha_-^1) \times \overline{M}_{d_-^1}(\alpha_-^1, \alpha_-^2) \times \cdots \times M(\alpha'_-, X, \alpha'_+)_{d'} \times \cdots \times \overline{M}_{d_+^0}(\alpha_+^1, \alpha_+^0)$$

where $d = d' + \sum d_+^i + \sum d_-^i$.

If there are bubbles, they can occur anywhere throughout our manifold X^+. In this case, we should have

$$d = d' + \sum d_-^i + \sum d_+^i + 8(\text{bubble} - \text{count}).$$

If $d < 8$ there never can be bubbles. Also, $d_-^i, d_+^i \ge 1$ and $d' \ge 0$.

For $d = 1$, we can compactify $M(\alpha_-, X, \alpha_+)$ by adjoining

$$\overline{M}(\alpha_-, \alpha_-^1)_1 \times M(\alpha_-^1, X, \alpha_+)_0 \quad \text{and} \quad M(\alpha_-, X, \alpha_+^1)_0 \times \overline{M}(\alpha_+^1, \alpha_+).$$

Furthermore, the compactification $\mathrm{Br}\, M(\alpha_-, M, \alpha_+)_1$ is a 1-manifold with boundary, where the boundary is as described above.

As before, let

$$n_{\alpha\beta}^- = \#\overline{M}(\alpha, \beta)_1 = \text{matrix entries of } \partial_{Y_-} \text{ for } I^w(Y_-),$$

and $n_{\alpha\beta}^+$ the same thing for Y_+. Also set

$$m_{\alpha_-, \alpha_+} = \#M(\alpha_-, X, \alpha_+)_0.$$

We can regard m_{α_-, α_+} as matrix entries for the linear map

$$m : C_*(Y_-) \to C_*(Y_+).$$

Lemma 20.1. *m is a chain map. That is, $\partial_{Y_+} \circ m = m \circ \partial_{Y_-}$.*

Proof. We can only check that matrix entries. Fix β_- and β_+, and if we count the boundary points, we get

$$0 = \# \bigcup_{\alpha_-} \overline{M}(\beta_-, \alpha_-)_1 \times M(\alpha_-, X, \beta_+)_0 \cup \bigcup_{\alpha_+} M(\beta_-, X, \alpha_+)_0 \times \overline{M}(\alpha_+, \beta_+)_1.$$

That is,

$$0 = \sum_{\alpha_-} m_{\alpha_-, \beta_+} n^-_{\beta_-, \alpha_-} + \sum_{\alpha_+} n^+_{\alpha_+, \beta_+} m_{\beta_-, \alpha_+}.$$

These are just the matrix entries. □

We put an auxiliary metric g_X and a perturbation π_0. Suppose we replace these with g'_X and π'_0, and suppose everything is still regular. Then we get a new chain map m' between the same complexes.

Lemma 20.2. *m and m' are chain homotopic. That is, there exists an $K : C(Y_-) \to C(Y_+)$ with*

$$m - m' = \partial_{Y_+} \circ K + K \circ \partial_{Y_-}.$$

We are essentially going to use the same idea. Consider a family of metrics $\{g_s\} = G$ on X, from g_X to g'_X where $s \in [0, 1]$. (We are actually doing the same thing with π_s, but we are going to omit this for now.) Consider

$$M(\alpha_-, X, \alpha_+)^G_{0+1} = \bigcup_{s \in [0,1]} M(\alpha_-, X, \alpha_+)^{g_s}_0.$$

Some transversality argument shows that if we choose G suitably, then the total space is going to be a 1-dimensional manifold. The endpoints at $s = 0$ is m_{α_-, α_+}, and the endpoints at $s = 1$ is m'_{α, α_+}. So it seems like at have proven

$$m = m'$$

modulo 2. But we have not done this, because $M(\alpha_-, X, \alpha_+)^G_{0+1}$ might not be compact. So we have

$$m - m' = \text{contributions from non-compactness in the interior of } (0, 1).$$

If we look at the proof of Uhlenbeck's compactness theorem, nothing bad happens for a one-dimensional family. Suppose we have $[A_n] \in M(\alpha_-, X, \alpha_+)^{g_{s_n}}_0$. The limits here are going to look like

$$\overline{M}_{Y_-}(\alpha_-, \alpha'_-)_1 \times M(\alpha'_-, X, \alpha_+)_{-1}$$

or similarly with the $+$ entries. Here, $M(\alpha'_-, X, \alpha_+)_{-1}$ looks like a -1-dimensional space, but in the space G, it really is

$$M(\alpha'_-, X, \alpha_+)^G_{-1+1}$$

a set of points.

Define

$$K : C(Y_-) \to C(Y_+); \quad K_{\alpha_-,',\alpha_+} = \#\text{points in } M(\alpha'_-, X, \alpha_+)^G_{-1+1}.$$

Then the number of boundary points in $M(\alpha_-, X, \alpha_+)^G_{0+1}$ being 0 means that

$$0 = \partial K + K\partial + m + m'.$$

21 March 19, 2018

There is one remaining thing to talk about instanton homology as a functor $I : \mathcal{C} \to \mathbb{F}_2-\mathsf{Vect}$, where $\mathcal{C}$ has (Y, w) as objects and isomorphism classes of cobordisms as morphisms. What we have not proved is that composites are sent to composites. Here, note that having w in the homology class isn't good enough, because then we don't know how to glue the cobordisms v together. We are going to need to specify w as an actual loop.

Now let (X_1, v_1) be a cobordism from (Y_0, w_0) to (Y_1, w_1) and (X_2, v_2) be a cobordism from (Y_1, w_1) to (Y_2, w_2). Let $(X, v) = (X_1, v_1) \cup (X_2, v_2)$, and denote

$$X^+ = (-\infty, 0] \cup X \cup [0, \infty).$$

Then the chain map $I^v(X)$ is going to be given by $M(\alpha_0, X^+, \alpha_2)$. To compare it with the composite, we put a metric g_R on X^+ so that the part between X_1 and X_2 looks like a long cylinder $[-R, R] \times Y_1$. For finite R, there is going to be a chain homotopy between g_1 and g_R. For $R \gg R_0$, you can show that

$$M^v(\alpha_0, X^+, \alpha_2)_0^{g_R}$$

is independent of R are homeomorphic to $\bigcup_{\alpha_1} M(X_1) \times M(X_2)$.

Example 21.1. Consider $Y = T^3$ and $w = * \times * \times S^1$. Then $\mathcal{R}^w(Y)$ had two points

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad c = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So what is the relative grading between these two α and β? The answer is that $\operatorname{ind}(\alpha, \beta) = 4$. We might have a cleaner way to see this, but there is a symmetry here. There is a flat line bundle ξ with holomony -1 along the c-loop such that $\alpha = \beta \otimes \xi$ as SU(2)-connections over $T^3 - w$. This will give an isomorphism $M(\alpha, \beta)_0 \cong M(\beta, \alpha)_0$, and anyways the index is either going to be 0 or 4. In any case, $\partial = 0$ nad so

$$I^w(T^3) = \mathbb{F}_2 \oplus \mathbb{F}_2.$$

This is really the only way case you will be able to compute instanton homology by hand, because the boundary map is going to be hard to compute.

21.1 Marked bundles

Now we are going to look at the hybrid between $\mathcal{G}^{\mathrm{SO}(3)}$ and $\mathcal{G}^{\mathrm{SU}(2)}$.

Definition 21.2. A **marking data** μ is a (open) subset $U_\mu \subseteq Y$ and a given SO(3)-bundle $E_\mu \to U_\mu$. A **μ-marked bundle** with connection on Y is a SO(3)-bundle E, a connection A in E, and an isomorphism $\sigma : E_\mu \to E|_{U_\mu}$.

So far, nothing has happend because $w_2(E)$ is not specified except that $w_2(E) = w_2(E_\mu)$ on U_μ. We then say that $(E, A, \sigma) \cong (E', A', \sigma)$ if there exists an isomorphism $\tau : E \to E'$ on Y such that $\tau^*(A') = A$ and $(\sigma')^{-1} \circ \tau \circ \sigma : E_\mu \to E_\mu$ on U_μ lifts to $G_{E_\mu}^{\mathrm{SU}(2)}$ on U_μ.

Example 21.3. Take T^3 with a, b, c generators. In the old story, we had w parallel to c. In the new picture, there is a torus $T_{a,b}$ transverse to the c-circle, and U_μ a neighborhood of $T_{a,b}$. We then take $E_\mu \to U_\mu$ the SO(3)-bundle with $w_2 \cdot T_{a,b} = 1$.

Here, marked flat connections (E, A, σ) form $\mathcal{R}(Y, \mu)$. At least when $w_2(E_\mu) = 0$, these are the conjugacy classes of SO(3)-representations of π_1, with lifts to SU(2) on $\pi_1(U_\mu)$.

Definition 21.4. The marking data is said to be **strong** if $\mathcal{R}(Y, \mu)$ contains no flat (E, A, μ) with non-trivial automorphisms.

Example 21.5. μ is strong if there exists some orientable $\Sigma^2 \subseteq Y$ with $\Sigma^2 \subseteq U_\mu$ and $w_2(E_\mu) \cdot \Sigma^2 = 1$.

22 March 21, 2018

Let Σ be a Riemann surface of genus g. Here are two ways you can look at the moduli space of these Riemann surfaces. The first one is to look at $\{(\Sigma, J)\}$ can quotient out by the equivalence relation $(\Sigma, J) \sim (\Sigma', J')$ if there exists a $\tau : \Sigma \to \Sigma'$ such that $\tau^*(J') = J$. The other way you can do is to fix the manifold Σ_0. Then we can look at the set of complex structures $\{J \text{ on } \Sigma_0\}$ and quotient out by the diffeomorphisms $\mathrm{Diff}(\Sigma_0)$.

The second approach allows you to do more general stuff easily. We can take a subgroup $H \subseteq \mathrm{Diff}(\Sigma_0)$ can quotient out by H instead of the entire $\mathrm{Diff}(\Sigma_0)$. This is equivalent to having some "marking". In terms of the first approach, we are taking the set of all diffeomorphisms $\sigma : \Sigma_0 \to \Sigma$ and divide by equivalence classes $\tau : \Sigma \to \Sigma'$ satisfying $\tau^*(J') = J$ and $(\sigma')^{-1} \circ \tau \circ \sigma \in H$.

22.1 Marking on a torus

Let's go back to the case of 3-manifolds. We have defined

$$\mathcal{B}(Y)^{\mathrm{SO}(3)} = \{(E, A) : A \text{ connection on } \mathrm{SO}(3)\text{-bundle } E\}/\sim$$

where $\sim$ means $\sigma : E \to E'$ and $\sigma^*(A') = A$. To fix only one E, we need to specify the topological type of E, which is determined by $w_2(E) = PD[w]$. Then we looked at

$$\mathcal{B}^w(Y)^{\mathrm{SU}(2)} = \{A \text{ connection on } P\}/\sim$$

where $A \sim A'$ if there exists $\tau : E \to E$ with $\tau^*(A') = A$ and τ lifts to $\mathfrak{G}^{\mathrm{SU}(2)} \to \mathcal{G}^{\mathrm{SO}(3)}$. This is equivalent to

$$\mathcal{B}^2(Y)^{\mathrm{SU}(2)} = \{\tilde{A} \text{ connection in } \tilde{P} : A = \mathrm{im}(\tilde{A}) \text{ extends to } P, \mathrm{Hol}(\tilde{A}) = -1 \in \mathrm{SU}(2)\}/\mathcal{G}^{\mathrm{SU}(2)}$$

where $\tilde{P}$ is the SU(2)-bundle lifting P on $Y \setminus w$. The marking is supposed to do $\mathcal{B}^w(Y)^{\mathrm{SU}(2)}$ on some subset of the manifold, and $\mathcal{B}(Y)^{\mathrm{SO}(3)}$ on the entire manifold.

A marking μ is a subset $U_\mu \subseteq Y$ with E_μ or P_μ a SO(3)-bundle on U_μ such that $\tilde{P}_\mu$ lifts to P_μ a SU(2)-bundle on $U_\mu \setminus w_\mu$, such that $w_\mu(P_\mu) = PD[w_\mu]$. A connection is then (E, A, σ) where (E, A) is an SO(3)-bundle with connection on Y, and $\sigma : E_\mu \to E$ is on U_μ. The equivalence relation $(E, A, \sigma) \sim (E', A', \sigma')$ means that there is a map $\tau : E \to E'$ such that $\tau^*(A') = A$ and $(\sigma')^{-1} \circ \tau \circ \sigma : E_\mu \to E_\mu$ lifting to $\mathcal{G}^{\mathrm{SU}(2)}$. Or we can say that our data is $(E, A, A_0, \sigma$ where A_0 is a connection in E_μ and the equivalence relation is as before bur also requiring $\tau^*(A) = A_0$ on U_μ.

Let us write $\mathcal{B}(Y;\mu)$ the μ-marked connections up to equivalence, and $\mathcal{B}^w(Y;\mu)$ those with given $w_2 = PD[w]$ on Y.

Example 22.1. Let us take $Y = T^3$ and $w = c$. If we take $\mathcal{B}^w(Y)^{\mathrm{SO}(3)}$, the representation variety is one point with automorphism group $\{1, i, j, k\}$. If we take $\mathcal{B}^w(Y)^{\mathrm{SU}(2)}$, this is two points with trivial automorphism group. If we take

U_μ the neighborhood of the 2-torus in the a, b direction, then the representation variety is one point with trivial automorphism group. Thus $I^w(Y;\mu)$ for $Y = T^3$ is going to be $\mathbb{F}_2$, which is 1-dimensional.

Recall that we say that μ is strong when $\mathcal{R}(Y;\mu)$ has only elements with trivial automorphism group. This happens when there is a $\Sigma \subseteq U_\mu$ such that $w_2(E)(U_\mu) = 1$.

Given any closed oriented 3-manifold Y, and look at the connected sum $Y\#T^3$. Take w to be the ordinary $(\mathrm{pt}) \times S^1$ inside the T^3 part, and also let us take μ to be the usual U_μ the neighborhood of a torus. We can then define

$$I^T(Y) = I^w(Y\#T^3, \mu).$$

Actually, to form $Y\#T$, we need a basepoint $y_0 \in Y$ and a frame e_1, e_2, e_3 at y_0. This is not just being pedantic, because we want diffeomorphisms of $(Y, y_0, \{e_1, e_2, e_3\})$ toe act on $I^T(Y)$ and also cobordisms to work functorially.

Example 22.2. By definition, $I^T(S^3) = \mathbb{F}_2$ as we have previously computed.

If we look at the representation variety $\mathcal{R}^w(Y\#T^3, \mu)$, this is going to be

- a representation $\rho : \pi_1(Y) \to \mathrm{SO}(3)$,
- a unique flat connection on T^3 with no automorphism,
- identified at the base point.

So these are the same as flat $\mathrm{SO}(3)$-connections (E, A) on Y together with a trivialization $E_{y_0} \to \mathbb{R}^3$. Although this looks complicated this is just the same as

$$\mathcal{R}^w(Y\#T^3;\mu) \cong \mathrm{Hom}(\pi_1(Y, y_0), \mathrm{SO}(3)),$$

not up to conjugation.

23 March 23, 2018

Let us look at our 3-manifold Y and its representation variety $\mathcal{R}$ as a SO(3)-representation variety without any decorations. Then

$$\mathcal{R} = \tilde{\mathcal{R}}/\operatorname{SO}(3); \quad \tilde{\mathcal{R}} = \operatorname{Hom}(\pi_1(Y), \operatorname{SO}(3)).$$

The stabilizer of $p \in \tilde{\mathcal{R}}$ is just the centralizer $C_{\operatorname{SO}(3)}(\operatorname{im}(p))$. So it is either 1, $(1, i)$, $(1, i, j, k)$, SO(2), O(2), or SO(3), up to conjugacy. Here we can compute

$$\begin{aligned} C_{\operatorname{SO}(3)} &\supseteq (1, i) & &\text{iff } \operatorname{im}(p) \subseteq \operatorname{O}(2), \\ C_{\operatorname{SO}(3)} &\supseteq (1, i, j, k) & &\text{iff } \operatorname{im}(p) \subseteq (1, i, j, k), \\ C_{\operatorname{SO}(3)} &\supseteq \operatorname{SO}(2) & &\text{iff } \operatorname{im}(p) \subseteq \operatorname{SO}(2), \\ C_{\operatorname{SO}(3)} &\supseteq \operatorname{O}(2) & &\text{iff } \operatorname{im}(p) \subseteq (1, i), \\ C_{\operatorname{SO}(3)} &\supseteq \operatorname{SO}(3) & &\text{iff } \operatorname{im}(p) = (1). \end{aligned}$$

If we take $Y = Y_1 \# Y_2$, then $\pi_1(Y) = \pi_1(Y_1) * \pi_1(Y_2)$, and we get $\tilde{\mathcal{R}}_1 \times \tilde{\mathcal{R}}_2 = \tilde{\mathcal{R}}$. But it is not the case that $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$. Imagine a (unrealistic) situation in which $\mathcal{R}$ is a orbit of a single p with trivial stabilzer. In this case, we will have

$$\mathcal{R} = \frac{\tilde{\mathcal{R}}_1 \times \operatorname{SO}(3)}{\operatorname{SO}(3)} = \tilde{\mathcal{R}}_1.$$

We were able to do this using markings. If we take $Y_1 \# T^3$, we used marking and non-trivial w_2 on T_3 to engineer $\tilde{\mathcal{R}}_{T^3} = \operatorname{SO}(3)$. These can be considered as bundles $E \to Y$ with a connection and an identification $E_y \cong \mathbb{R}^3$. Then we are looking at the space $\mathcal{A}/\mathcal{G}^\circ$. Here, $\mathcal{G}^\circ$ is the gauge transformations $u : E \to E$ with $u(y) = 1_{E_y}$.

23.1 Orbifolds

Definition 23.1. An **orbifold** is a topological space with charts

$$\tilde{U}_y/H_y \xrightarrow{\varphi} U_y$$

around each point y, such that $\tilde{U}_y \subseteq \mathbb{R}^n$ and H_y is a finite group acting faithfully and linearly on $B^n \cong \tilde{U}_y$ fixing $\varphi^{-1}(y) = 0$. It is called **orientable** if $H_y \subseteq \operatorname{SO}(n)$. We want this to satisfy some compatibility condition, like if $y' \in U_y$ then $\varphi_y^{-1}(y') \in \tilde{U}_y$ has stabilizer $H_{y'} \cong H' \subseteq H_y$.

For $\check{Y}$ an orbifold, we take its singular locus $\operatorname{sing}(\check{Y}) = \{y : H_y \neq 1\}$ and $\check{Y}^\circ = \check{Y} \setminus \operatorname{sing}(\check{Y})$. In the oriented case, sing is going to have codimension at least 2. So, in dim $= 2$ the model is $\mathbb{R}^2/\langle g \rangle$ where $g^k = 1$. We say that it is a **bifold** if $k = 2$ at all singular points.

Example 23.2. In dimension 3, there are going to be codimension 2 strata and codimension 3 strata. The codimension 2 strata is going to look like $\mathbb{R}\times(\mathbb{R}^2/\langle g\rangle)$, which is a cone. There are also going to be codimension 3 strata. For instance, it can look like U_y homeomorphic to a cone on $S^2/\mathbb{I}$ where $\mathbb{I}$ is the icosahedral group. This look like a cone on $\check{S} \cong S^2$ where $\hat{S}$ has three singular points of order $2, 3, 5$. Topologically U_y looks like a ball, but it has three 1-dimensional singular strata coming out of the singular point, with order $2, 3, 5$.

The three cases of codimension 3 strata are $\mathbb{I}$ (icosahedral group, $2, 3, 5$), $\mathbb{O}$ (octahedral group, $2, 3, 4$), $\mathbb{T}$ (tetralhedral group, $2, 3, 3$), $\mathbb{D}_{2k}$ (dihedral group, $2, 2, k$).

So you can think of a 3-dimensional orbifold as a 3-dimensional manifold and inside it specifying a trivalent embedded graph $K \subseteq Y$, possibly with no vertices and only loops. Then each edge is labeled by natural numbers $k \geq 2$, and we require that each vertex looks like one of $\mathbb{I}, \mathbb{O}, \mathbb{T}, \mathbb{D}$. We say that a 3-dimensional orbifold is a **bifold** if $k = 2$ on all edges.

Example 23.3. Let's think about 4-dimensional bifolds. The codimension 2 strata will be $\mathbb{R}^2 \times (\mathbb{R}^2/\langle g\rangle)$, and codimension 3 strata should look like $\mathbb{R} \times (\mathbb{R}^3/V_4)$. For the codimension 4 strata, we only allow one case: B^4/V_8. Then this looks likes a cone on the bifold coming from (S^3, K). This is a edges of a tetrahedron, so we will get a tetrahedral point, which looks like coning off the tetrahedron at the center.

24 March 26, 2018

So we were looking at closed oriented "bifolds". These $\check{Y}$ corresponded to (Y, K) where Y is a 3-manifold and K is a trivalent graph K or a "web", which is the singular set. The edges are locally modeled on $B^3/\langle 1, -1, -1\rangle$, and the vertices are modeled on B^3/V_4.

24.1 Examples of bifolds

Example 24.1. A $K \subseteq \mathbb{R}^3 \subseteq S^3$ can be

(i) an unknot, or any knot or link,

(ii) the θ-web with two vertices,

(iii) the tetrahedron with 4 vertices,

(iv) the cube with 8 vertices,

(v) the dodecahedron with 20 vertices,

(vi) handcuffs with 2 vertices,

(vii) start with any two-component link and take any arc between them, and get a different embedding of a handcuff with 2 vertices (e.g., Hopf's handcuffs by doing this on the Hopf link),

(viii) start with a trivial link and take an interesting arc, etc.

If you take S^3/C_2 by $(1, 1, -1, -1)$, then you will get an unknot. If you take S^3/V_4, you will get a θ-web. If you take T^3/H, you will get a cube, and the dodecahedron will come from some quotient of $\mathbb{H}^3$. All these are finite quotients of compact 3-manifolds, except for the (vi).

In 4-dimensions, we said that $\check{X}$ corresponds to (X, S) where S is the singular set, which is a 2-dimensional structure (or a "foam").

Example 24.2. A foam in $\mathbb{R}^4 \subseteq S^4$ can be

(i) the standard $S^2 \subseteq \mathbb{R}^4$, coming from $\mathbb{R}^4/C_2$, or knotted or linked surfaces,

(ii) the Θ-foam, which is a $D^2 \cup S^2$,

(iii) to the Θ-foam, adding an additional vertical half-disc in the northern hemisphere, which is the double of a cone on a tetrahedron.

We can talk about oriented 4-bifolds with boundary, $\partial\check{X} = \check{Y}$. So we may consider a category where

- objects are closed oriented 3-bifolds,
- morphisms are isomorphism classes of 4-bifolds as cobordism.

For example, we can think of a special case where the objects are webs in $\mathbb{R}^3 \subseteq S^3$, and forms with boundary are in $[0, 1] \times \mathbb{R}^3$.

Example 24.3. A closed 4-form is a morphism from $\emptyset$ to $\emptyset$. If we chop up the Θ-foam into half, we are going to get a morphism $\theta \to \emptyset$.

Note that foams don't have to be oriented.

Example 24.4. If we quotient $\mathbb{C}P^2$ by complex conjugation σ, we are going to get $\mathrm{Fix}(\sigma) = \mathbb{R}P^2$. Then $\mathbb{C}P^2/\sigma = (S^4, S)$ where S is an $\mathbb{R}P^2$.

24.2 Analysis on bifolds

We want to do analysis on orbifolds. First of all, what is a smooth function $f : \check{X} \to \mathbb{R}$? This should mean that if $\varphi_x : \tilde{U}_x \to \mathbb{R}$ is a chart, then $f \circ \varphi_x$ is smooth for all x. Another way to say this is that it is a smooth function

$$f : \check{X}^\circ = \check{X} \setminus \mathrm{Sing}(\check{X})$$

such that each $f \circ \varphi_x$ extends to a smooth function on $\tilde{U}_x$. More generally, we can say that a smooth p-form on $\check{X}$ is a $\omega \in \Omega^p(\check{X}^\circ)$ such that $\varphi_x^*(\omega)$ on $\tilde{U}_x \setminus \varphi_x^{-1}(\mathrm{Sing})$ extends smoothly to $\tilde{\omega} \in \Omega^p(\tilde{U}_x)$. Similarly we can define Riemannian metrics.

A bundle with connection is a (E, A) on $\check{X}^\circ$ such that for all x, the bundle $(\varphi_x^{-1}(E), \varphi_x^*(A))$ on $\tilde{U}_x \setminus \varphi_x^{-1}(\mathrm{sing})$ extends to a bundle with connection $(\tilde{E}, \tilde{A})$ on $\tilde{U}_x$. This makes sense, because if we have a bundle with connection away from some bad set, there is a unique way to extend it. Here, the extended bundle $(\tilde{E}, \tilde{A})$ will have a H_x-action, so H_x should act on $\tilde{E}_x$. In other words, orbifold bundles or rank r come with local data

$$H_x \to G = \mathrm{SO}(r) \text{ or other groups.}$$

We will look at SO(3)-bundles with connection on $\check{X}^4$ and $\check{Y}^3$. Here, we require that for x with $H_x = C_2$, the map $H_x = C_2 \to \mathrm{SO}(3)$ is nontrivial. We are going to call them **bifold bundles with connection**.

In a web, let us see what happens at a vertex. From a bifold bundle, we get a homomorphism

$$\epsilon : H_x = V_4 = \langle 1, a, b, c \rangle \to \mathrm{SO}(3)$$

This should satisfy the requirement that $\epsilon(a), \epsilon(b), \epsilon(c)$ are not 1. Then up to conjugation, it has to be

$$a \mapsto \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

So let us consider

$$\mathcal{B}(Y) = \text{all isom. classes of bifold bundles with connection } (E, A) \text{ on } \check{Y}.$$

Inside there, there is the representation variety $\mathcal{R}(\check{Y}) \subseteq \mathcal{B}(\check{Y})$ of flat conections, which is

$$\mathcal{R}(\check{Y}) = \tilde{\mathcal{R}}(\check{Y}) / \mathrm{SO}(3).$$

Here, we have

$$\tilde{\mathcal{R}}(\check{Y}) = \{\rho : \pi_1(\check{Y}^\circ) \to \mathrm{SO}(3) : \text{ for every edge } e, \rho(m_e) \text{ has order } 2\}$$

where m_e is the meridian around the edge ϵ. Note that we don't have to worry about things happening at vertices, because

$$\langle a, b, c = ab \mid a^2 = b^2 = c^2 = 1 \rangle$$

is the Klein four group.

25 March 28, 2018

We have a web $K \subseteq \mathbb{R}^3 \subseteq S^3$, and we are looking at representations

$$\tilde{\mathcal{R}}(K) = \{\rho : \pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SO}(3) : \rho(m_e) \text{ have order 2 for all } e\}.$$

This sits inside

$$\mathrm{Hom}\Big(\frac{\pi_1(\mathbb{R}^3 \setminus K)}{m_e^2 = 1}, \mathrm{SO}(3)\Big) = \mathrm{Hom}(\pi_1(\check{Y}), \mathrm{SO}(3)).$$

25.1 Fundamental group of a bifold

This $\pi_1(\check{Y})$ is the orbifold fundamental group, which classifies coverings in the orbifold sense. It is going to look like

$$\pi_1(\check{Y}) = \langle m_e \mid m_e^2 = 1, m_{e_1} m_{e_2} m_{e_3} = 1 \text{ at vertices}\rangle.$$

This something similar to the Wirtinger presentation of a knot complement. First we project the K onto $\mathbb{R}^2$, but keep track of over-crossings and under-crossings. Then $\pi_1(\check{Y})$ has generators m_e for each “arc” of the diagram, with relations

$$m_e^2 = 1, \quad m_{e_1} m_{e_2} m_{e_3} = 1, \quad m_{e'} = m_f^{-1} m_e m_f$$

if e and e' is an under crossing of f.

Here, rotations in SO(3) of order 2 are determined by the axis of rotation. So what we have is actually a map

$$\{\text{arcs}\} \to \mathbb{R}P^2.$$

The condition $m_{e_1} m_{e_2} m_{e_3} = 1$ means that the three points $m_{e_1}, m_{e_2}, m_{e_3}$ to three orthogonal points. The condition $m_{e'} = m_f^{-1} m_e m_f$ is going to mean that the axes m_e and $m_{e'}$ have equal spacing between m_f, with them lying on a great circle.

Example 25.1. So let's look at some examples of $\tilde{R}(K)$.

(1) If $K = \emptyset$, then it is $\{1\}$.

(2) If K is an unknot, then it is $\mathbb{R}P^2$. If it is an unlink, it is just $(\mathbb{R}P^2)^{\# c}$.

(3) If K is the θ-web, then $\tilde{\mathcal{R}}(K)$ is the space of 3 orthogonal axes, which is $\mathrm{SO}(3)/V_4 = F_3$.

(4) If K is the tetrahedral web, then $\tilde{\mathcal{R}} = F_3$ because actually the non-meeting arcs are going to be sent to equal points in $\mathbb{R}P^2$.

(5) If K is the Hopf handcuffs, then the complement is homotopic to a punctured torus. Then the representation variety is a pair of axes x_1, x_2 such that the commutator has order 2. This just means that the angle between them is 45°. Then $\tilde{\mathcal{R}} = \mathrm{SO}(3)/\{1, i\} = L(4,1)$ is the space of unit tangent vectors on $\mathbb{R}P^2$.

(6) If K is the "twisted handcuffs" on a unlink, this one is interesting because there is a representation with image the octahedral group. Then $\tilde{\mathcal{R}} = \mathrm{SO}(3)$.

(7) If K is the cube web, we can put i, j, i, j on the outer square. Then this forces four edges to be k, and in the inner square, we can put any i', j', i', j' where i', j', k are orthogonal. But there are three ways to do this. So at the end, you are going to get something like $\mathcal{R} \cong [0,1] \amalg_0 [0,1] \amalg_0 [0,1]$.

(8) If K is just the Hopf link, the two m_{e_1} and m_{e_2} commute. So either $x_1 \perp x_2$ or $x_1 = x_2$. So $\tilde{\mathcal{R}} = F_3 \amalg \mathbb{R}P^2$.

25.2 Anti-self-duality equation on an orbifold

We want to look at anti-self-duality equations on these. Let $\check{X}$ be a closed oriented 4-dimensional bifold, and $S = \mathrm{sing}(\check{X})$. These look like (X, S) where S is a foam. More generally, we can take $\check{X}$ to be an orbifold.

Consider $S^4 \subseteq \mathbb{R}^2 \times \mathbb{R}^3$ and the cyclic group C_m acting on $\mathbb{R}^2$. Then the fixed point set is $S^2 \subseteq \mathbb{R}^3$. The quotient is still going to be a 4-sphere, with $S = S^2$, but with order m. (This is going to be a bifold if $m = 2$.) We know what an orbifold bundle with connection (E, A) is. Put the orbifold-Riemannian metric on $\check{X}$, coming from the round metric. Now we can look at the anti-self-duality equation $F_A^+ = 0$ on $\check{X}$. Define

$$\kappa = \frac{1}{C} \int \mathrm{tr}(F \wedge F)$$

normalized so that $\kappa = 1$ for the standard 1-instanton. Then we have

$$\delta_A = d_A^+ \oplus d_A^* : \Omega^1(X, \mathfrak{g}_E) \to \Omega^+ \oplus \Omega^0,$$

and the index is the formal dimension of the moduli space M, which is $\dim H_A^1 - \dim H_A^0 + \dim H_A^2$.

Example 25.2. Consider the trivial connection. Then $\kappa = 0$, and

$$\mathrm{ind}(\delta_A) = 0 - \dim(\text{stabilizer}) = 0 - 1 = -1.$$

Example 25.3. Now consider the standard instanton, divided by C_m. Here, we get a solution on $\check{X}$, and it is going to have $\kappa = \frac{1}{m}$. Now,

$$\mathrm{ind}(\delta_A) = \dim(\text{moduli}) = 3$$

because we are looking at the C_m-invariants of the original moduli space of instanton, which is B^5.

The reason we are going this is to find out whether we still have this equation

$$\mathrm{ind}(\delta_A) = 8\kappa + i(\check{X}).$$

But we see that κ increases $1/m$, while the index increases by 4. So we are going to have this kind of a thing only for $m = 2$.

26 March 30, 2018

We had looked at various examples of bifolds S^4/H. If $H = 1$, we get the trivial S^4. If $H = C_2$, we get (S^4, S^2). If $H = V_4$, we get a θ-foam. If $H = V_8$, we get a tetrahedral foam, which is like a suspension of a tetrahedral web. A bifold connection is a given (E, A) on S^4 and H-action such that E has fiber $\mathbb{R}^3$, and for $h \in H$ of order 2, on each $s \in \mathrm{Fix}(\langle h \rangle)$, h acts on $E_s = \mathbb{R}^3$ as an actual order 2 element.

26.1 Index between points

We have the 0-instanton on S^4, which is flat on S^4, and also the 1-instanton on S^4, which is $\Lambda^- \to S^4$. Then we can tabulate the index and κ for each of the cases above. Note that $\mathrm{ind}(\delta_A)$ is the dimension of the moduli space minus the dimension of the stabilizer.

H	0-instanton		1-instanton	
	$\mathrm{ind}(\delta_A)$	κ	$\mathrm{ind}(\delta_{A'})$	κ'
1	-3	0	5	1
C_2	-1	0	3	1/2
V_4	0	0	2	1/4
V_8	0	0	1	1/8

Table 1: Dimension and κ

Here, you can notice that

$$\mathrm{ind}(\delta_{A'}) - \mathrm{ind}(\delta_A) = 8(\kappa' - \kappa)$$

in all cases. In general, we claim that in general, if we compare (E, A) and (E', A') on $\check{X} = (X, S)$ then this still holds.

Let me sketch the prove assuming $w_2(E) = w_2(E')$. Note that we can take the orbifold connected sum

$$\check{X} = \check{X}_0 \check{\#} \check{X}_1$$

of two orbifolds. Now assume that

$$(E, A) = (E_0, A_0)\check{\#}(E_1, A_1), \quad (E'_0, A'_0)\check{\#}(E_1, A_1).$$

Then "excision" for index says that

$$\mathrm{ind}(\Delta_{A'}) - \mathrm{ind}(\delta_A) = \mathrm{ind}(\delta'_{A_0}) - \mathrm{ind}(\delta_{A_0}).$$

So we apply it to the case

$$\check{X}_0 = S^4/H, \quad E_0 = (\text{0-instanton})/H, \quad E'_0 = (\text{1-instanton})/H.$$

Because a bundle is determined by its w_2 and c_2, we can change the bundle by these operations. Also, excision and the verification we did above shows that the comparison works out at each step.

Similarly, we can discuss Uhlenbeck compactness for $M_k(X, S)$. This is the same as in the smooth manifold case, with additional point. If the bubble is at $x \in \check{X}$, for $x \in \mathrm{Sing}(\check{X})$, the change in κ should be a multiple of $\frac{1}{|H_x|} = 1, 1/2, 1/4, 1/8$.

Consider now a 3-dimensional bifold $\check{Y}^3 = (Y, K)$. We want to do Morse theory on the space $\mathcal{B}(\check{Y})$ of bifold connections modulo gauge, with the Chern–Simons functional. Take two paths z, z' from α to β. Then we have the relative index $\mathrm{ind}_{z'}(\alpha, \beta) - \mathrm{ind}_z(\alpha, \beta)$. This will be essentially the change in the Chern–Simons functional

$$\mathrm{ind}_{z'}(\alpha, \beta) - \mathrm{ind}_z(\alpha, \beta) = c(\Delta_{z'}(CS) - \Delta_z(CS))$$

up to some constant c. So we will also have, on a closed loop z,

$$\mathrm{spct.\,flow}(z) = 8(\text{change in Morse function}).$$

Now we can start defining instanton Floer homology. Let $\check{Y} = (Y, K)$ be a bifold equipped witha Riemannian orbi-metric. Then we can look at

$$\mathcal{B}(\check{Y}) = \frac{\text{bifold connections}}{(\mathrm{SO}(3)\text{ gauge}}.$$

Then we can look at the Chern–Simons functional $\mathcal{B}(\check{Y}) \to S^1$ with critical points being flat bifold connections $\mathcal{R}(Y, K)$.

Suppose for now that $\mathcal{R}(Y, K)$ consists of connections all of which have trivial stabilizer. (For instance, the twisted handcuffs.) You can check that holonomy perturbations are Morse–Smale. We now define

$$(C, \partial) = \bigoplus_{\alpha} \mathbb{F}_2, \quad \partial\alpha = \sum_{\beta} n_{\alpha\beta}\beta$$

where

$$n_{\alpha\beta} = \#(\overline{M}(\alpha, \beta)_1).$$

We need to check that $\partial^2 = 0$. Here we need to look at the boundary of $\overline{M}(\alpha, \gamma)_2$. Here, we now have to worry about additional noncompactness, because now bubbles can have dimension drop that is 2.

27 April 2, 2018

We want to check that $\partial^2 = 0$. Here, the issue is with bubbling.

27.1 Anti-self-duality on connected sum

Consider $X = X_0 \cup_Y X_1$ where there is a cylindrical neck Y times an interval of length R. As this neck grows longer, anti-self-dual connections will look like A_0 on X_0^+ and A_1 on X_1^+. Here, the connection will look like

$$[A_0|_t] \to [B] \text{ as } t \to \infty, \quad [A_1|_t] \to [B] \text{ as } t \to -\infty.$$

Here, B is going to be a flat connection. This is non-degenerate, but might have stabilizer Γ_B in $\mathcal{G}_Y$, like the trivial connection on S^3.

For $R \gg 0$, we can construct solutions on X by gluing solutions on each side. Choose a gauge so that $A = B + a_0$ where a_0 is exponentially decaying. Now take a cut-off function β and take

$$A_0' = B + \beta_0(t)a_0, \quad A_1' = B + \beta_1(t)a_1.$$

Then we can actually glue the two connections together to get a connection $[A']$, and using Newton's method, we can find an anti-self-dual connection $[A]$ near $[A']$ on X.

Here, we made a choice when we glued these two connection. This gluing map ψ is an element of $\Gamma_B = \mathrm{Aut}(E, B)$.

Proposition 27.1. *A neighborhood of $[A]$ in $M(X, g_R)$ (the moduli space of Riemannian metrics) is isomorphic to $N_0 \times (\Gamma_B) \times N_1$ where N_i is a neighborhood of $[A_i]$ in $M(X_i^+, [B])$ (assuming $\Gamma_{A_i} = \{1\}$, so that the choice of an automorphism on the B side fixes the automorphism on the X_i side).*

Example 27.2. Consider $X = X_0 \# X_1$ and $X_0 = S^4$. Then this is conformal to having a thin neck, and $\Gamma_B = \mathrm{SO}(3)$.

Example 27.3. We can do a similar thing for orbifold connected sum. We can write a 4-bifold $\check{X}$ as

$$\check{X} = (S^4/C_2)\check{\#}\check{X}.$$

Here, we're gluing along $\check{Y} = (S^3, S^1)$, so that the neck will look like $\mathbb{R} \times \check{Y}$. Then the gluing parameter is $\Gamma_B = S^1 = C_{\mathrm{SO}(3)}(C_2)$.

Example 27.4. We can also take connected sum along a seam. We can take

$$\check{X} = (S^4/V_4)\check{\#}\check{X}$$

at a seam of X. Then $\check{Y} = (S^3/V_4)$ is a θ-web, and then $\Gamma_B = V_4 = C_{\mathrm{SO}(3)}(V_4)$.

Consider now an 1-instanton on $\check{X}_0 = S^4/V_4$, which is just the standard 1-instanton divided by V_4. A neighborhood of $[A]$ is then homeomorphic to

$$N_0 \times \Gamma_B \times N_1 = D^2 \times (4 \text{ point}) \times N_1,$$

where N_1 is a neighborhood of $[A_1]$. Conformally, for $R \gg 1$, this looks like a bubble (on a seam) with small radius, and $\kappa = \frac{1}{4}$ at the bubble.

We apply this to a cylindrical $\check{X}$, and here consider $\partial^2 = 0$. Take $\check{X} = \mathbb{R} \times \check{Y}$ and look at the moduli $M_z(\alpha_0, \alpha_2)$ that is 2-dimensional and $\overline{M}_z(\alpha_0, \alpha_2)$ that is 1-dimensional. Here, we regard $\overline{M} \subseteq M$ by normalizing so that the center of mass of $|F_!|^2$ is at $t = 0$.

Recall the old proof of $\partial^2 = 0$. Here, the space $\overline{M}_z(\alpha_0, \alpha_2)$ was an 1-manifold with a compactification

$$\bigcup_{\alpha_1} \overline{M}_{z_0}(\alpha_0, \alpha_1) \times \overline{M}_{\zeta_1}(\alpha_1, \alpha_2).$$

Now we have new ends, with $\kappa = 1/4$ bubbles. In this case, the weak limit is

$$M_{a'}(\alpha_0, \alpha_2)$$

where the index is $\mathrm{ind}_{z'}(\alpha_0, \alpha_2) = \mathrm{ind}_z(\alpha_0, \alpha_2) - 8(\frac{1}{4}) = 0$. This means that $\alpha_2 = \alpha_2$ and z' is the constant path (with the pulled-back flat connection). Then locally,

$$M_z(\alpha_0, \alpha_2) \cong D^2 \times M_{\mathrm{const}}(\alpha_0, \alpha_0) \times V_4.$$

So new solutions actually come in groups of 4. Then we are find because we are working modulo 2. Maybe $\partial^2 = 4\,\mathrm{id}_C$ if we work with $\mathbb{Z}$-coefficients.

27.2 Instanton homology on bifolds is a functor

There is a similar issue if you try to show that 4-dimensional cobordisms give chain maps that are identity on homology. Again, the category is given as objects are closed oriented 3-dimensional bifolds $\check{Y} = (Y, K)$ with the constraint that $\mathcal{R}(\check{Y})$ consists of bifold SO(3)-connections with $\Gamma = (1)$. Morphisms are 4-dimensional oriented bifold cobordisms $\check{X} = (X, S)$ up to isomorphism relative to the boundary.

Proposition 27.5. *Instanton homology (with* SO(3)*-gauge groups) gives a functor* $J : C \to \mathsf{Vect}_{\mathbb{F}_2}$.

A special case of $\check{Y}$ is webs $K \subseteq \mathbb{R}^3 \subseteq S^3$. Then we can consider the category of webs and foams.

Example 27.6. Take the twisted handcuffs THC. Given $K \subseteq \mathbb{R}^3$, we may consider $K \amalg THC$. Because

$$\mathcal{R}(THC) = 1 \text{ point with trivial stablizer},$$

we have

$$\mathcal{R}(K \amalg THC) \cong \tilde{\mathcal{R}}(K) = \{\rho : \pi_1^{\mathrm{orb}}(\mathbb{R}^3, K) \to \mathrm{SO}(3) \mid \rho(m_e) \text{ order } 2\}$$

Now $\mathcal{R}(K \amalg THC)$ always satisfies the condition that $\mathcal{R}(\check{Y})$ has trivial stabilizers. So we can look at a new category

$$\mathcal{C} = \text{all webs in } \mathbb{R}^3$$

with all morphisms beings forms. On this category, we have a functor

$$J^{\#} : \mathcal{C} \to \mathsf{Vect}_{\mathbb{F}_2}.$$

Example 27.7. $J^{\#}(\emptyset) = J(THC) = \mathbb{F}_2$. This is because the complex has one generator.

28 April 4, 2018

Let's pick up where we were. For K or a bifold $\check{Y}$, we have $J(K)$ or $J(\check{Y})$ defined if $\Gamma = (1)$ for all elements of $\mathcal{R}$. But this is not always satisfied. So we used $\check{Y}_{\mathrm{THC}} = (S^3, THC)$ and defined

$$J^{\#}(K) = J(K \amalg THC).$$

Here is a more general formulation. Given a general bifold $(\check{Y}, y_0)$ with basepoint in the smooth part, $y_0 \in \check{Y} \setminus \mathrm{sing}$, and a framing of $T_{y_0}\check{Y}$, we can take $\check{Y} \# \check{Y}_{\mathrm{THC}}$. Then we are defining

$$J^{\#}(\check{Y}) = J(\check{Y} \# \check{Y}_{\mathrm{THC}}).$$

Then $J^{\#}(S^3) = \mathbb{F}_2$. The key part in this definition is that $\check{Y}_{\mathrm{THC}}$ is "atomic", that is, $\mathcal{R}$ is a point and the stabilizer is the point so that $\tilde{\mathcal{R}} = \mathrm{SO}(3)$.

28.1 Atomic bifolds (with markings)

There are other things with this atomic property. We could use T^3 with marking as before, but this is a bit cumbersome.

Example 28.1. Consider θ inside S^3. The representation variety $\mathcal{R}$ is going to be a point, given by $\pi_1(\check{Y}) \to V_4$, but the stabilizer is the stabilizer of V_4, which is V_4.

Now take a marking $\mu = (U_\mu, E_\mu)$ where $U_\mu \cong B^3$ is a neighborhood of θ and E_μ is a trivial bundle. Then we are looking at lifts $\tilde{i}, \tilde{j}, \tilde{k} \in \mathrm{SU}(2)$ such that $\tilde{i}\tilde{j}\tilde{k} = 1$. This now has trivial stabilizer. So this $(\check{Y}_\theta, \mu) = ((S^3, \theta), \mu)$ is a alternative "atom". That is, we may define

$$J^{\#}_\theta(\check{Y}) = J(\check{Y} \# \check{Y}_\theta, \mu).$$

Example 28.2. Another example is the Hopf ring H in S^3. Here, we take the marking data as $U_\mu \cong B^3$ a neighborhood of H and E_μ the bundle which has Stiefel–Whitney the Poincaré dual of a arc w joining the two components. Then the representation variety is a representation variety for the 2-torus with w being a point, which is $\tilde{i}, \tilde{j} \in \mathrm{SU}(2)$ with $[\tilde{i}, \tilde{j}] = -1$. Then this is one point with trivial stabilizer. So we can also define

$$J^{\#}_H(\check{Y}) = J(\check{Y} \# \check{Y}_H, \mu).$$

It turns out that

$$\dim J^{\#}_\theta = \dim J^{\#}_H = \dim J^{\#}$$

for all $\check{Y}$. This is something that works for all atomic bifolds, by general nonsense.

28.2 Grading of instanton Floer homology

In the smooth manifold case, we had a 8-periodic grading on the instanton Floer homology. Let Y be a smooth 3-manifold, and look at $\mathcal{B}(Y)$ the SO(3)-connections up to gauge transformation. For z a closed loop, we looked at the spectral flow of Hess(CS) along z. If $SF(z) \in d\mathbb{Z}$ for all z, then the index ind(α, β) is well-defined modulo d. This happened to be 8 for SU(2)-connections, because

$$\Delta_z(CS) = c \int_{[0,1]\times Y} \mathrm{tr}(F \wedge F) = \kappa.$$

Then by proportionality, we had

$$SF(z) = 8\kappa(z) = 8\Delta_z(CS).$$

On a closed 4-manifold X, we have for $\tilde{E}$ an SU(2)-bundle the Chern class $c_2 = k$. If E an associated SO(3)-bundle, we have $p_1 = -4c_2(\tilde{E})$ if it comes from a SU(2)-bundle. So for an SO(3)-bundle, we have $\kappa = -\frac{1}{4}p_1(E)$.

For an SO(3)-bundle on X, we have $p_1(E)$ and $w_2(E) \in H^2(X; \mathbb{Z}/2)$. There is something called Pontryagin square that lifts to

$$w_2^2 \in H^4(X; \mathbb{Z}/4\mathbb{Z}).$$

Then we have $p_1(E)[X] = w_2(E)^2[X]$ modulo 4.

Corollary 28.3. *If Q_X is even then $p_1(E)$ is even and $\kappa \in \frac{1}{2}\mathbb{Z}$ (otherwise $\frac{1}{4}\mathbb{Z}$). If $Q_X = 0$, then $p_1(E)$ is a multiple of 4 and $\kappa \in \mathbb{Z}$.*

Now Q_X is even if $X = S^1 \times Y$. So $\kappa(z) \in \frac{1}{2}\mathbb{Z}$ and so $SF(z) \in 8\frac{1}{2}\mathbb{Z} = 4\mathbb{Z}$ for all loops z in $\mathcal{B}(Y)$. This means that SO(3) instanton Floer homology only has a $\mathbb{Z}/4\mathbb{Z}$-grading.

But what about bifolds $S^1 \times \check{Y}$? What is the smallest $\kappa(z)$ for z a loop?

Example 28.4. Let's first look at the case where $\check{Y} = (S^3, \text{unknot})$. We have a connection $[A]$ on $S^1 \times \check{Y} = S^1 \times (S^3/C_2)$. Then we can pull back to $[A^*]$ on $S^1 \times S^3$, and here $\kappa(A^*) \in \mathbb{Z}$ because $Q_X = 0$. So $\kappa(A) \in \frac{1}{2}\mathbb{Z}$ and this is $\mathbb{Z}/4$-graded.

Example 28.5. Consider the θ-graph $\check{Y} = S^3/V_4$. Using a similar argument, we similarly get $\kappa \in \frac{1}{4}\mathbb{Z}$ and so it is going to be $\mathbb{Z}/2$-graded.

Example 28.6. In the tetrahedral graph $\check{Y} = S^3/V_8$, we only have $\kappa(z) \in \frac{1}{8}\mathbb{Z}$ and its now $\mathbb{Z}/1$-graded.

29 April 6, 2018

Consider $\check{Y} = (S^3, K)$. For K an unknot, this was $\mathbb{Z}/4$-graded. For K a θ-graph, this was $\mathbb{Z}/2$-graded. For K a tetrahedral graph, this was $\mathbb{Z}/1$-graded. This is related to the fact that θ is bipartite where as the tetrahedral graph is not bipartite. Consider (S^3, H, μ) where $w_2(E_\mu)$ is the Poincaré dual of w an arc connecting the two components. Then $\mathcal{B}(S^3, H, \mu)$ is $\mathbb{Z}/4$-graded.

Let us use now $\check{Y}_0 = (S^3, H, \mu)$ and

$$J^{\#}(\check{Y}) = J(\check{Y} \# \check{Y}_0, \mu).$$

Recall that the dimension of the homology theories are going to be the same. But this gives more grading. For instance, if $\check{Y} = (S^3, \text{knot})$, this is going to be $\mathbb{Z}/4$-graded.

Let us write $\mathcal{R}^{\#}(\check{Y}) = \mathcal{R}(\check{Y} \# \check{Y}_H; \mu) \cong \tilde{\mathcal{R}}(\check{Y})$. Similarly, let us write $\mathcal{B}^{\#}(\check{Y}) = \mathcal{B}(\check{Y} \# \check{Y}_H; \mu)$.

Example 29.1. Let us compute $J^{\#}(\text{unknot})$. Recall that

$$\mathcal{R}^{\#}(\text{unknot}) = \tilde{\mathcal{R}}(\text{unknot}) = \text{Hom}(\pi_1(\check{Y}), \text{SO}(3) : \text{ord}\, m = 2) \cong \mathbb{R}P^2.$$

Now the set of critical points is not discrete. So we need to choose a holonomy perturbation f and look at $CS + f$ instead. We are going to be make this so that $f = O(\epsilon)$ and $f|_{\mathbb{R}P^2}$ is the standard Morse function on $\mathbb{R}P^2$, which has three critical points. After perturbation, there will be three critical points $\alpha_0, \alpha_1, \alpha_2$. You can see that there will be two flowlines from α_2 to α_1 and two flowlines from α_1 to α_0 as in $\mathbb{R}P^2$.

Because the grading is $\mathbb{Z}/4$, we have that

$$\text{ind}_z(\alpha_j, \alpha_i) = j - i + 4d.$$

But $j - i \geq -2$ and so $j - i + 4d = 1$ only when $d = 0$. This means that index 1 trajectories have $\Delta(CS + f) = O(\epsilon)$. Uhlenbeck compactness then tells us that the only trajectories are flowlines of $\text{grad}(f)$. The upshot of this is that

$$J^{\#}(\text{unknot}) \cong H_*(\mathbb{R}P^2; \mathbb{F}_2) = \mathbb{F}_2^3.$$

29.1 Operators on $J^{\#}(K)$

Consider finite-dimensional Morse theory. Suppose I have a double cover $\phi : \hat{B} \to B$, which is a sphere bundle of a $\mathbb{R}$-bundle. Then with $\mathbb{F}_2$-coefficients, I have a Gysin sequence

$$H_j(B) \to H_j(\hat{B}) \to H_j(B) \xrightarrow{u} H_{j-1}(B) \to \cdots$$

where u is given by cap product with $[w_1(L)]$.

Let's describe this in the Morse complex. Take a cycle $\sum n_\alpha \alpha$ and look at $p^{-1}(\alpha) = \{\hat\alpha, \hat\alpha'\}$ in $\hat{\mathcal{B}}$. Pick one and consider $(\partial \sum n_\alpha \hat\alpha) \in C_{j-1}(\hat B)$. Then we have $\partial \sum n_\alpha \hat\alpha = \sum m_\beta(\hat\beta + \hat\beta')$. Then we can define

$$u\Big(\sum n_\alpha \hat\alpha\Big) = \sum m_\beta \beta.$$

Now let us look at $\mathcal{B}(\check{Y})^*$, which is the $\Gamma = 1$ locus. We use ν a marking data, consisting of U_μ a neighborhood around some point $x \in e$, and E_μ the trivial bundle. Then we have

$$\mathcal{B}(\check{Y}, \nu)^* \to \mathcal{B}(\check{Y})^*.$$

This is a double cover, so we get for each $x \in e \subseteq K$, an operator

$$u_e : J^\#(K) \to J^\#(K)$$

of degree -1. Here, C_2 acts on $\tilde{E}_x$ by $(-1, -11)$ so there are $+1$-eigenspaces $\tilde{L}_x \subseteq \tilde{E}_x$.

Using this, we can construct a line bundle

$$\mathbb{L}_2 \to \mathcal{B}(\check{Y})^\#$$

with fiber $\tilde{L}_x$ at $[E, A]$. The double cover is just going to be $S(\mathbb{L}_x)$.

Given a vertex on K with edges e_1, e_2, e_3, each of them gives u_1, u_2, u_3 on $J^\#(K)$. This gives three line bundle $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, and $\mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$ is an SO(3)-bundle. So you can check that $\sum w_i(\mathbb{L}_i) = 0$. Therefore

$$u_1 + u_2 + u_3 = 0.$$

Example 29.2. Let's see what happens to $K =$ unknot. There is only one line bundle L, and it's not hard to see that L restricted to $\mathbb{R}P^2$ is nontrivial. So we are taking cap product with $w_1(L) \in H^1(\mathbb{R}P^1; \mathbb{F}_2)$. This is going to send the generator α_2 to α_1, and the generator α_1 to α_0.

Suppose we have a functor S from K to K'. Given a point $x \in \mathrm{Face}(S)$, we can take the composite

$$T : J^\#(K) \to J^\#(K_{1/2}) \xrightarrow{u_x} J^\#(K_{1/2}) \to J^\#(K').$$

This composite depends on the face only. So "foams with dots" provide morphisms in an enriched category.

30 April 11, 2018

For Θ the Θ-foam, we are trying to evaluate

$$J^{\#}(\Theta) : J^{\#}(\emptyset) = \mathbb{F}_2 \to J^{\#}(\emptyset) = \mathbb{F}_2.$$

More generally, for a, b, c the number of dots on the three faces, we want to evaluate $J^{\#}(\Theta(a, b, c)) \in \mathbb{F}_2$.

30.1 Computation of the operators

Let $\check{Y}_0 = ((S^3, H); \mu)$ be an atom. Then we are looking at $\check{X}^+ = (\mathbb{R} \times \check{Y}_0)\#(S^4, \Theta)$. Then the moduli space is $M_\kappa = M_\kappa(\alpha_0, \check{X}, \alpha_0)$ for the unique α_0 and $\kappa \in \frac{1}{4}\mathbb{Z}$. Then we can write

$$J^{\#}(\Theta(a, b, c)) = \#(M_\kappa \cap W_1 \cap \cdots \cap W_{a+b+c})$$

where each W_i is codimension 1 which is Poincaré dual to $w_1(\mathbb{L}_i)$.

M_0 is the flat connections, and this is

$$\begin{aligned} &\{\rho \in \mathrm{Hom}(\pi_1(S^4 \setminus \Theta), \mathrm{SO}(3)) : \rho(m) \text{ order } 2, \dots\} \\ &\quad = \{(i, j, k) : \text{ orthogonal lines in } \mathrm{SO}(3)\} = \mathrm{SO}(3)/V_4 = F(3). \end{aligned}$$

If any of $a, b, c \geq 3$, then we get $J^{\#}(\Theta(a, b, c)) = 0$ by the knot argument we gave last time. For $a + b + c$, we have tautological line bundles L_1, L_2, L_3 on $F(3)$, corresponding to classes w_1, w_2, w_3. Then direct analysis gives

$$J^{\#}(\Theta(a, b, c)) = (w_1^a w_2^b w_3^c)[F(3)].$$

Because $w_j(L_1 \oplus L_2 \oplus L_3) = 0$, we can compute these.

Corollary 30.1. *$J^{\#}(\Theta(a, b, c)) = 1$ if $(a, b, c) = (2, 1, 0)$ or permutations, and 0 otherwise.*

We can do a finer analysis. We can decompose the operator into the first half and the second half, and consider $J^{\#}(\Theta(a, b, c))$ as a composition of

$$J^{\#}(\Theta^+(a, b, c)) \in J^{\#}(\theta), \quad J^{\#}(\Theta^-(a', b', c')) \in J^{\#}(\Theta)^*.$$

Then the paring is given by $\Theta(a + a', b + b', c + c')$. You can check that $J^{\#}(\Theta^+(a, b, c))$ for $0 \leq a \leq 2$ and $0 \leq b \leq 1$ and $c = 0$ are independent.

Corollary 30.2. $\dim J^{\#}(\theta) \geq 6$.

But actually we also know that the dimension is at most 6. The reason is that

$$\mathcal{R}^{\#}(S^3, \theta) \cong \tilde{\mathcal{R}}(S^3, \theta) \cong F(3)$$

and there is a Morse function $F(3) \to \mathbb{R}$ with 6 critical points.

Corollary 30.3. *$J^{\#}(\theta) \cong H_*(F(3);\mathbb{F}_2)$ as vector spaces. Moreover, u_i acts as $\frown w_i$ for $i = 1, 2, 3$.*

More generally, let S be a foam-cobordism from K_0 to K_1, and let f_1, f_2, f_3 be faces incident along a seam. Let $S(a, b, c)$ be S with dots a, b, c on these three faces. Then

- $S(3, 0, 0) = 0$,
- $S(1, 0, 0) + S(0, 1, 0) + S(0, 0, 1) = 0$,
- $S(1, 1, 0) + S(1, 0, 1) + S(0, 1, 1) = 0$,
- $S(1, 1, 1) = 0$,
- all of these after adding auxiliary dots.

Proof. We know them for $S = \Theta^+$ and $K_0 = \emptyset$ and $K_1 = \theta$. In the general case, we can manipulate the cobordism so that instead of $K_0 \to K_{1/2} \to K_1$ we have $K_0 \to K_{1/2} \amalg \Theta \to K_1$. The first map can be thought of as $J^{\#}(S) \otimes J^{\#}(\Theta(a, b, c))$. Then we check this for $\Theta(a, b, c)$. □

30.2 Nonvanishing theorem for webs

Theorem 30.4. *We have $J^{\#}(K) \neq 0$ for any web $K \subseteq \mathbb{R}^3$, provided that K is not 1-splittable (i.e., there does not exist a 2-sphere meeting K transversely at one point).*

This 1-splitting is clearly necessary, because if there is this 1-splitting edge, then the meridian should be order 2 and trivial, so $\mathcal{R}(K) = \emptyset$. This dimension $\dim J^{\#}(K)$ has some properties in common with the number of **Taie colorings** of K. These are edge 3-colorings

$$\{(c : \text{Edges} \to \{1, 2, 3\}) : c(e) \neq c(e') \text{ if } e, e' \text{ meet at a vertex}\}.$$

Theorem 30.5. *If K is planar, i.e., $K \subset \mathbb{R}^2 \subset \mathbb{R}^3$, then $\dim J^{\#}(K) \geq$ #Tait colorings.*

31 April 13, 2018

Given a bifold $\partial P^4 = Q^3$ with boundary, we can puncture it can consider it as

$$J^\#(P \setminus B^4) : \mathbb{R}_2 \to J^\#(Q).$$

So we can write this as $J^\#(P) \in J^\#(Q)$.

31.1 Relations between operators

Given a cobordism X_1 between Y_1 and Y_1', and a cobordism X_2 between Y_2 and Y_2', suppose that there is a cobordism between $Y = Y_1 \# Y_2$ and $Y' = Y_1' \# Y_2'$ that comes from gluing X_1 and X_2 together. Then excision tells us that

$$\begin{array}{ccc} J^\#(Y) & \xrightarrow{\quad X \quad} & J^\#(Y') \\ \downarrow \cong & & \downarrow \cong \\ J^\#(Y_1) \otimes J^\#(Y_2) & \xrightarrow{X_1 \otimes X_1'} & J^\#(X_1) \otimes J^\#(X_2) \end{array}$$

commutes.

So if we have a cobordism $X : Y \to Y'$, and P is a region inside X with $Q = \partial P$, we can factor it as

$$Y = S^3 \# Y \xrightarrow{Z} Q \# Y \to Y'$$

where Z is a split cobordism to $S^3 \to Q$ and $Y \to Y$.

Corollary 31.1. *$J^\#(X)$ factors through $J^\#(P) \otimes 1 : \mathbb{F}_2 \otimes J^\#(Y) \to J^\#(Q) \otimes J^\#(Y)$.*

Corollary 31.2. *If $J^\#(P) = 0$ in $J^\#(Q)$ then $J^\#(X) = 0$.*

In particular, if the cobordism X contains dots, we see that $\sum J^\#(P_i) = 0 \in J^\#(Q)$ implies $\sum J^\#(X_i) = 0$.

Suppose our bifold (X, S) is such that the foam S contains some $I \times S^1 \subseteq B^3 \subseteq B^4 \subseteq X$. Then we can cut this neck and call that S'. We can put dots on S' on either sides, and look at $S'(n, m)$.

Proposition 31.3. *$J^\#(S) = J^\#(S'(2, 0)) + J^\#(S'(1, 1)) + J^\#(S'(0, 2))$.*

Proof. We can check this for $S = I \times S^1$ in B^4. We can check this manually. □

There is also the bubble busting relation. Suppose S looks locally like a soap bubble with $S^2 \cup (\mathbb{R}^2 \times \{0\} \setminus B^2$. We want to compare this to just $\mathbb{R}^2 \setminus \{0\}$, so that we are left with a burst bubble. We can look at $S(k_1, k_2)$ which has k_1 dots on the upper hemisphere and k_2 dots on the lower hemisphere, and we can also look at $S'(k)$ which has k dots on the plane. Then the relations are

- $S(1, 0) = S'(0)$

- $S(2,0) = S'(1)$
- $S(2,1) = S'(0)$
- $S(1,1) = 0$
- $S(0,0) = 0$.

To check these relations, we can again just check equality in $J^{\#}(L_1)$ where L_1 is the unlink. We know that $J^{\#}$ is a 3-dimensional vector space, so we check the pairing with the dual basis, which are given by $D(n)$. Then we can manually check them.

Using these, we can now come back to the case of webs. Let K be a web that at some point looks locally like $S^1 \cup (\mathbb{R} \times \{0\} \setminus B^2)$. Then we can compare this with K' that just looks like $\mathbb{R} \times \{0\}$.

Proposition 31.4. $J^{\#}(K) \cong J^{\#}(K') \oplus J^{\#}(K')$.

This is why $J^{\#}(\Theta)$ has dimension 6 for 3-dimensional $J^{\#}(\text{unknot})$. If we look at two bigons on an unknot, we can directly see from the representation variety that $\dim \leq 12$. But this actually shows that the dimension is equal to 12.

Consider cobordisms $A, B : K \to K'$ with A no dots and B one dot on the front foam. Likewise consider $C, D : K' \to K$ such that C has one dot on the back and D has no dots. Then we may consider maps

$$J^{\#}(K') \oplus J^{\#}(K') \xrightarrow{\begin{pmatrix} C & D \end{pmatrix}} J^{\#}(K) \xrightarrow{\begin{pmatrix} A \\ B \end{pmatrix}} J^{\#}(K') \oplus J^{\#}(K').$$

We check that they are mutually inverses. To check that the composition is the identity on $J^{\#}(K') \oplus J^{\#}(K')$, we use the bubble bursting relations. The other thing we need to check is that $AC + BD$ is the identity on $J^{\#}(K)$. But we can check this on $J^{\#}$ of the two-bigon unknot. Because we know how to generate the dual of $J^{\#}$, we can check this.

32 April 16, 2018

Last time we talked about the bigon relation, which says that the dimension doubles when you introduce a bigon. There are similar relations.

32.1 More relations

Proposition 32.1 (square relation)**.** *For K a that locally looks like a square, we have*

$$J^{\#}(K) = J^{\#}(K') \oplus J^{\#}(K'')$$

where K' is one that removes two edges of the square and K'' is one that removes other two edges of the square.

Proposition 32.2 (triangle relation)**.** *For K that locally looks like a triangle, we have*

$$J^{\#}(K) = J^{\#}(K')$$

where K' is one that removes an edge of a triangle.

Note that for the triangle relation, we have $\mathcal{R}^{\#}(K) \cong \mathcal{R}^{\#}(K')$. For the square relation, we again define maps

$$A' : J^{\#}(K) \to J^{\#}(K'), \quad A'' : J^{\#}(K) \to J^{\#}(K'')$$

by looking at a cobordism and likewise B', B' by their mirror images. The claim is that

$$\begin{pmatrix} A' \\ A'' \end{pmatrix} \begin{pmatrix} B' & B'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} B' & B'' \end{pmatrix} \begin{pmatrix} A' \\ A'' \end{pmatrix} = 1.$$

The first you can verify explicitly using bubble bursting and neck cutting. Then for the other direction, we look at bounds on the dimension as we did for the bigon relations. Recall that the representation variety of the cube web looks like $I \amalg_0 I \amalg_0 I$. If you look at a perturbation, there are going to be four critical points, and you can work this out pretty explicitly. Then $\mathcal{R}^{\#}(K)$ is going to be 4 copies of $F(3)$, and so there will be 24 critical points.

All these relations hold for Tait colorings of trivaletn graphs as well. If we denote by $\mathrm{Tait}(K)$ the number of Tait colorings, we have the

- bigon relation,
- triangle relation,
- square relation.

These rules alone do not determine the Tait colorings. It is determined by

- multiplicative for disjoint unions,
- 0 for K which is 1-splitable,
- the unknot has 3,
- the Tutte relation.

The Tutte relation does not hold, because then $\dim J^{\#}(K) = \text{Tait}(K)$ for all K, which can't be true because the number of Tait colorings do not depend on the embedding. What is possibly true is that the Tutte relation holds if K lies in a plane. In fact, one can prove that $\dim J^{\#}(K) \geq \text{Tait}(K)$ for planar K.

33 April 18, 2018

The following is Floer's version of excision.

33.1 Excision

Suppose Y' and Y'' are two T^3 with T_1 and T_2 be two T^2 sections. Then we can form Y by cutting Y' and Y'' along T_1 and T_2, and then gluing.

Consider X that is a cobordism from Y to Y', and assume that T^3 is embedded in X. Inside $T^3 = T^2 \times \gamma$, there is a path $w_{T^2} = \text{point} \times \gamma$. Then we can look at the representation variety

$$\mathcal{R}(T^2 \times \gamma, \mu)^w = \text{point, no automorphism}$$

where $U_\mu = T^3$ and the representation variety will be $[\tilde{a}, \tilde{b}] = -1$ in SU(2) and $\tilde{c} = \pm 1$ in SU(2). Now inside X, we have $T^2 \times V \subseteq U_\nu$ a neighborhood of $T^2 \times \gamma$. Here, w is points times Σ in this neighborhood. The surface $T^2 \times \gamma$ cuts V into two parts, and let us call these two pieces V^+ and V^- in U_ν.

In the normal situation we have a map

$$J(X; \nu)^w : J(Y; \mu)^w \to J(Y'; \mu')^w.$$

But if we cut along $T^2 \times \gamma$ (which is T^3) we get a cobordism from $T^3 \amalg T^3 \amalg Y$ to Y'. Then we get a map

$$J(Y; \mu)^w \otimes (J(T^3; \mu_{T^3})^{w_{T^3}})^{\otimes 2} \to J(Y'; \mu')w.$$

We can also think of it as the operator after we attach caps to the two copies of T^3, by $\partial(T^2 \times D) \cong T^3$. Let us call this new manifold X_1 (which is a result of some surgery on X) with the unique extension of marking data ν_1. Then we can write

$$J(X, \nu)^w = J(X_1, \nu_1)^w.$$

Now you can see why this is called excision. Suppose Y is a 3-manifold and it looks like the disjoint union of $Y_a \cup Y_b$ attached along T^2 and also $Y_c \cup Y_d$ attached along T^2. Assume the marking region is U_μ which are small neighborhoods of the attaching T^2, with w being paths transversing along the region U_μ. Now consider Y' which looks similarly, but with $(Y_a \cup Y_d) \amalg (Y_b \cup Y_c)$.

Theorem 33.1 (excision). *$J(Y; \mu)^w = J(Y_a \cup Y_b) \otimes J(Y_c \cup Y_d) = J(Y_a \cup Y_d) \otimes J(Y_b \cup Y_c) = J(Y'; \mu')$.*

So we see that $\dim J(Y_a \cup Y_b) / \dim J(Y_c \cup Y_b)$ is independent of Y_b. To prove this, we consider the following cobordism. First take an octagon Σ^+, with four disjoint edges labeled a, b, c, d in an appropriate order, and take

$$X^+ = (T^2 \times \Sigma^+) \cup (I \times Y_a) \cup (I \times Y_b) \cup (I \times Y_c) \cup (I \times Y_d).$$

This is a cobordism from $Y = (Y_a \cup Y_b) \amalg (Y_c \cup Y_d)$ to Y'. The marking data is going to be U_{μ^+} being the entire region and w extended appropriately. Then we get

$$J(X^+; \mu^+) : J(Y) \to J(Y'), \quad J(X^-; \mu^-) : J(Y') \to J(Y).$$

The composite is going to be given by $X = X^+ \cup X_-$. Here, we use the marking data $\tilde{\mu}_X \subseteq \mu_X = \mu^- \cup \mu^+$ given by deleting the neighborhood of $T^2 \times \delta$. Then by the observation for 4-manifold we have made, we get

$$J(X^-; \mu^-)^w \circ J(W^+; \mu^+)^w = J(X; \tilde{\mu}_X) = J(X_1; \tilde{\mu}_1)^w = \mathrm{id}\,.$$

Note that if we take (S^3, H) as Y, and look at the w connecting the two components, this is precisely our atom $\check{Y}_H = (S^3, H, \mu)$. So we get

$$J^\#(\emptyset) \otimes J^\#(K_1 \amalg K_2) \cong J^\#(K_1) \otimes J^\#(K_2)$$

by excision.

34 April 20, 2018

We recall that for the Hopf link and w connecting the two, $J^\#$ is $\mathbb{Z}/4\mathbb{Z}$-graded.

34.1 Deformations of $J^\#$

Consider $S = \mathbb{F}_2[[t]]$. We consider a complex

$$(C^\#(Y)_S, \partial_S)$$

of free S-modules. Put $t = 0$ in this complex, and we can recover the original chain complex $(C^\#(Y), \partial)$ giving $J^\#$.

Consider a local system Γ of R-modules. This means that R is a ring (of characteristic 2) and for each $b \in B$, there is Γ_b an R-module. For ζ a path from a to b, there is an isomorphism $\Gamma_\zeta : \Gamma_a \to \Gamma_b$ only depending on the homotopy class, that is compatible with composition.

In Morse theory, we can define

$$C(f; \Gamma) = \bigoplus_\alpha \Gamma_\alpha.$$

If we look at a flowline $z : \alpha \to \beta$, we can look at the induced map $\Gamma_z : \Gamma_\alpha \to \Gamma_\beta$. Using this, we can define the differential as the matrix with $\alpha \to \beta$ entry given by

$$\sum_z \Gamma_z(\#\overline{M}_z(\alpha, \beta))$$

where z runs over index 1.

Example 34.1. Consider $R = \mathbb{F}_2[T, T^{-1}]$ and let us look at a local system over $S^1 = \mathbb{R}/\mathbb{Z}$. Then a local system is the same as a map $\pi_1(S^1) \to R^\times$, and in particular let us take the generator to T. Here is an explicit description. The ring $R = \mathbb{F}_2[\mathbb{Z}]$ is a subring of $S = \mathbb{F}_2[\mathbb{R}] = \sum a_r T^r$ for $r \in \mathbb{R}$. Then for $r \in \mathbb{R}$, the set $T^r \cdot R \subseteq S$ is a free rank-1 R-module. On $S^1 = \mathbb{R}/\mathbb{Z}$, we define $\Gamma_{\bar{x}} = T^x \cdot R$. Given a path z from $\bar{x}$ to $\bar{y}$, there is a well-defined change in the coordinate. Define $\Gamma_z : \Gamma_a \to \Gamma_b$ by multiplication by T^{y-x}.

We can look at

$$\mathcal{B}^\#(Y) = \mathcal{B}(Y \# Y_\theta, \mu_\theta)$$

where Y_θ and the marking data is the θ-atom. We are going to define a map $h : \mathcal{B}^\#(Y) \to S^1$ and pull back Γ.

Let us construct h for $\mathcal{B}^\#(S^3) = \mathcal{B}(Y_\theta, \mu_\theta)$. This is the V_4-quotient of S^3 withe orthogonal axes in $\mathbb{R}^3 \cup \{\infty\}$. If we take e_1, the bundle $\tilde{E}$ restricted to $\tilde{e}_1$, it is going to split as $\tilde{E}|_{\tilde{e}_1} = \mathbb{R} \oplus \mathbb{K}$. Then we look at the holomony

$$\mathbb{K}_0 \to \mathbb{K}_\infty$$

and look at the rotation angle. This define a map $h : \mathcal{B}^\#(Y) \to S^1$.

35 April 23, 2018

We defined Floer homology with coefficients in a local system $J^\#(K;\Gamma)$, where Γ is pulled back from S^1 along the map $h : \mathcal{B}^\# \to S^1$. For $\check{Y}_0 = (S^3, \theta)$ the θ-graph, edges e_1, e_2, e_3, we defined h by looking at the holonomy of $\mathbb{R}\oplus\mathbb{K}$ along the edges $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$. Consider $h_i : \mathcal{B}^\# \to S^1$ which looks at the holonomy around $\tilde{e}_i$. Then we set $h = h_1 + h_2 + h_3$.

35.1 Properties of homology with local coefficients

For a path ζ in $\mathcal{B}^\#$, this can be thought of as a connection A_ζ on $\theta \times \mathbb{R}$. It can be thought of as a connection $\tilde{A}_\zeta$ in $\mathbb{R} \oplus \mathbb{K}$, and then we get

$$\Gamma_\zeta = T^{\Delta_\zeta h} = \prod_1^3 T^{\Delta_\zeta} h_i$$

where $\Delta_\zeta h_i$ is the change in holonomy $\frac{i}{2\pi}\int_{R\times\tilde{e}_i} F_{\mathbb{K}}$.

Using this, we can construct $(C^\#(K;\Gamma), \partial)$. This satisfies:

- $\partial^2 = 0$,
- $J^\#(\text{unknot};\Gamma)$ is a free rank 3 module,
- $J^\#(\theta;\Gamma)$ is a free rank 6 module,
- $u^3 + Pu = 0$ where $P = T^3 + T^{-1}$ (on unknot and so at any edge),
- $u_1 + u_2 + u_3 = 0$, $u_1u_2 + u_2u_3 + u_3u_1 = P$, $u_1u_2u_3 = 0$ (on θ so at any vertex).

To show that $\partial^2 = 0$ with the local coefficients, recall that we had these normal ends that are broken trajectories and also the bubbles. The broken ends contribute to

$$\sum_z (\#\text{ of ends of } \overline{M}_z)\Gamma_z = 0$$

and so there is nothing on the broken ends. For the bubbles, we only need to worry about $\kappa = \frac{1}{4}$. But it can only happen for $\alpha_0 = \alpha_2$ with counting by $\overline{M}_2(\alpha_0, \alpha_2)_2$. This at least shows that ∂^2 is a diagonal operator with the standard basis. If we can show that this diagonal entry is universal, it suffices to compute the coefficient only for $K = \emptyset$.

For $u^3 + Pu = 0$, again we only need to check this for the unknot. But because u is an operator on a rank 3-module, and u is odd, we have $u^3 + Qu = 0$ for some $Q \in R$. For an unknot, we have $u(u^2 + P) = 0$, and in the field of fractions R' or R, we have $1 = \pi + \pi' = \frac{1}{P}(u^2 + P) + (\frac{u}{P})u$. Each of these terms lie in $\ker(u)$ and $\operatorname{im}(u)$.

For the three θ-relations, the first one is elementary. Recall that at a vertex, $\tilde{E}$ looks locally like $\tilde{E} = L_1 \oplus L_2 \oplus L_3$. Here, $u_i = w_1(L_i)$ formally, and $u_1u_2 + u_2u_3 + u_3u_1 = w_2(\tilde{E})$ formally. You can actually compute and then you get $P = T^3 + T^{-1}$.

36 April 25, 2018

Last time we had Γ a system of $R = \mathbb{F}_2[T, T^{-1}]$-modules, where $R' = \mathrm{Frac}(R)$. Then we define $J^{\#}(K, \Gamma)$ as an R-module, and $J^{\#}(K, \Gamma \otimes R')$ as an R'-vector space.

36.1 Instanton Floer homology with coloring

We can write $J^{\#} \otimes R' = V \oplus V'$ where $V = \ker(u_e)$ and $V' = \ker(u_e^2 + P)$. check
Given any subset $S \subseteq \mathrm{Edges}(K)$, we can define

$$W(S) = \bigcap_{e \in S} V_e \cap \bigcap_{e \notin S} V'_e$$

so that

$$J^{\#}(K, \Gamma \otimes R') = \bigoplus_S W(S).$$

We say that for u_1, u_2, u_3 around a vertex, $J^{\#} \otimes R' = V_1 \oplus V_2 \oplus V_3$. Here, $V'_1 = V_2 \oplus V_3$ and so on. In particular, we have

$$V'_1 \cap V'_2 \cap V'_3 = 0, \quad V_1 \cap V_2 = 0, \ldots.$$

This shows that in order for $W(S)$ to be nonzero, at each vertex, we need that there is exactly one edge incident to it that is in S. That is, S has to be a 1-set of the graph, and its complement has to be a 2-set. These are the possibly nonzero parts.

If we look at the bigon relation, there are going to be three ways we can color the bigon. There are two ways you can color the picture so that the ends are both not in S, and this will correspond to just the simple edge that is not in S. So if we analyze this picture, we have that in R'-coefficients, $W(K) \cong W(K')$ where K' is obtained from K by attaching an S-edge onto a non-S-edge.

Similarly, we can show that if you have 4 points and S-edges between two of them, you can move the edges around and get another way of trying S-edges between the 4 points. The W for both of these graphs are going to be equal. One way you can do this to use a 3-periodic long exact sequence and check that the third term corresponds to 0 in the appropriate W. Another way you can do this is to look at the corresponding cobordism and show that it is the identity map.

In any case, this new relation shows that for the S-edges, only its homology class matters. At each non-S-cycle, we can make sure there is at most one S-edge sticking out, because we can cancel off two edges using the relations described above. In particular, if K is planar, no non-S-cycles are going to be linked to each other. Note that if there is at least one non-S-cycle with one S-edge, then we have a splitable handcuff and so $W(S) = 0$. Otherwise we are going to have a disjoint union of non-S-cycles, and the dimension of $W(S)$ is going to be 2^n where n is the number of components.

Proposition 36.1. *If K is planar, we have $W(S) = 0$ unless S is an even 1-set (i.e., the non-S-edges form even cycles). In this case* $\dim W(S) = 2^n$ *where n is the number of non-S components.*

This shows that for planar K,

$$\dim J^{\#}(K; \Gamma \otimes R') = \sum_{\text{even 1-set } S} \dim W(S) = \sum_{S} 2^{n(S)}.$$

The right hand side can be recognized as the number of Tait colorings of K, because $2^{n(S)}$ is the number of ways to color the non-S-edges into two colors.

So $J^{\#}(K; \Gamma \otimes R')$ categorifies Tait colorings of planar graphs. It is unknown if dimension of $J^{\#}(K)$ is the number of Tait colorings. There is a map $R \to \mathbb{F}_2$ given by $T \mapsto 1$. Then $J^{\#}(K; \Gamma)$ is going to look like R^{rank} plus some torsion parts. If you work out, you will see that

$$J^{\#}(K; \mathbb{F}_2) = \mathbb{F}_2^{\text{Tait}(K)} \oplus (\mathbb{F}_2)^{2\ell}$$

for some $\ell \geq 2$. The question is whether this interesting torsion behavior does appear.

Index